TWO-SIDED ASYMPTOTIC BOUNDS FOR THE COMPLEXITY OF CYCLIC BRANCHED COVERINGS OF TWO-BRIDGE LINKS

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Abstract. We consider closed orientable 3-dimensional hyperbolic manifolds which are cyclic branched coverings of the 3-sphere, with branching set being a two-bridge knot (or link). As the order of the covering tends to infinity we establish two-sided linear asymptotic bounds for the Matveev complexity of the covering manifold. The lower estimate uses the hyperbolic volume and results of Cao-Meyerhoff and Lackenby, while the upper estimate is based on an explicit triangulation, which also allows us to give a bound on the Delzant T-invariant of the fundamental group of the manifold.

1. Definitions, motivations and statements

Complexity Using simple spines (a technical notion from piecewise linear topology that we will not need to recall in this paper), Matveev [15] introduced a notion of *complexity* for compact 3-dimensional manifolds. If M is such an object, its complexity $c(M) \in \mathbb{N}$ is a very efficient measure of "how complicated" M is, because:

- every 3-manifold can be uniquely expressed as a connected sum of prime ones (this is an old and well-known fact, see [10]);
- c is additive under connected sum;
- if M is closed and prime, $c(M)$ is precisely the minimal number of tetrahedra needed to triangulate M.

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In the last item the notion of triangulation is only meant in a loose sense, namely just as a gluing of tetrahedra along faces, and an exception has to be made for the four prime M's for which $c(M) = 0$, that is S^3, $\mathbb{RP}^3$, $S^2 \times S^1$, and $L(3,1)$.

Computing exactly the complexity $c(M)$ of any given 3-manifold M is theoretically very difficult, even if quite easy experimentally, using computers [17]. In the closed prime case the state of the art is as follows:

- A computer-aided tabulation of the closed M's with $c(M) \leqslant 12$ has been completed in various steps [13, 17, 18] (see also [16]);
- A general lower bound for $c(M)$ in terms of the homology of M was established in [19];
- Asymptotic two-sided bounds for the complexity of some specific infinite series of manifolds were obtained in [20, 23];
- A conjectural formula for the complexity of any Seifert fibred space and torus bundle over the circle was proposed (and proved to be an upper bound) in [14].

Several other results, including exact computations for infinite series, have been obtained in the case of manifolds with non-empty boundary, see [2, 7, 8, 9]. Since we will stick in this paper to the closed case, we do not review them here.

Using the hyperbolic volume and deep recent results of Lackenby [12], together with explicit triangulation methods to be found [21, 22], we will analyze in this paper the complexity of cyclic coverings of the 3-sphere branched along two-bridge knots and links. More specifically, we will prove asymptotic two-sided linear estimates for the complexity in terms of the order of the covering. Before giving our statements we need to recall some terminology.

Two-bridge knots and links If p, q are coprime integers with $p \geqslant 2$ we denote by $K(p,q)$ the two-bridge link in the 3-sphere S^3 determined by p and q, see [4, 11, 22]. It is well-known that $K(p,q)$ does not change if a multiple of $2p$ is added to q, so one can assume that $|q| < p$. In addition $K(p,-q)$ is the mirror image of $K(p,q)$. Therefore, since we will not care in the sequel about orientation, we can assume $q > 0$.

Summing up, from now on our assumption will always be that the following happens:

$$(1) \qquad p,q \in \mathbb{Z}, \qquad p \geqslant 2, \qquad 0 < q < p, \qquad (p,q) = 1.$$

We recall that if p is odd then $K(p,q)$ is a knot, otherwise it is a 2-component link; moreover, two-bridge knots and links are alternating [4, p. 189]. Planar alternating projections of $K(p,q)$ will be shown below. Since we are only interested in the topology of the branched coverings of $K(p,q)$, we regard it as an unoriented knot (or link), and we define it to be equivalent to some other $K(p',q')$ if there is an automorphism of S^3, possibly an orientation-reversing one, that maps $K(p,q)$ to $K(p',q')$. It is well-known (see [4, p. 185]) that $K(p',q')$ and $K(p,q)$ are equivalent if and only if $p' = p$ and $q' \equiv \pm q^{\pm 1} \pmod p$.

Under the current assumption (1), the two-bridge knot (or link) $K(p,q)$ is a torus knot (or link) precisely when q is 1 or $p-1$, and it is hyperbolic otherwise. The simplest non-hyperbolic examples are the Hopf link $K(2,1)$, the left-handed trefoil knot $K(3,1)$ and its mirror image $K(3,2)$, the right-handed trefoil (but we are considering a knot to be equivalent to its mirror image, as just explained). The easiest hyperbolic $K(p,q)$ is the figure-eight knot $K(5,2)$.

Branched coverings If $K(p,q)$ is a knot (*i.e.* p is odd) and $n \geqslant 2$ is an integer, the n-fold cyclic covering of S^3 branched along $K(p,q)$ is a well-defined closed orientable 3-manifold that we will denote by $M_n(p,q)$. One way of defining it is as the metric completion of the quotient of the universal covering of $S^3 \setminus K(p,q)$ under the action of the kernel of the homomorphism $\pi_1(S^3 \setminus K(p,q)) \to \mathbb{Z}/_{n\mathbb{Z}}$ which factors through the Abelianization $\pi_1(S^3 \setminus K(p,q)) \to H_1(S^3 \setminus K(p,q))$ and sends a meridian of $K(p,q)$, which generates $H_1(S^3 \setminus K(p,q))$, to $[1] \in \mathbb{Z}/_{n\mathbb{Z}}$.

