On the existence of branched coverings between surfaces with prescribed branch data, II

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November 10, 2006

Abstract

If $\widetilde{\Sigma} \to \Sigma$ is a branched covering between closed surfaces, there are several easy relations one can establish between $\chi(\widetilde{\Sigma})$, $\chi(\Sigma)$, orientability of Σ and $\widetilde{\Sigma}$, the total degree, and the local degrees at the branching points, including the classical Riemann-Hurwitz formula. These necessary relations have been shown to be also sufficient for the existence of the covering except when Σ is the sphere $\mathbb{S}$ (and when Σ is the projective plane, but this case reduces to the case $\Sigma = \mathbb{S}$). For $\Sigma = \mathbb{S}$ many exceptions are known to occur and the problem is widely open.

Generalizing methods of Baránski, we prove in this paper that the necessary relations are actually sufficient in a specific but rather interesting situation. Namely under the assumption that $\Sigma = \mathbb{S}$, that there are three branching points, that one of these branching points has only two preimages with one being a double point, and either that $\widetilde{\Sigma} = \mathbb{S}$ and that the degree is odd, or that $\widetilde{\Sigma}$ has genus at least one, with a single specific exception. For the case of $\widetilde{\Sigma} = \mathbb{S}$ we also show that for each even degree there are precisely two exceptions.

MSC (2000): 57M12.

*Partially supported by the INTAS YS fellowship 03-55-1423

**Supported by the INTAS project "CalcoMet-GT" 03-51-3663

1 Problem and new realizability results

A thorough description of the problem faced below, including several remarks on its relevance to other areas of mathematics and a comprehensive survey on the partial solutions obtained over the time, was given in the first part of the present paper [5]. For this reason we confine ourselves here to the notation and known facts strictly needed to state and prove our new results.

Basic definitions A *branched covering* is a map $f : \widetilde{\Sigma} \to \Sigma$, where $\widetilde{\Sigma}$ and Σ are closed connected surfaces and f is locally modelled on maps of the form $\mathbb{C} \ni z \mapsto z^k \in \mathbb{C}$ for some $k \geqslant 1$. The integer k is called the *local degree* at the point of $\widetilde{\Sigma}$ corresponding to 0 in the source $\mathbb{C}$. If $k > 1$ then the point of Σ corresponding to 0 in the target $\mathbb{C}$ is called a *branching point*. The number of branching points, which is finite by compactness, will be denoted by n. We define d as the degree of the genuine covering that is obtained by removing the branching points in Σ and all their pre-images in $\widetilde{\Sigma}$ and taking the restriction of f. We also denote the number of pre-images of the i-th branching point on Σ by m_i and the local degrees at these pre-images by $(d_{ij})_{j=1}^{m_i}$. It is easy to see that $(d_{ij})_{j=1}^{m_i}$ forms a partition of d. In the sequel we will always assume that in a partition $(d_1, \ldots, d_m)$ of d we have $d_1 \geqslant \ldots \geqslant d_m$.

Branch data Suppose we are given closed connected surfaces $\widetilde{\Sigma}$ and Σ, integers $n \geqslant 0$ and $d \geqslant 2$, and for $i = 1, \ldots, n$ a partition $(d_{ij})_{j=1}^{m_i}$ of d. The 5-tuple $\big(\widetilde{\Sigma}, \Sigma, n, d, (d_{ij})\big)$ will be called the *branch datum* of a candidate branched covering.

Compatibility A branch datum is called *compatible* if the following conditions hold:

1. $\chi(\widetilde{\Sigma}) - \widetilde{n} = d \cdot (\chi(\Sigma) - n)$;
2. $n \cdot d - \widetilde{n}$ is even;
3. If Σ is orientable then $\widetilde{\Sigma}$ is also orientable;
4. If Σ is non-orientable and d is odd then $\widetilde{\Sigma}$ is also non-orientable;

5. If Σ is non-orientable but $\widetilde{\Sigma}$ is orientable then each partition $(d_{ij})_{j=1}^{m_i}$ of d refines the partition $(d/2, d/2)$.

The meaning of Condition 5 is that $(d_{ij})_{j=1}^{m_i}$ is obtained by juxtaposing two partitions of $d/2$ and reordering. Note that d is even by Condition 4.

The problem It is not too difficult to show that if a branched covering $\widetilde{\Sigma} \to \Sigma$ exists then the corresponding branch datum, with $n, d, (d_{ij}), \widetilde{n}$ defined as above, is compatible (see [5] for an explanation of Condition 2, the other ones are obvious). The so-called *Hurwitz existence problem* is the question of which compatible branch data are actually realized by some branched covering.

Thanks to the contributions of many authors, among which we will only mention the fundamental one by Edmonds, Kulkarni and Stong [2], we now know that the Hurwitz existence problem has a positive solution whenever $\chi(\Sigma) \leqslant 0$, and that the case where Σ is the projective plane reduces to the case where Σ is the sphere $\mathbb{S}$. For this reason we will assume henceforth that $\Sigma = \mathbb{S}$.

After being neglected for many years, the problem was picked up again recently by Baránski [1], who proved some existence results for $\widetilde{\Sigma} = \Sigma = \mathbb{S}$, and by Zheng [6], who introduced a new approach to face the question and obtained many interesting experimental results. In this paper we extend the technique employed by Baránski to prove the following results (with $\mathbb{T}$ denoting the torus in the second statement and $g\mathbb{T}$ denoting the orientable surface of genus g in the third one):

Theorem 1.1. *The compatible and non-realizable branch data of the form* $\big(\mathbb{S}, \mathbb{S}, 3, d, (d-2, 2), (d_{2j}), (d_{3j})\big)$ *are precisely those of the following types:*

- $\big(\mathbb{S}, \mathbb{S}, 3, 2k, (2k-2, 2), (2, \ldots, 2), (2, \ldots, 2)\big)$ *with* $k > 2$*;*
- $\big(\mathbb{S}, \mathbb{S}, 3, 2k, (2k-2, 2), (2, \ldots, 2), (k+1, 1, \ldots, 1)\big)$.