If $K(p,q)$ is a link and a generator $[m]$ of $\mathbb{Z}/_{n\mathbb{Z}}$ is given, a similar construction defines the *meridian-cyclic* branched covering $M_{n,m}(p,q)$ of S^3 along $K(p,q)$, by requiring the meridians of the two components of $K(p,q)$ to be sent to $[1]$ and $[m] \in \mathbb{Z}/_{n\mathbb{Z}}$ respectively. Note that meridian-cyclic coverings are also called *strongly cyclic* in [28]. Since the two components of $K(p,q)$ can be switched, we do not need to specify which meridian is mapped to [1] and which to $[m]$. Since in the sequel we will prove estimates on the complexity of $M_{n,m}(p,q)$ which are asymptotic in n and apply to every $M_{n,m}(p,q)$, with a slight abuse we will simplify the notation and indicate by $M_n(p,q)$ an arbitrary meridian-cyclic n-fold covering of S^3 branched along $K(p,q)$. This will allow us to give a unified statement for knots and links. We recall that $M_2(p,q)$ is the lens space $L(p,q)$.

Continued fractions In the sequel we will employ continued fractions, that we define as follows:

$$[a_1, a_2, \ldots, a_{k-1}, a_k] = a_1 + \cfrac{1}{a_2 + \cdots + \cfrac{1}{a_{k-1} + \cfrac{1}{a_k}}}.$$

Given p,q satisfying (1), we now recall [11, p. 25] that there is a unique minimized expansion of p/q as a continued fraction with positive entries, namely an expression as $p/q = [a_1, \ldots, a_k]$ with $a_1, \ldots, a_{k-1} > 0$ and $a_k > 1$. (The expansion is called *minimized* because if $a_k = 1$ then $[a_1, \ldots, a_{k-1}, 1] = [a_1, \ldots, a_{k-1} + 1]$, as one easily sees.) We then define $\ell(p,q)$ to be k if $a_1 > 1$ and $k-1$ if $a_1 = 1$.

This apparently original definition of $\ell(p,q)$ is explained by the following result established below (see also the proof of Proposition 2.5):

Proposition 1.1. *$\ell(p,q)$ is the minimum of the lengths of positive continued fraction expansions of rational numbers p'/q' such that $K(p',q')$ is equivalent to $K(p,q)$.*

Remark 1.2. *$\ell(p/q) = 1$ if and only if $K(p,q)$ is a torus knot (or link).*

Main statements The following will be established below:

Theorem 1.3. *Let $K(p,q)$ be a given two-bridge knot (or link) and let $(M_n(p,q))_{n=2}^{\infty}$ be a sequence of meridian-cyclic n-fold branched coverings of S^3, branched along $K(p,q)$. Then:*

$$\frac{c(M_n(p,q))}{n} \leqslant p - 1 \quad \forall n. \tag{2}$$

If in addition $K(p,q)$ is hyperbolic, then:

$$\liminf_{n\to\infty} \frac{c(M_n(p,q))}{n} \geqslant \max\{2, \ell(p,q)-2\}. \tag{3}$$

Remark 1.4. *There is some overlapping between the cases covered by Theorem 1.3 and those treated in* [20], *and the corresponding estimates can differ. For instance, choosing $p=5$ and $q=2$ we have the figure-eight knot, and Theorem 1.3 gives the upper estimate*

$$\frac{c(M_n(5,2))}{n} \leqslant 4 \quad \forall n$$

whereas [20] *contains the following better one:*

$$\frac{c(M_n(5,2))}{n} \leqslant 3 \quad \forall n.$$

Moreover $\ell(5,2)=2$, because $5/2 = 2+\frac{1}{2}$, and $K(5,2)$ is hyperbolic, therefore Theorem 1.3 gives the lower estimate

$$\liminf_{n\to\infty} \frac{c(M_n(5,2))}{n} \geqslant 2,$$

and this is the same estimate implied by [20].

To state our next result, we recall that the T-invariant $T(G)$ of a finitely presented group G was defined in [6] as the minimal number t such that G admits a presentation with t relations of length 3 and an arbitrary number of relations of length at most 2. A presentation with this property will be called *triangular*.

Proposition 1.5. *For $n \geqslant 2$ let $M_n(p,q)$ be a meridian-cyclic n-fold branched coverings of S^3, branched along a two-bridge knot (or link) $K(p,q)$. Then:*

$$T(\pi_1(M_n(p,q))) \leqslant n(p-1).$$

We note that some connections between the Matveev complexity of a closed 3-manifold and the T-invariant of its fundamental group were already discussed in [23].

The proofs of inequalities (2) and (3) are completely independent of each other. We will begin with the latter, which is established in Section 2, and then prove the former (together with Proposition 1.5, which follows from the same argument) in Section 3. The special case where $\ell(p,q)$ equals 2 is discussed in detail in Section 4.

2. Hyperbolic volume, orbifolds, and twist number: The lower estimate

We begin by recalling that a manifold is hyperbolic if it has a Riemannian metric of constant sectional curvature -1. We will use in the sequel many facts from hyperbolic geometry without explicit reference, see for instance [1, 3, 26].