In particular, if d *is odd then all the data of the relevant form are realizable.*

Theorem 1.2. *With the single exception of* $\big(\mathbb{T}, \mathbb{S}, 3, 6, (4, 2), (3, 3), (3, 3)\big)$, *every compatible branch datum of the form* $\big(\mathbb{T}, \mathbb{S}, 3, d, (d-2, 2), (d_{2j}), (d_{3j})\big)$ *is realizable.*

Theorem 1.3. *If* $g \geqslant 2$ *then every compatible branch datum of the form* $\big(g\mathbb{T}, \mathbb{S}, 3, d, (d-2, 2), (d_{2j}), (d_{3j})\big)$ *is realizable.*

Remark 1.4. The odd-degree part of Theorem 1.1 was already established in an earlier version of this paper. Later an interesting preprint of Pakovich [4], using an entirely different approach, reproved the result and extended it to the case of branch data of form $\big(\mathbb{S}, \mathbb{S}, n, d, (d-k, k), (d_{2j}), \ldots, (d_{nj})\big)$, which actually contains the whole of our Theorem 1.1. Given the independence of the techniques and the fact that they prompt to extensions in different directions, we have decided to include the proof of Theorem 1.1 anyway.

To explain our motivation for considering the case where one of the partitions has the special form $(d-2, 2)$, we mention the fact that "small" partitions were already considered in the literature. The following is for instance known (see [2] and the discussion in [5]):

Theorem 1.5. *If one of the partitions of d in a compatible branch datum is given by (d) only then the datum is realizable*

The paper [2] also contains some results for the case where one of the partitions is $(d-1, 1)$ and more generally $(*, 1, \ldots, 1)$. In particular, the following statement is established:

Proposition 1.6. *The compatible and non-realizable branch data of the form $\big(\mathbb{S}, \mathbb{S}, n, d, (d-1, 1), (d_{2j}), \ldots, (d_{nj})\big)$ are precisely those of the following types:*

- $\big(\mathbb{S}, \mathbb{S}, n, 4, (3, 1), (2, 2), \ldots, (2, 2)\big)$ *with* $n \geqslant 2$*;*
- $\big(\mathbb{S}, \mathbb{S}, 3, 2k, (2k-1, 1), (2, \ldots, 2), (2, \ldots, 2)\big)$.

Under this perspective, the case $(d-2, 2)$ that we consider is then the next natural one to face.

2 A geometric criterion for existence

In this section we prove that the Hurwitz existence problem with $\Sigma = \mathbb{S}$ has a geometric equivalent in terms of the existence of certain families of graphs on the putative covering surface $\widetilde{\Sigma}$. Our result extends that proved by Baránski [1] for $\widetilde{\Sigma} = \mathbb{S}$. A little machinery has to be developed to give the statement.

Minimal checkerboard graphs We begin with a notion also used in [5]. We call *checkerboard graph* a finite 1-subcomplex of the surface $\widetilde{\Sigma}$ whose complement consists of open discs each bearing a color black or white, so that each edge separates black from white. We regard checkerboard graphs up to homeomorphism of $\widetilde{\Sigma}$ and switching of colors. A checkerboard graph is called *minimal* if at every vertex all the black germs of discs are contained in the same global disc, and similarly for white.

Lemma 2.1. *If G is minimal then $\widetilde{\Sigma}\setminus G$ consists of one black and one white disc.*

Proof. If x_0 and x_1 are vertices of G joined by an edge, then there is a black disc incident to both x_0 and x_1, so there is only one global black disc incident to x_0 and x_1, and similarly for white. Now the conclusion follows from the remark that G is connected, because its complement consists of discs. $\square$

The case $\widetilde{\Sigma}=\mathbb{S}$ and vertices of minimal graphs If $\widetilde{\Sigma}=\mathbb{S}$ then of course a minimal checkerboard graph is an embedded circle, as in [1], that we consider not to have any vertex at all. On the contrary, if $\widetilde{\Sigma}$ has positive genus, every minimal checkerboard graph has vertices of valence greater than 2, and we can disregard those of valence 2, which we will do henceforth.

Minimal graphs from permutations We will now prove that for any g there are finitely many minimal checkerboard graphs on the closed orientable surface $g\mathbb{T}$ of genus $g>0$, and that they can be constructed algorithmically. Fix two copies B and W of the unit disc of $\mathbb{C}$, and for every integer $p>1$ denote by $\mathcal{R}_p^B$ (respectively, $\mathcal{R}_p^W$) the set of p-th roots of unity on ∂B (respectively, ∂W).

Proposition 2.2. *Suppose that $g>0$. Then:*

1. *Let $p>1$ and $f\in\mathfrak{S}(\mathcal{R}_p^B)$. Define $\widetilde{f}\in\mathfrak{S}(\mathcal{R}_p^B)$ as $\widetilde{f}(z)=f(e^{2\pi i/p}\cdot z)$. Suppose that:*
   - *f consists of q cycles each having length at least 2;*
   - *$\widetilde{f}$ is a full cycle;*
   - *$p-q=2g$.*

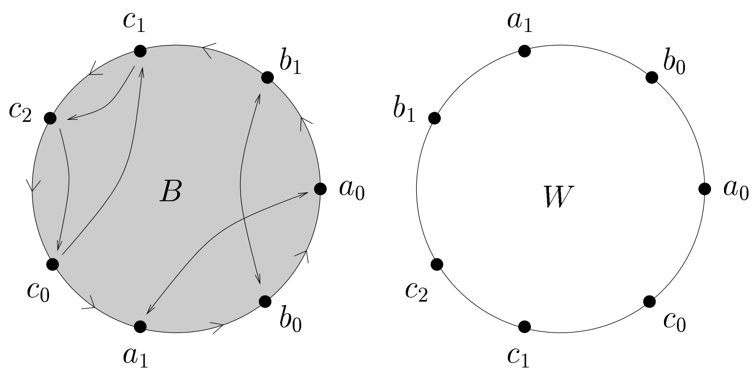

Figure 1: A permutation of the black 7-th roots of unity and the associated white permutation.

Then define $h : \partial W \to \partial B$ as

$$h(\mathrm{e}^{2\pi i(k+t)/p}) = \mathrm{e}^{2\pi i t/p} \cdot (\widetilde{f})^k(1), \qquad k = 0, \ldots, p-1, \quad 0 \leqslant t < 1,$$

and note that h is bijective. Define $f' \in \mathfrak{S}(\mathcal{R}_p^W)$ as $f' = h^{-1} \circ f \circ h$ and remark that h induces a homeomorphism $\overline{h} : \partial B/_f \to \partial W/_{f'}$. Then:

- *The gluing of $B/_f$ to $W/_{f'}$ along $\overline{h}$ gives $g\mathbb{T}$, with $g > 0$;*
- *The image in $g\mathbb{T}$ of ∂B, which is equal to the image of ∂W, is a minimal checkerboard graph on $g\mathbb{T}$;*

2. *All minimal checkerboard graphs on $g\mathbb{T}$ arise as above for suitable $p > 1$ and $f \in \mathfrak{S}(\mathcal{R}_p^B)$.*

Before turning to the proof of this result, we provide an alternative description of the statement and an example which show that the construction, despite its apparent complication, is actually straight-forward. Let the cycles of f have lengths $\ell_1, \ldots, \ell_q$. Give labels $a_j^{(i)}$ to the points of $\mathcal{R}_p^B$ for $i = 1, \ldots, q$ and $j = 0, \ldots, \ell_i - 1$, in such a way that $f(a_j^{(i)}) = a_{j+1}^{(i)}$ with indices modulo ℓ_i. An example is shown for $p = 7, q = 3, \ell_1 = \ell_2 = 2, \ell_3 = 3$ in Fig. 1-left, where for simplicity we have replaced $a^{(1)}, a^{(2)}, a^{(3)}$ by a, b, c. The definition of $\widetilde{f}$ is then as follows: from any point of $\mathcal{R}_p^B$, first go to the next point on ∂B in the counter-clockwise order, and then jump to the next point according to f. In our case, we get the full cycle $a_0 \mapsto b_0 \mapsto a_1 \mapsto b_1 \mapsto c_2 \mapsto c_1 \mapsto c_0$. If

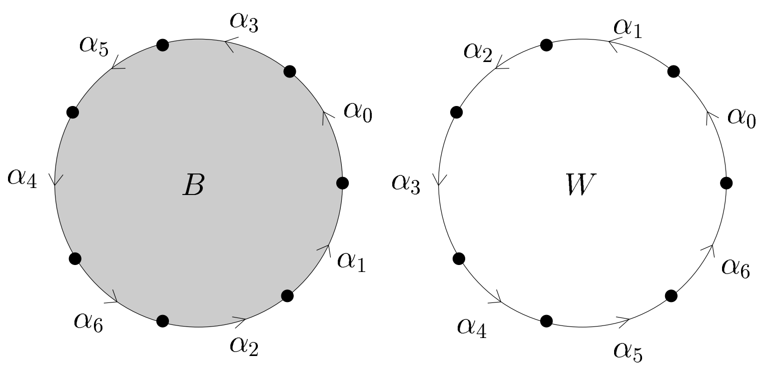

Figure 2: Example continued: identifications of arcs on ∂B and ∂W.

we label the points of $\mathcal{R}_p^W$ according to the labels in this cycle $\widetilde{f}$, as in Fig. 1-right, the definitions of f' and h are now immediately understood: again $f'(a_j^{(i)}) = a_{j+1}^{(i)}$, while $h : \partial W \to \partial B$ maps each open arc of $\partial W \setminus \mathcal{R}_p^W$ to the open arc of $\partial B \setminus \mathcal{R}_p^B$ having *first* endpoint with the same label. See Fig. 2 for the identifications of arcs in our example, and Fig. 3 for the resulting checkerboard coloring of the genus-2 surface.

Proof. Suppose a minimal checkerboard graph G is given. Since g is positive, G has vertices. Suppose there are q of them, and label them by $a^{(i)}$ for $i = 1, \ldots, q$. If $a^{(i)}$ has valence $2\ell_i$, label the germs of white and black discs around $a^{(i)}$ in positive order as

$$a_0^{(i)}, a_0^{(i)}, a_1^{(i)}, a_1^{(i)}, \ldots, a_{\ell_i-1}^{(i)}, a_{\ell_i-1}^{(i)}$$

starting from an arbitrary *white* germ. Recalling that there is only one white disc, label the edges of G as $\alpha_1, \ldots, \alpha_p$ while following the boundary of the white disc in the *negative* fashion. Now read the labels of the germs of discs and of the edges on the boundary of the abstract white and black discs W and B with respect to the obvious immersions from W and B to the surface. The maps f and f' of the definition are now those telling that the vertices $a_j^{(i)}$, for $j = 0, \ldots, \ell_i - 1$, become the single vertex $a^{(i)}$ in the surface, and describing the way the germs are cyclically arranged around $a^{(i)}$. Similarly, the map h says that each arc α_k on ∂W is glued to the corresponding α_k on ∂B. The cell decomposition of the surface associated to G consists of q vertices, p edges, and 2 discs, whence $p - q = 2g$, which implies that G arises from $f \in \mathfrak{S}(\mathcal{R}_p^B)$ as described in point 1.