Inequality (3) is readily deduced by combining the following three propositions. Here and always in the sequel $v_3 = 1.01494\ldots$ denotes the volume of the regular ideal tetrahedron in hyperbolic 3-space $\mathbb{H}^3$, and "vol(M)" is the hyperbolic volume of a manifold M.

Proposition 2.1. *If M is a closed orientable hyperbolic manifold then*

$$\mathrm{vol}(M) < c(M) \cdot v_3.$$

Proposition 2.2. *If $K(p,q)$ is hyperbolic then $M_n(p,q)$, as defined in the statement of Theorem 1.3, is hyperbolic for $n \gg 0$, and*

$$\lim_{n\to\infty} \frac{\mathrm{vol}(M_n(p,q))}{n} = \mathrm{vol}(S^3 \setminus K(p,q)). \tag{4}$$

Proposition 2.3. *If $K(p,q)$ is hyperbolic then*

$$\mathrm{vol}(S^3 \setminus K(p,q)) \geqslant v_3 \cdot \max\{2, \ell(p,q) - 2\}. \tag{5}$$

We begin proofs by establishing the general connection between complexity and the hyperbolic volume:

Proof of Proposition 2.1. Set $k = c(M)$. Being hyperbolic, M is prime and not one of the exceptional manifolds S^3, $\mathbb{RP}^3$, $S^2 \times S^1$, or $L(3,1)$, so there exists a realization of M as a gluing of k tetrahedra. If Δ denotes the abstract tetrahedron, this realization induces continuous maps $\sigma_i : \Delta \to M$ for $i = 1, \ldots, k$ given by the restrictions to the various tetrahedra of the projection from the disjoint union of the tetrahedra to M. Note that each σ_i is injective on the interior of Δ but maybe not on the boundary. Since the gluings used to pair the faces of the tetrahedra in the construction of M are simplicial, it follows that $\sum_{i=1}^{k} \sigma_i$ is a singular 3-cycle, which of course represents the fundamental class $[M] \in H_3(M; \mathbb{Z})$.

We consider now the universal covering $\mathbb{H}^3 \to M$. Since Δ is simply connected, it is possible to lift σ_i to a map $\widetilde{\sigma}_i : \Delta \to \mathbb{H}^3$. We then define the simplicial map $\widetilde{\tau}_i : \Delta \to \mathbb{H}^3$ which agrees with $\widetilde{\sigma}_i$ on the vertices, where geodesic convex combinations are used in $\mathbb{H}^3$ to define the notion of "simplicial". We also denote by $\tau_i : \Delta \to M$ the composition of $\widetilde{\tau}_i$ with the projection $\mathbb{H}^3 \to M$. It is immediate to see that $\sum_{i=1}^{k} \tau_i$ is again a singular 3-cycle in M. Using this and taking convex combinations in $\mathbb{H}^3$, one can actually check that the cycles $\sum_{i=1}^{k} \sigma_i$ and $\sum_{i=1}^{k} \tau_i$ are homotopic to each other. Therefore, since the first cycle represents $[M]$, the latter also does, which implies that $\bigcup_{i=1}^{k} \tau_i(\Delta)$ is equal to M, otherwise $\sum_{i=1}^{k} \tau_i$ would be homotopic to a map with 2-dimensional image.

Next we note that $\widetilde{\tau}_i(\Delta)$ is a compact geodesic tetrahedron in $\mathbb{H}^3$, so its volume is less than v_3, see [3]. Moreover the volume of $\tau_i(\Delta)$ is at most equal to the volume of $\widetilde{\tau}_i(\Delta)$, because the projection $\mathbb{H}^3 \to M$ is a local isometry, and the volume of M is at most the sum of the volumes of the $\tau_i(\Delta)$'s, because we have shown above that M is covered by the $\tau_i(\Delta)$'s (perhaps with some overlapping). This establishes the proposition. $\square$

We next use hyperbolic orbifolds:

Proof of Proposition 2.2. Let us denote by $O_n(p,q)$ the 3-orbifold with underlying space S^3 and singular locus $K(p,q)$ of order n. The orbifold version of Thurston's hyperbolic Dehn filling theorem [26] implies that $O_n(p,q)$ is hyperbolic for $n \gg 0$, and that

$$\lim_{n\to\infty} \mathrm{vol}(O_n(p,q)) = \mathrm{vol}(S^3 \setminus K(p,q)). \tag{6}$$

We then note that $M_n(p,q)$ is also hyperbolic by the same theorem, and

$$\mathrm{vol}(M_n(p,q)) = n \cdot \mathrm{vol}(O_n(p,q)).$$

This readily implies (4). $\square$

Remark 2.4. *Even if we will not use this, we can actually prove that* $\mathrm{vol}(O_n(p,q))$ *tends to* $\mathrm{vol}(S^3\setminus K(p,q))$ *monotonically from below in* (6). *We achieve this by defining* $C^{(\alpha)}(p,q)$ *as the cone manifold supported by* S^3 *with singular locus* $K(p,q)$ *and cone angle* α *along it, so* $O_n(p,q) = C^{(2\pi/n)}(p,q)$. *One knows that* $C^{(\alpha)}(p,q)$ *is hyperbolic when* α *varies in some interval* $(0,\varepsilon)$. *Moreover by the Shläffli variation formula* [24, p. 512],

$$\frac{\mathrm{d}\ \mathrm{vol}(C^{(\alpha)}(p,q))}{\mathrm{d}\ \alpha} = -\frac{L(\alpha)}{2},$$

where $L(\alpha)$ *is the length of* $K(p,q)$ *in* $C^{(\alpha)}(p,q)$. *This readily implies the desired conclusion.*

Before turning to the proof of Proposition 2.3 we establish the characterization of $\ell(p,q)$ stated in the first section:

Proof of Proposition 1.1. Under assumption (1), we know that the relevant pairs (p',q') are those with p' equal to p and q' equal to either $p-q$ or r or $p-r$, where $1 \leqslant r \leqslant p-1$ and $q \cdot r \equiv 1 \pmod{p}$.