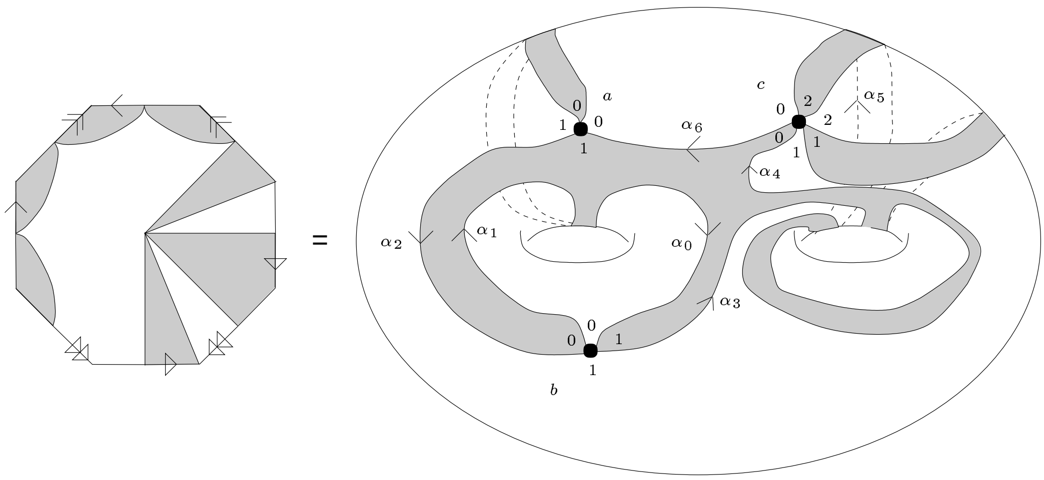

Figure 3: Example continued: a minimal checkerboard coloring of the genus-2 surface.

Now suppose $f \in \mathfrak{S}(\mathcal{R}_p^B)$ is given. For each orbit $a_0^{(i)}, \ldots, a_{\ell_i - 1}^{(i)}$ of f we identify the points of the orbit to a single point $a^{(i)}$ and we give to the quotient space a planar structure near $a^{(i)}$ so that the germs of discs arising from the $a_j^{(i)}$'s are cyclically arranged in the order $j = 0, \ldots, \ell_i - 1$. With this choice a system of attaching loops for white discs is well-defined. The condition that $\widetilde{f}$ is a full cycle translates the fact that a single white disc is glued to the black disc, and the condition that $p - q = 2g$ means that the result of the gluing is the orientable genus-g surface. $\square$

Corollary 2.3. *On $g\mathbb{T}$ there are finitely many minimal checkerboard graphs.*

Proof. With the notation of the previous proposition, we have that $q \leqslant p/2$ from the assumption that f has no fixed points. Whence $p \leqslant 4g$ from the condition $p - q = 2g$. The conclusion follows. $\square$

Genus 0 **and** 1 As already noticed, the only minimal checkerboard graph on $\mathbb{S}$ is a plain circle, which splits $\mathbb{S}$ into an embedded black and an embedded white disc. This corresponds in Proposition 2.2 to the limit case $p = q = 0$.

For genus 1, the argument proving Corollary 2.3 shows that $p \leqslant 4$. Moreover $p - q = 2$. Now we note the following easy general fact:

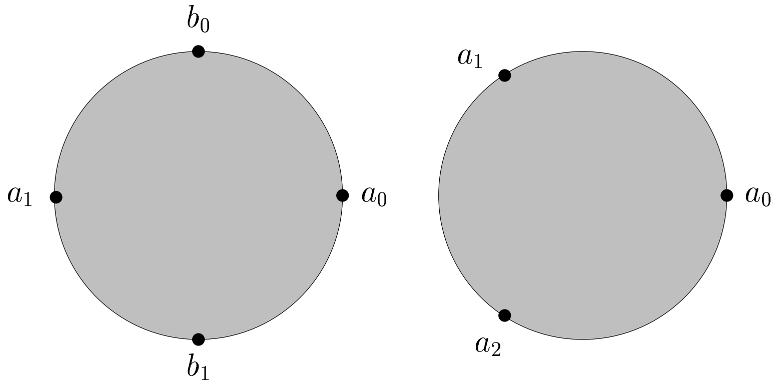

Figure 4: Permutations of roots of unity relevant for the torus.

Remark 2.4. Let $f \in \mathfrak{S}(\mathcal{R}_p^B)$ and suppose that f maps some point in $\mathcal{R}_p^B$ to the point preceding it in the counter-clockwise order on ∂B. Then the corresponding $\widetilde{f}$ as in Proposition 2.2 has a fixed point, so it is not a full cycle.

According to this result, only the p's and f's as in Fig. 4 can be relevant for $g = 1$. Both these permutations f satisfy the condition that the corresponding $\widetilde{f}$ as in Proposition 2.2 is a full cycle. The associated minimal checkerboard graphs (and colorings) of the torus are shown in Fig. 5.

Existence of coverings We now prove our extension of Baránski's criterion for existence of a branched covering onto $\mathbb{S}$. We denote by $V(G)$ the set of vertices of a graph G and by $\mathrm{val}_G(w)$ the valence in G of a vertex w.

Theorem 2.5. *A branch datum $\big(g\mathbb{T}, \mathbb{S}, n, d, (d_{ij})\big)$ is realizable if and only if there exists on $g\mathbb{T}$ a minimal checkerboard graph G associated to $f \in \mathfrak{S}(\mathcal{R}_p^B)$ as in Proposition 2.2, a set S of $n \cdot d - p$ points of $G \setminus V(G)$, a labelling in $\{1, \ldots, n\}$ of $V(G) \cup S$ and a collection $\Gamma = \bigcup \Gamma_{ij}$ of graphs on $g\mathbb{T}$ such that:*

- *The Γ_{ij}'s are pairwise disjoint trees;*
- *The set of vertices of Γ is equal to $G \cap \Gamma$ and is also equal to $V(G) \cup S$;*
- *Pulling back the labelling of the points through the projection $\partial B \to G$ (or, equivalently, $\partial W \to G$) we find $1, \ldots, n, \ldots, 1, \ldots, n$, with each string $1, \ldots, n$ appearing d times, in counter-clockwise order;*

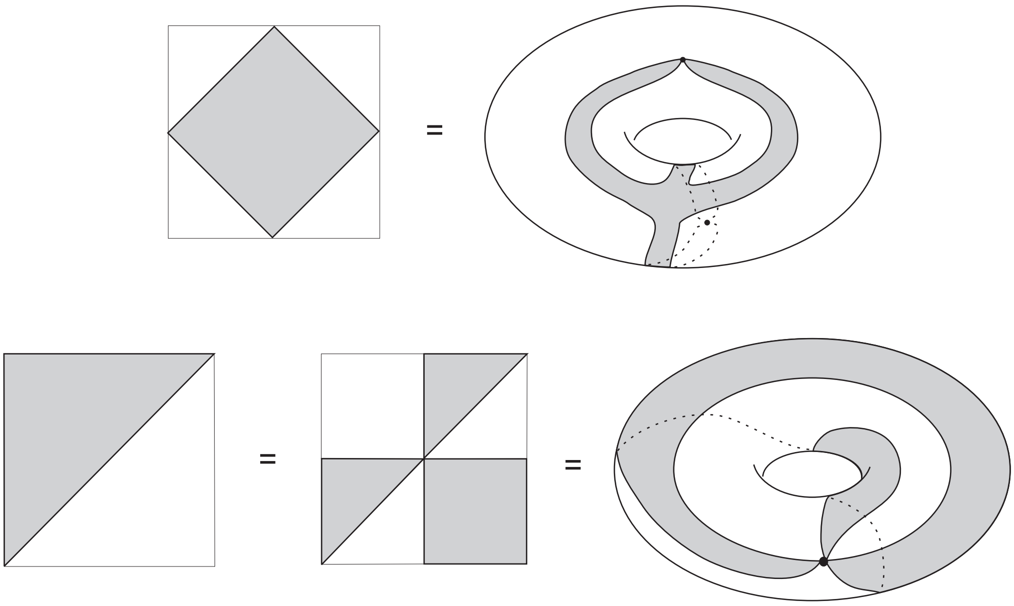

Figure 5: Minimal checkerboard graphs on the torus.

- $\Gamma_{ij} \cap G$ *consists of points having label* i*;*
- *The number of edges of* Γ_{ij} *is given by*

$$d_{ij} - 1 - \sum_{w \in V(G) \cap \Gamma_{ij}} \left(\frac{1}{2} \mathrm{val}_G(w) - 1 \right).$$

Proof. Suppose the labelling of points and the trees Γ_{ij} exist. Let us assign to all the points of Γ_{ij} the label i, and let us call *i-portion* of a set X a connected component of the set of points of X having label i. We claim the following assertion:

> *The complement in $g\mathbb{T}$ of $G \cup \Gamma$ consists of d black and d white discs. On each black (respectively, white) disc the labelled portions of Γ cyclically found on the boundary in the positive (respectively, negative) order have labels $1, \ldots, n$.*

Assuming the assertion for a moment, let us explain how to construct the desired covering. Since the Γ_{ij}'s are disjoint trees, we can collapse them to

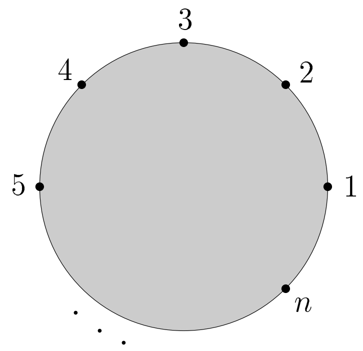

Figure 6: The sphere as a union of a black and a white disc.

points without changing the topology of $g\mathbb{T}$. After this collapse, G gives a graph G' on $g\mathbb{T}$ with labelled vertices and the complement consisting of d black and d white discs. Moreover on each black (respectively, white) disc the labels of the vertices cyclically found in the positive (respectively, negative) order on the boundary are $1, \ldots, n$. And each edge of G' separates discs of different colors, because this is true for G. The last two statements easily imply that the discs of $g\mathbb{T} \setminus G'$ have embedded closures.

Consider now the analogous decomposition of $\mathbb{S}$ as shown in Fig. 6, with one black and one white disc. We map the closure of each black (respectively, white) disc on $g\mathbb{T} \setminus G'$ to the closure of the black (respectively, white) disc on $\mathbb{S}$, matching the labels of the vertices. These maps can be arranged to coincide on the edges of G (which are shared by white and black discs). The result is of course a branched covering, and to conclude we only need to prove that the local degree at the point of $g\mathbb{T}$ arising from the collapse of Γ_{ij} is d_{ij}. This local degree is half of the valence of the vertex in G'. We now show that half of this valence is given by the number of edges of Γ_{ij} plus

$$1 + \sum_{w \in V(G) \cap \Gamma_{ij}} \left(\frac{1}{2} \mathrm{val}_G(w) - 1 \right)$$

which implies the conclusion by the assumption. The formula is of course correct when Γ_{ij} has no edges, because it must consist of a single point of $S \cup V(G)$. Adding an edge with endpoint w to an already existing Γ_{ij}, half of the valence in G' increases by $\frac{1}{2}\mathrm{val}_G(w) = 1 + \left(\frac{1}{2}\mathrm{val}_G(w) - 1\right)$. The conclusion follows.

Let us now prove the claimed assertion. Note first that, by the second assumption and the last one, the total number of edges in Γ is

$$\sum_{i=1}^{n}\sum_{j=1}^{m_i}(d_{ij}-1)-\sum_{w\in V(G)}\left(\frac{1}{2}\mathrm{val}_G(w)-1\right).$$

The first sum gives $n\cdot d-\widetilde{n}$ while the second sum gives $p-q$. Since $n\cdot d-\widetilde{n}=2d-2+2g$ and $p-q=2g$, the total number of edges is $2d-2$. Before adding edges, *i.e.* in $g\mathbb{T}\setminus G$, there are two discs, and each time we add an edge we increase by one the number of discs, therefore the total number of discs in $g\mathbb{T}\setminus G'$ is indeed $2d$.

To proceed, let us call *G-arc* a connected component of $G\setminus(S\cup V(G))$ and let us assign the label i to all the points of a G-arc having ends i and $i+1$ in counter-clockwise order with respect to B. If U is a component of $g\mathbb{T}\setminus(G\cup\Gamma)$ and we are given distinct points of ∂U, we can speak of the two halves into which the points split ∂U, even if we do not know that the closure of U is embedded. We claim the following: *for every component U of $g\mathbb{T}\setminus(G\cup\Gamma)$, for all $i\neq i'$ and for every two distinct points of label i' in ∂U, at least one of the halves into which the points split ∂U contains a G-arc of label i. The other half is either entirely labelled with i' or it also contains a G-arc of label i.* We prove this claim again by imagining that Γ has been constructed adding one edge after the other. The claim is of course true before adding edges, *i.e.* in G. Assume the claim is true up to some stage. When we add the next edge inside a certain region U, we replace U by two regions U' and U'', and both $\partial U'$ and $\partial U''$ differ from ∂U as follows: a (possibly immersed) segment having ends of a certain label i' is replaced by an embedded segment entirely labelled by i'. One easily sees that the property stated in the claim is preserved by such a transformation (recall that Γ consists of trees).