We begin by noting that if we take positive continued fraction expansions of p/q and $p/(p-q)$ we find 1 as the first coefficient in one case and a number greater than 1 in the other case. Supposing first that $p/q = [1, a_2, a_3, \ldots, a_k]$ it is now easy to see that $p/(p-q) = [a_2+1, a_3, \ldots, a_k]$, so the minimized positive expansion of $p/(p-q)$ has length $k-1$. The same argument with switched roles shows that if the first coefficient a_1 of the minimized positive expansion of p/q is larger than 1 then the length of the expansion of $p/(p-q)$ is $k+1$. Therefore the minimal length we can obtain using q and $p-q$ is indeed $\ell(p,q)$.

Supposing $p/q = [a_1, \ldots, a_k]$, we next choose s with $1 \leqslant s \leqslant p-1$ and $q \cdot s \equiv (-1)^{k-1} \pmod{p}$, and we note that $\{s, p-s\} = \{r, p-r\}$. Now it is not difficult to see that p/s has a positive continued fraction expansion $p/s = [a_k, a_{k-1}, \ldots, a_2, a_1]$. Note that this may or not be a minimized expansion, depending on whether a_1 is greater than 1 or equal to 1, but the length of the minimized version is $\ell(p,q)$ anyway, thanks to the definition we have given. By the above argument, since $a_k > 1$, the length of the minimized positive expansion of $p/(p-s)$ is 1 more than that of p/s, and the proposition is established. $\square$

Our proof of Proposition 2.3 will be based on results of Cao-Meyerhoff [5] and Lackenby [12]. Note that (5) is equivalent to the two inequalities

$$\mathrm{vol}(S^3 \setminus K(p,q)) \geqslant 2v_3 \tag{7}$$

$$\mathrm{vol}(S^3 \setminus K(p,q)) \geqslant v_3 \cdot (\ell(p,q) - 2). \tag{8}$$

Now, Cao and Meyerhoff have proved in [5] that the figure-eight knot complement (namely $S^3 \setminus K(5,2)$ in our notation) and its sibling manifold (which can be described as $(5,1)$ Dehn surgery on the right-handed Whitehead Link) are the orientable cusped hyperbolic 3-manifolds of minimal volume, and they are the only such 3-manifolds. Each has volume equal to $2v_3 = 2.02988\ldots$, which implies inequality (7) directly.

To establish (8) we need to recall some terminology introduced by Lackenby in [12]. A *twist* in a link diagram $D \subset \mathbb{R}^2$ is either a maximal collection of bigonal regions of $\mathbb{R}^2 \setminus D$ arranged in a row, or a single crossing with no incident bigonal regions. The *twist number* $t(D)$ of D is the total number of twists in D. Moreover D is called *twist-reduced* if it is alternating and whenever $\gamma \subset \mathbb{R}^2$ is a simple closed curve meeting D transversely at two crossing only, one of the two portions into which γ separates D is contained in a twist. (This is not quite the definition in [12], but it is easily recognized to be equivalent to it for alternating diagrams.)

Lackenby proved in [12] that if D is a prime twist-reduced diagram of a hyperbolic link L in S^3 then

$$v_3 \cdot (t(D) - 2) \leqslant \mathrm{vol}(S^3 \setminus L) \leqslant 10 \cdot v_3 \cdot (t(D) - 1), \tag{9}$$

where v_3 is the volume of the regular ideal tetrahedron.

Using (9), the next result implies (8), which completes the proof of Proposition 2.3 and hence of inequality (3) in Theorem 1.3:

Proposition 2.5. *The link $K(p,q)$ has a twist-reduced diagram with twist number $\ell(p,q)$.*

Proof. The required diagram D is simply given by the so-called Conway normal form of $K(p,q)$ associated to the minimized positive continued fraction expansion $[a_1, \ldots, a_k]$ of p/q. The definition of the Conway

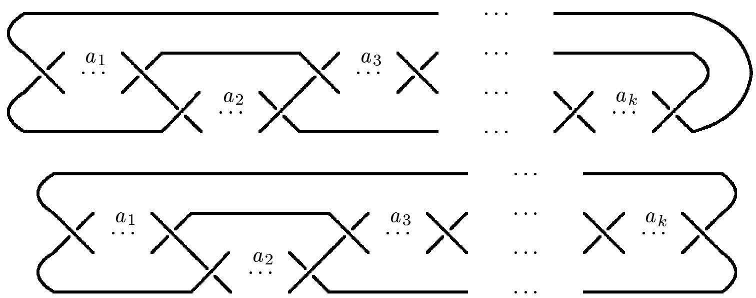

FIGURE 1. The Conway normal form of a two-bridge link. The number of half-twists of the appropriate type in the j-th portion of the diagram is given by the positive integer a_j. The upper picture refers to the case of even k and the lower picture to the case of odd k.