The claim implies that each component of $g\mathbb{T}\setminus(G\cup\Gamma)$ contains at least one G-arc of each label. But there are $2d$ components, n labels and $n\cdot d$ arcs, and each G-arc is shared by 2 components, so the boundary of each component of $g\mathbb{T}\setminus(G\cup\Gamma)$ contains precisely one G-arc for each label. For the same reasons, there are d black and d white components. The cyclic order of the labels of the G-arcs is preserved at each insertion of an edge, so it is $1,\ldots,n$ in positive (respectively, negative) order on the black (respectively, white) discs. To conclude the proof of the assertion we only need to note that a G-arc of label i and one of label $i+1$ are always separated by an i-portion of Γ. The assertion, and hence the existence of the covering, are eventually established.

Having proved the sufficiency of the conditions stated in the theorem, let us turn to the opposite implication. We then assume that the desired branched covering $g\mathbb{T} \to \mathbb{S}$ exists and we suppose, which of course we can, that the branching points on $\mathbb{S}$ are the labelled points in Fig. 6.

Let H be the pre-image in $g\mathbb{T}$ of the circle in Fig. 6 and $\Delta = (\Delta_{ij})$ be the collection of pre-images of the branching points. We give a label i to Δ_{ij} and a black or white color to each disc in $g\mathbb{T} \setminus H$, according to the color of its projection. We consider moves on the pair $(H, (\Delta_{ij}))$ which preserve the following properties:

- H is a checkerboard graph on $g\mathbb{T}$;
- The Δ_{ij}'s are mutually disjoint trees;
- The set of vertices of each Δ_{ij} is given by its intersection with H;
- The number of edges of Δ_{ij} plus a contribution of $\frac{1}{2}\mathrm{val}_H(v) - 1$ for each vertex v of H contained in Δ_{ij} is $d_{ij} - 1$.

Note that the conditions are of course met at the beginning. There are two moves that we employ, a black one and a white one. The white one applies if at some vertex of H there are two distinct white discs. The move itself is described in Fig. 7 where we assume that $R_1 \neq R_2$. Note that there is some arbitrariness in the move, related to the choice of the vertex where it is applied, to the white discs of $g\mathbb{T} \setminus H$ which are merged, and to the new position of the edges in Δ coming from the previous steps. The black move is completely analogous to the white one.

Let us now apply the moves as long as possible and call $(G, (\Gamma_{ij}))$ the final $(H, (\Delta_{ij}))$. By definition G is a minimal checkerboard graph (to ensure the minimality, we stipulate that the vertex set of G consists of the points with link of cardinality at least three). Let us define S to be the set of points of $G \cap \Gamma$ which are not vertices of G, and let us assign the label i to a point of $S \cup V(G)$ if it belongs to some Γ_{ij}. It is now a routine matter to repeat the above arguments in the reverse order to show that the assumptions of the theorem are verified. $\square$

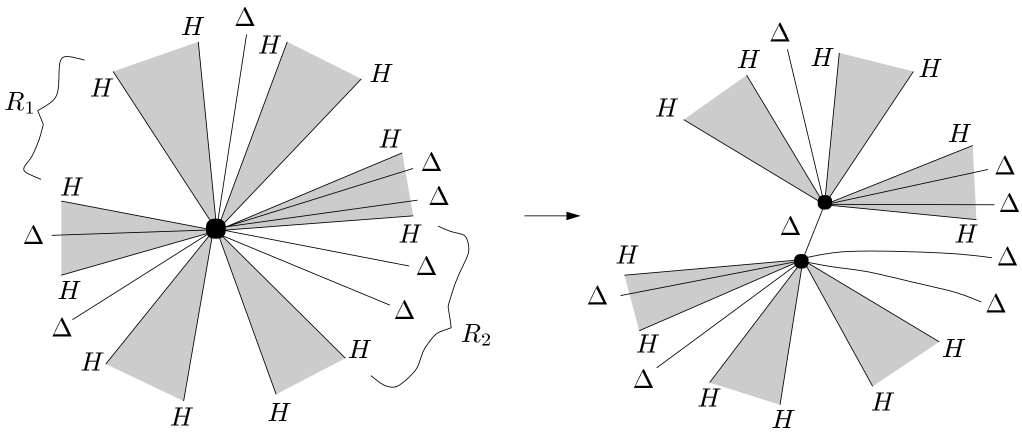

Figure 7: The white move.

Useful minimal checkerboard graphs According to Theorem 2.5, only the minimal checkerboard graphs supporting the further structures described in the theorem are relevant to the Hurwitz existence problem, so *a priori* some graph may be dispensable. However we have the following:

Proposition 2.6. *If G is a minimal checkerboard graph for $g\mathbb{T}$ then there exists a branch datum with $n = 3$ and suitably large d which is realized according to Theorem 2.5 using the graph G.*

Proof. Let us give label 1 to all the vertices of G. Let S consist of two points (with labels 2 and 3) on each edge of G. Now consider in B and W the trees as shown in Fig. 8. These trees give a branched covering of type $(d), (d), (*)$. $\square$

This result does not imply, however, that all minimal checkerboard graphs are necessary to construct the realizable branch data, because a branch datum could be constructed using different graphs. The next remark shows that this is what happens in the case of the torus.