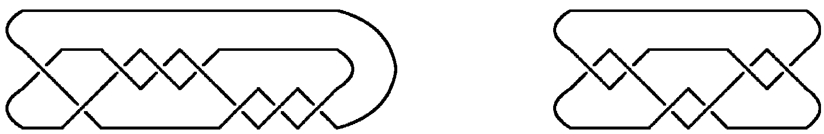

FIGURE 2. Conway diagrams of $K(23, 13)$ and $K(12, 5)$. Note that the required expansions are $23/13 = [1, 1, 3, 3]$ and $12/5 = [2, 2, 2]$.

normal form differs for even and odd k, and it is described in Fig. 1. Two specific examples are also shown in Fig. 2.

Since the a_j's are positive, it is quite obvious that the Conway normal diagram D always gives an alternating projection, besides being of course prime. The twists of this diagram are almost always the obvious ones obtained by grouping together the first a_1 half-twists, then the next a_2, and so on. An exception has to be made, however, when a_1 equals 1, because in this case the first half-twist can be grouped with the next a_2 to give a single twist (as in Fig. 2-left). Note that $a_k > 1$ by assumption, so no such phenomenon appears at the other end. Since our definition of $\ell(p, q)$ is precisely k if $a_1 > 1$ and $k - 1$ if $a_1 = 1$, we see that indeed the diagram always has $\ell(p, q)$ twists.

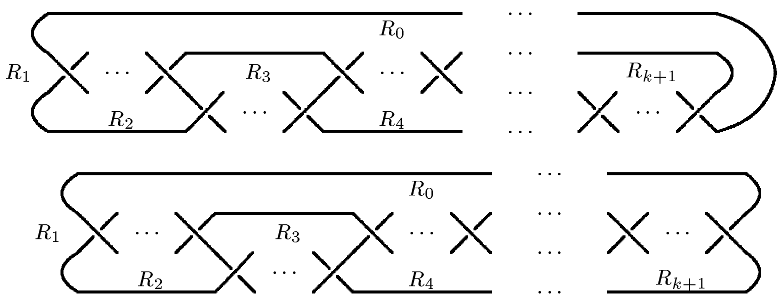

FIGURE 3. Labels for the non-bigonal regions of the complement of a Conway normal diagram.

Before proceeding we note that if $a_1 = 1$ then the Conway normal form for $K(p,q)$ is actually the same, as a projection, as the mirror image of the Conway normal form for $K(p,p-q)$. The picture showing this assertion gives a geometric proof of the fact that if $p/q = [1, a_2, \ldots, a_k]$ then $p/(p-q) = [a_2+1, a_3, \ldots, a_k]$, used in Proposition 1.1. So we can proceed assuming that $a_1 > 1$. In particular, each bigonal region of $S^2 \setminus D$ is one of the $(a_1-1)+(a_2-1)+\ldots+(a_k-1)$ created when inserting the $a_1, a_2, \ldots, a_k$ half-twists of the normal form.

To prove that D is twist-reduced, let us look for a curve γ as in the definition, namely one that intersects D transversely at two crossings. Near each such intersection, γ must be either horizontal or vertical (see Fig. 1). Let us first show that if it meets some crossing c of D horizontally then c is the crossing arising from the single half-twist that corresponds to some coefficient a_j equal to 1. If this is not the case, then either to the left or to the right of c there is a bigonal region of $S^2 \setminus D$. Then γ must meet horizontally the crossing at the other end of this bigonal region, which readily implies that γ cannot meet the diagram in two points only.

Having shown that γ can only be vertical when it intersects vertices, except at the vertices arising from the a_j's with $a_j = 1$, let us give labels $R_0, R_1, \ldots, R_k, R_{k+1}$ to the non-bigonal regions of $S^2 \setminus D$, as in Fig. 3, and let us construct a graph Γ with vertices $R_0, R_1, \ldots, R_k, R_{k+1}$ and an edge joining R_i to R_j for each segment through a crossing of D going from R_i to R_j. By assumption γ must correspond to a length-2 cycle in Γ. Now for odd k the connections existing in Γ are precisely as follows:

- an a_{2j-1}-fold connection between R_0 and R_{2j} for $j = 1, \ldots, (k+1)/2$;
- an a_{2j}-fold connection between R_1 and R_{2j+1} for $j = 1, \ldots, (k-1)/2$;
- a single connection between R_j and R_{j+2} if $2 \leqslant j \leqslant k-1$ and $a_j = 1$.

Then the only length-2 cycles are the evident ones either between R_0 and some R_{2j} or between R_1 and some R_{2j+1}, and the curve γ corresponding to one of these cycles does bound a portion of a twist of D, as required by the definition of twist-reduced diagram.

A similar analysis for even k completes the proof. □

Proof of inequality (3). Combining (7) and the first inequality in (9) with Proposition 2.5 we get

$$v_3 \cdot \max\{2, \ell(p,q) - 2\} \leqslant \mathrm{vol}(S^3 \setminus K(p,q)).$$

Together with (4), this formula implies that

$$v_3 \cdot \max\{2, \ell(p,q) - 2\} \leqslant \liminf_{n\to\infty} \frac{\mathrm{vol}(M_n(p,q))}{n}$$

and the conclusion now readily follows from Proposition 2.1. □

Remark 2.6. *Giving general lower bounds for the hyperbolic volume is known to be hard, so Lackenby's inequality* (8) *is definitely praiseworthy, considering in particular its very broad range of application. However the inequality is numerically not very effective in some cases. For instance in the case of hyperbolic two-bridge knots and links the lower bound on the volume given by* (8) *seems to be very far from the actual value. As an example we will discuss below the case* $p/q = k + 1/m$, *where* (8) *does not give any lower bound, since* $\ell(p,q) = 2$. *Using arguments close to those in* [12] *specialized to the particular case of two-bridge knots and links one could possibly get better estimates.*

Remark 2.7. *On the basis of some computer experiments, we conjecture that the Whitehead link complement (namely* $S^3 \setminus K(8,3)$, *with*

$\mathrm{vol}(S^3 \setminus K(8,3)) = 3.66386\ldots$*) has the smallest volume among all two-bridge two-component links.*

3. Minkus polyhedral schemes and triangulations: The upper estimate

The proof of (2) and Proposition 1.5 is based on a realization of $M_n(p,q)$ as the quotient of a certain polyhedron under a gluing of its faces. This construction extends one that applies to lens spaces and it is originally due to Minkus [21]. We will briefly review it here following [22].

Let us begin from the case where $K(p,q)$ is a knot, *i.e.* p is odd, whence $M_n(p,q)$ is uniquely defined by p,q,n. Recall that by the assumption (1), $0 < q < p$. Then we consider the 3-ball

$$B^3 = \{(x,y,z) \in \mathbb{R}^3 : \ x^2 + y^2 + z^2 \leqslant 1\}$$

and we draw on its boundary n equally spaced great semicircles joining the north pole $N = (0,0,1)$ to the south pole $S = (0,0,-1)$. This decomposes ∂B^3 into n cyclically arranged congruent lunes $L_0, \ldots, L_{n-1}$. Now we insert $p-1$ equally spaced vertices on each semicircle, thus subdividing it into p identical segments, which allows us to view each lune L_i as a curvilinear polygon with $2p$ edges. Next, we denote by P_i the vertex on the semicircle $L_i \cap L_{i-1}$ which is q segments down from N, and by P'_i the vertex which is q segments up from S (indices are always meant modulo n). We then draw inside L_i an arc of great semicircle joining P_i to P'_{i+1}, thus bisecting L_i into two regions that we denote by R_i and R'_{i+1}, with R_i incident to N and R'_{i+1} incident to S. Fig. 4 illustrates the resulting decomposition of ∂B^3, which is represented as $\mathbb{R}^2 \cup \{\infty\}$ with $S = \infty$. In the picture we assume $q > p/2$.

Summing up, we have subdivided ∂B^3 into $2n$ curvilinear polygons R_i, R'_i for $i = 0, \ldots, n-1$, each having $p+1$ edges. The polygons R_i are around N and the polygons R'_i are around S, and there is a marked vertex P_i shared by R_i and R'_{i+1} (we will not need to use P'_i again). It is now possible to show that the manifold $M_n(p,q)$ is obtained from B^3 by identifying R_i with R'_i on ∂B^3 for $i = 0, \ldots, n-1$ through an orientation-reversing simplicial homeomorphism which matches the vertex P_i of R_i with the vertex P_{i-1} of R'_i.

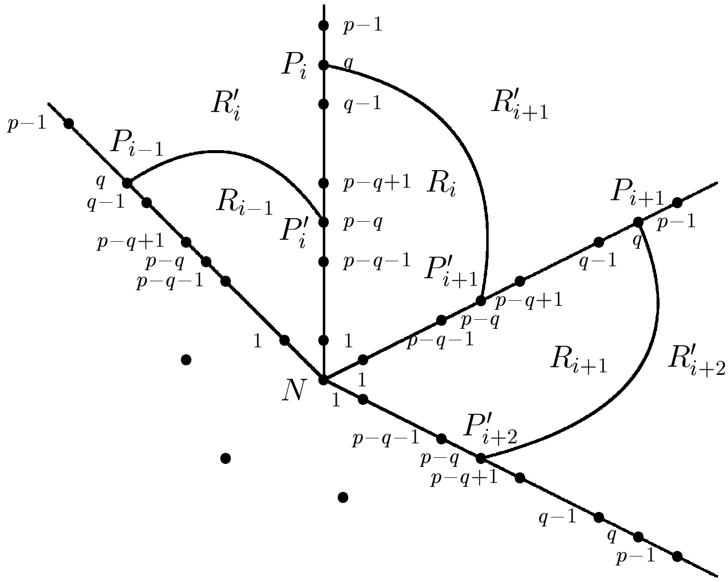

FIGURE 4. The Minkus polyhedral scheme for $M_n(p, q)$.

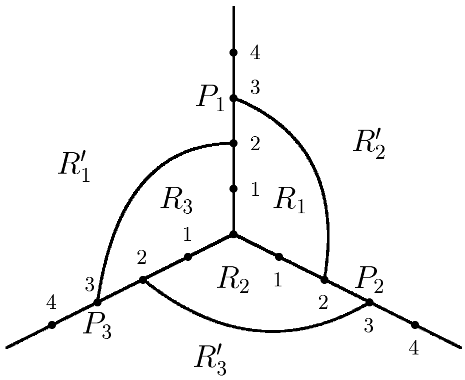

FIGURE 5. The Minkus polyhedral scheme for $M_3(5, 3)$.

As an example, Fig. 5 shows the Minkus polyhedral construction of the Hantzsche-Wendt manifold, that is $M_3(5, 3)$ in our notation.