Remark 2.7. Let us denote by G_1 and G_2 the minimal checkerboard graphs on the torus that are shown in Fig. 5-top and in Fig. 5-bottom, respectively. The graph G_1, labels and trees in Fig. 9 give a covering realizing the datum $\big(\mathbb{T}, \mathbb{S}, 4, 2, (2), (2), (2), (2)\big)$, whose non-trivial automorphism is the hyperelliptic involution, which cannot be obtained using the graph G_2. On the

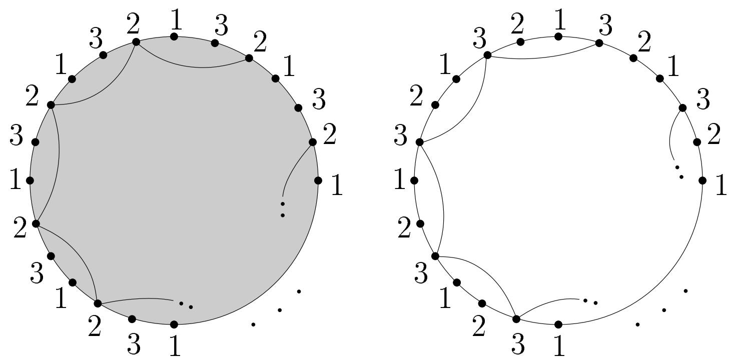

Figure 8: Trees showing that all minimal checkerboard graphs give rise to some covering.

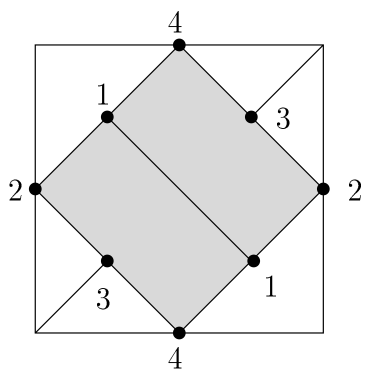

Figure 9: Realization of a covering $\mathbb{T} \to \mathbb{S}$.

contrary, we will now show that any datum that can be realized using G_2 can also be realized using G_1. Indeed, suppose that a realization of some datum is given by a family of graphs with labelled vertices $G_2 \cup \Gamma \subset \mathbb{T}$, according to Theorem 2.5. We first claim that in the white disc W there is an edge e with ends in internal points of different edges of G_2. If this is not the case then there is a disc in $W \setminus (G_2 \cup \Gamma)$ doubly incident to v, but we have shown within the proof of Theorem 2.5 that (even after collapsing each component of Γ to a point) these discs have embedded closures. The claim is proved and we can now collapse e to a point and perform a white move at v, which turns G_2 into G_1 and Γ into another union of trees realizing the same covering.

3 Dessins d'enfants

In this short section we review a technique introduced in [5] to investigate existence of branched coverings. It is based on the notion of dessin d'enfant due to Grothendieck [3]. This notion in its original form was only relevant to the case of $n = 3$ branching points, but in [5] it was extended to arbitrary $n \geqslant 4$. In this paper, however, we will only need it for $n = 3$, so we quickly present it in its simplified form here.

To begin we recall that a graph (a finite 1-complex) is called *bipartite* if its set of vertices is split as $V_1 \sqcup V_2$ and each edge has one endpoint in V_1 and one in V_2.

Definition 3.1. A *dessin d'enfant* on the surface $\widetilde{\Sigma}$ is a bipartite graph $D \subset \widetilde{\Sigma}$ such that $\widetilde{\Sigma} \setminus D$ consists of open discs. The *length* of one of these discs is the number of edges of D along which its boundary passes (with multiplicity).

Proposition 3.2. *The realizations of a branch datum* $\left(\widetilde{\Sigma}, \mathbb{S}, 3, d, (d_{ij})\right)$ *correspond to the dessins d'enfants* $D \subset \widetilde{\Sigma}$ *with set of vertices split as* $V_1 \sqcup V_2$ *such that for* $i = 1, 2$ *the vertices in* V_i *have valences* $(d_{ij})_{j=1}^{m_i}$*, and the discs in* $\widetilde{\Sigma} \setminus D$ *have lengths* $(2d_{3j})_{j=1}^{m_3}$.

Proof. Suppose a realization $f : \widetilde{\Sigma} \to \mathbb{S}$ exists, let the branching points be p_1, p_2, p_3, choose an arc α joining p_1 to p_2 and avoiding p_3, and define D as $f^{-1}(\alpha)$. Setting $V_i = f^{-1}(p_i)$ for $i = 1, 2$, it is clear that D is a bipartite graph with partition of vertices $V_1 \sqcup V_2$ with the required valences. Now

$\mathbb{S} \setminus \alpha$ is an open disc and the restriction of f to any component of $\widetilde{\Sigma} \setminus D$ is a covering onto this disc with a single branching point. Such a covering is always modelled on the covering $z \mapsto z^k$ of the open unit disc onto itself, so the components of $\widetilde{\Sigma} \setminus D$ are open discs. More precisely, there is one such disc for each element of $f^{-1}(p_3)$, and it is easy to see that the j-th one has length $2d_{3j}$.

Reversing this construction is a routine matter that we can leave to the reader. $\square$

To apply Proposition 3.2 we will often switch the viewpoint: instead of trying to embed a dessin D in the surface $\widetilde{\Sigma}$, we will try to thicken a given bipartite graph D with certain prescribed valences of vertices to a surface with boundary, so to get $\widetilde{\Sigma}$ by capping off the boundary circles.

4 Existence results

In this section we establish Theorems 1.1 to 1.3. Our proofs will repeatedly employ Theorem 1.5. We begin with some technical notions.

Diagrams and accessible trees To investigate the branch data involving the partition $(d-2, 2)$ we will apply the geometric existence criterion Theorem 2.5, of which we will use the notation throughout. For the sake of brevity, a triple G, S, Γ as in the statement of this theorem will be called a *diagram* realizing the corresponding branch datum. A connected component of $\widetilde{\Sigma} \setminus (G \cup \Gamma)$ will be called a *face* of this diagram. Recalling that each tree $\Gamma_{ij} \in \Gamma$ corresponds to some entry d_{ij} in one of the partitions of the relevant branch datum, we will call *degree* of Γ_{ij} the associated d_{ij}. The collection of degrees of the trees incident to a given face (*i.e.* having non-empty intersection with its closure in $\widetilde{\Sigma}$) will be called the *type* of the face.