Proof of inequality (2) *for odd* p. Referring to the above polyhedral construction of $M_n(p, q)$, we subdivide each R_i into $p - 1$ triangles by taking diagonals from the north pole N, and each R'_i so that the gluing between R_i and R'_i matches the subdivision. Note that the "diagonals" are only meant in a combinatorial sense, they cannot be taken as arcs

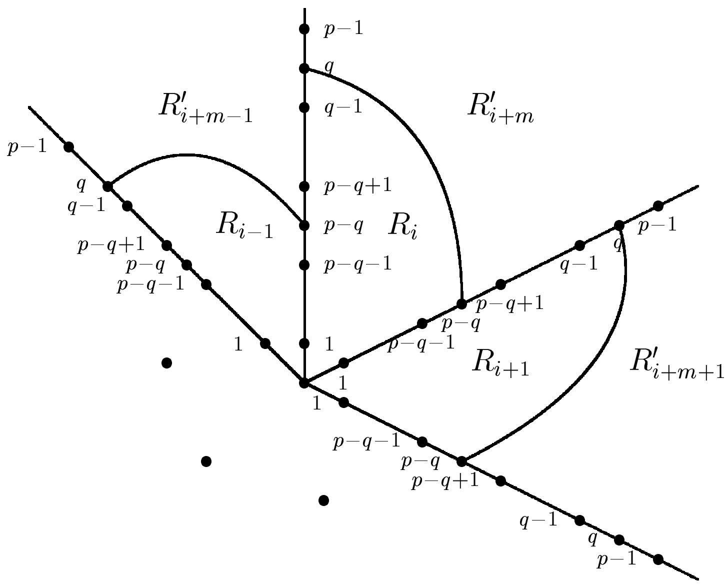

FIGURE 6. The Minkus polyhedral scheme for $M_{n,m}(p,q)$.

of great circles. We can then take (combinatorial) cones with vertex at N and bases at the triangles contained in the R'_i, thus getting a subdivision of B^3 into $n(p-1)$ tetrahedra. By construction the gluings on ∂B^3 restrict to gluings of the faces of these tetrahedra, therefore $M_n(p,q)$ has a (loose) triangulation made of $n(p-1)$ tetrahedra, and the proof is now complete. □

Proof of inequality (2) *for even* p. To establish (2) for even p, *i.e.* for 2-component two-bridge links, we extend to this case the Minkus polyhedral construction, see [22]. The way to do this is actually straightforward: to realize the meridian-cyclic covering $M_{n,m}(p,q)$ of S^3 branched along $K(p,q)$ we subdivide ∂B^3 precisely as above, but we denote by R_i and R'_{i+m} the two regions into which the lune L_i is bisected. Then we glue R_i to R'_i by an orientation-reversing simplicial homeomorphism matching the vertex P_i of R_i with the vertex P_{i-m} of R'_i. This construction is illustrated in Fig. 6. This realization of $M_{n,m}(p,q)$ again induces a triangulation with $n(p-1)$ tetrahedra, which proves (2) also in this case. □

Proof of Proposition 1.5. Let us carry out only the "first half" of the subdivision we did above of the Minkus polyhedral realization of $M_n(p,q)$. Namely, we subdivide the regions R_i, R_i' on ∂B^3 into triangles, but then we do not add anything inside B^3. This yields a cellularization of $M_n(p,q)$ with 2-cells being triangles and with a single 3-cell. Therefore there is a triangular presentation of $\pi_1(M_n(p,q))$ with precisely the same number of relations as the number of triangles in this cellularization. And this number is $n(p-1)$, because there are $2n(p-1)$ triangles on ∂B^3, but they get glued in pairs. □

4. An Example: $p/q = k + 1/m$

As already noticed, in the case of hyperbolic two-bridge knots and links the lower bound on the volume given by (8) does not seem to provide very effective estimates. We consider in this section the case where p/q has the form $k + 1/m = (km+1)/m$ and $K(p,q)$ is hyperbolic. Then $\ell(p,q) = 2$, so the estimate provided by (8) is void. However, denoting by $\mathcal{B}$ the Borromean rings, a hyperbolic 3-component link (depicted below) with volume $7.32772\ldots$, we can establish the following:

Proposition 4.1. *If $k, m > 1$ then $K(km+1, m)$ is hyperbolic and*

$$\lim_{k,m\to\infty} \mathrm{vol}(S^3 \setminus K(km+1,m)) = \mathrm{vol}(S^3 \setminus \mathcal{B}) = 7.32772\ldots$$

Proof. We set $p = km+1$ and $q = m$ and note that $K(p,q)$ is hyperbolic by the discussion in Section 1. For the limit of the volume, the proof is always based on hyperbolic surgery, but it differs according to whether k and m are even or odd.

Case 1: $k = 2i$ and $m = 2j$. In this case $K(p,q) = K(4ij+1, 2j)$, shown in Fig. 7-left, can be obtained by Dehn surgery on the Borromean rings $\mathcal{B} = 6^3_2$ pictured in Fig. 7-right, doing a $(1/i)$-surgery on the first small circle and a $(1/j)$-surgery on the another one, see [25, Ch. 9] for details. Labels for links, here and in the sequel, are also taken from [25]. Thurston's hyperbolic Dehn surgery theorem then implies that

$$\lim_{i,j\to\infty} \mathrm{vol}(S^3 \setminus K(4ij+1,2j)) = \mathrm{vol}(S^3 \setminus \mathcal{B}) = 7.32772\ldots$$

as required.

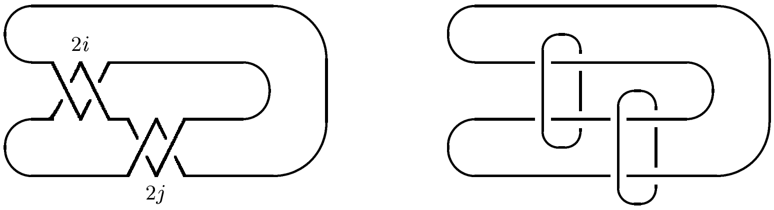

FIGURE 7. $K(4ij+1, 2j)$ and $\mathcal{B} = 6^3_2$.

CASE 2: $k = 2i+1$ AND $m = 2j$. In this case $K(p, q) = K(4ij + 2j + 1, 2j)$, shown in Fig. 8-left, can be obtained by Dehn surgery on the 3-component link $\mathcal{B}' = 8^3_9$ pictured in Fig. 8-right, doing a $(1/i)$-surgery on the first small circle and a $(1/j)$-surgery the another one. By the same arguments as in [3, p. 269-270] we have $\mathrm{vol}(S^3 \setminus \mathcal{B}') = \mathrm{vol}(S^3 \setminus \mathcal{B})$. Hence again

$$\lim_{i,j\to\infty} \mathrm{vol}(S^3 \setminus K(4ij + 2j + 1, 2j)) = \mathrm{vol}(S^3 \setminus \mathcal{B}) = 7.32772\ldots$$

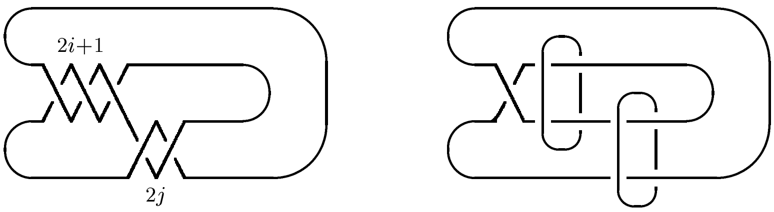

FIGURE 8. $K(4ij+2j+1, 2j)$ and $\mathcal{B}' = 8^3_9$.

CASE 3: $k = 2i$ AND $m = 2j+1$. In this case $K(p, q) = K(4ij + 2i + 1, 2j + 1)$ is equivalent to $K(p, q') = K(4ij + 2i + 1, 2i)$ because $(2j+1)(2i) \equiv -1 \pmod{p}$. The conclusion then follows from Case 2.

CASE 4: $k = 2i+1$ AND $m = 2j+1$. In this case $K(p, q) = K(4ij + 2i + 2j + 2, 2j + 1)$, shown in Fig. 9-left, is a 2-component link that can be obtained by Dehn surgery on the two small circles of the 4-component link $\mathcal{B}''$ pictured in Fig. 9-right, doing a $(1/i)$-surgery and a $(1/j)$-surgery. Since as above $\mathrm{vol}(S^3 \setminus \mathcal{B}'') = \mathrm{vol}(S^3 \setminus \mathcal{B})$, we have

$$\lim_{i,j\to\infty} \mathrm{vol}(S^3 \setminus K(4ij + 2i + 2j + 2, 2j + 1)) = 7.32772\ldots$$

again. □

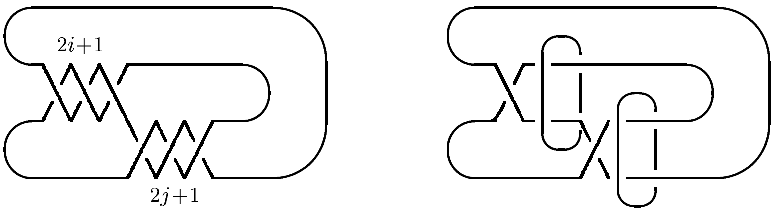

FIGURE 9. $K(4ij+2i+2j+2, 2j+1)$ and $\mathcal{B}''$.

Propositions 2.1, 2.2 and 4.1 imply that

$$\liminf_{n,k,m\to\infty} \frac{c(M_n(km+1,m))}{n} \geqslant \frac{\mathrm{vol}(S^3\setminus\mathcal{B})}{v_3} = 7.21985\ldots$$

Fixing small k and m, letting only n tend to infinity, and using the computer program SnapPea [27] to calculate the volume of $K(km+1,m)$, one gets more specific asymptotic lower bounds (with smaller constants). For instance, with notation again taken from [25], one can consider $K(5,2)=4_1$, the figure-eight knot, $K(7,3)=5_2$, and $K(9,4)=6_1$, getting the estimates

$$\begin{aligned}
\liminf_{n\to\infty} \frac{c(M_n(5,2))}{n} &\geqslant \frac{\mathrm{vol}(S^3\setminus K(5,2))}{v_3} = 2\\
\liminf_{n\to\infty} \frac{c(M_n(7,3))}{n} &\geqslant \frac{\mathrm{vol}(S^3\setminus K(7,3))}{v_3} = \frac{2.81812\ldots}{v_3} = 2.77663\ldots\\
\liminf_{n\to\infty} \frac{c(M_n(9,4))}{n} &\geqslant \frac{\mathrm{vol}(S^3\setminus K(9,4))}{v_3} = \frac{3.16396\ldots}{v_3} = 3.11738\ldots
\end{aligned}$$

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