For the rest of the paper we fix branch data as in the assumptions of Theorems 1.1, 1.2, and 1.3, *i.e.* we will henceforth assume that $m_1 = 2$ and $(d_{11}, d_{12}) = (d-2, 2)$. Moreover, in all our diagrams the trees Γ_{11} and Γ_{12} will correspond to this partition of d. For $i = 2, 3$ we will say that a tree Γ_{ij} of a diagram is *accessible* if there is a G-arc with one endpoint in Γ_{ij} and the other endpoint in Γ_{11}. Recall that a G-arc is a connected component of $G \setminus (S \cup V(G))$. A diagram is called *accessible* if all Γ_{ij}'s with $i = 2, 3$ are accessible.

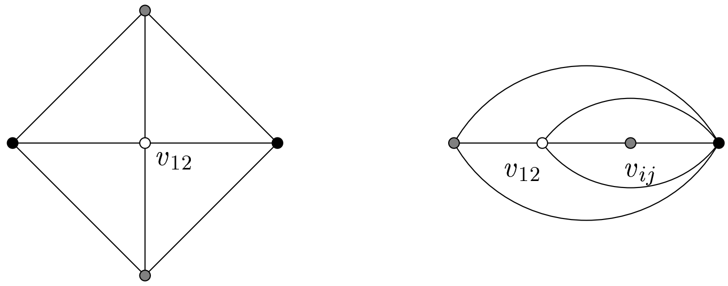

Figure 10: The triangles incident to v_{12}.

Lemma 4.1. *In a diagram realizing our branch datum there is at most one inaccessible tree, which must be a point that is not a vertex of G.*

Proof. Recall the collapse used in the proof of Theorem 2.5, where each Γ_{ij} is shrunk to a point v_{ij} labelled $i \in \{1,2,3\}$, and G is transformed to some graph G'. The valence of v_{ij} is $2d_{ij}$, so v_{11} has valence $2d-4$ and v_{12} has valence 4. The complement of G' consists of $2d$ triangles, each with embedded closure and vertices labelled $1,2,3$. So there are $2d-4$ triangles incident to v_{11}, and, if Γ_{ij} is inaccessible, then v_{ij} does not belong to any of them. So it belongs to the interior of the union U of the other 4 triangles, which are all incident to v_{12}. The only two possibilities for U are as shown in Fig. 10, and the conclusion follows. $\square$

Odd-degree coverings by the sphere In the case $\widetilde{\Sigma} = \mathbb{S}$ it will be convenient to investigate the data with odd degree d first. Namely, we establish the following:

Proposition 4.2. *If d is odd then every compatible branch datum of the form $\big(\mathbb{S}, \mathbb{S}, 3, d, (d-2,2), (d_{2j}), (d_{3j})\big)$ is realizable.*

Proof. Note first that compatibility means that $m_2 + m_3 = d$ and recall that, in a diagram G, S, Γ realizing the datum, G is a plain circle, S consists of $3d$ points, and Γ_{ij} is a tree with $d_{ij}-1$ edges. So Γ_{11} has $d-3$ edges and Γ_{12} is just one edge.

By Theorem 1.5 the datum is realizable if $(d_{ij})_{j=1}^{m_i} = (d)$ for some $i \in \{2,3\}$. We will then assume henceforth that this is not the case, which easily implies that $d \geqslant 5$ (recall that d is odd). To prove the proposition the general strategy will now be to proceed by induction on d, starting from the

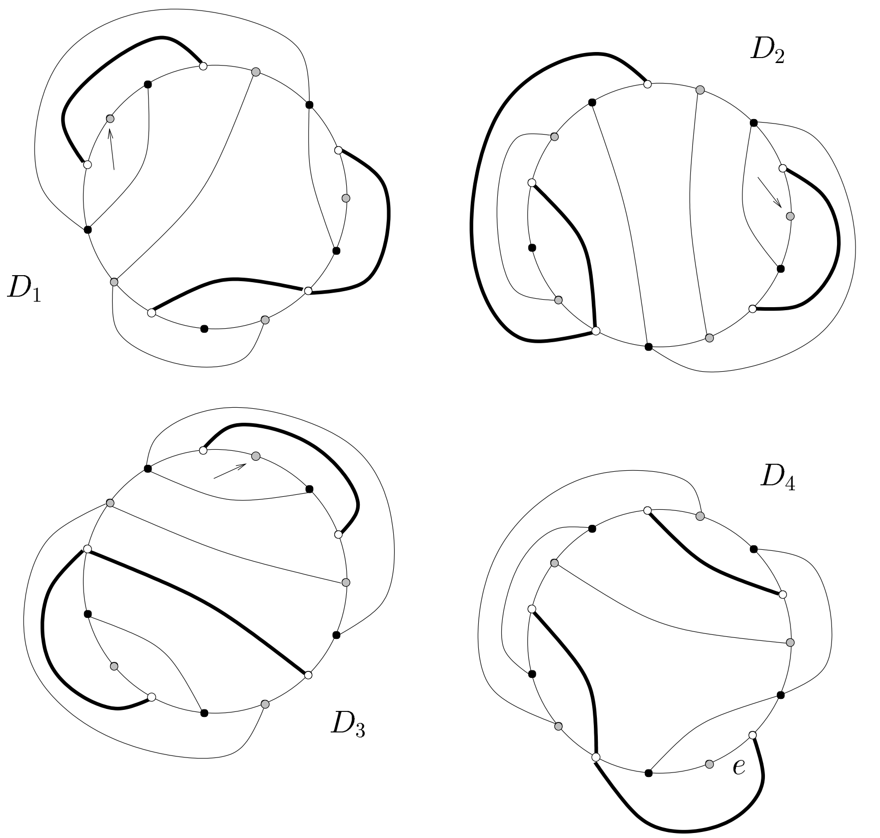

Figure 11: The diagrams $D_1, \ldots, D_4$, with Γ_{11} and Γ_{12} marked by thicker lines. In D_1, D_2, D_3 a small arrow indicates the only inaccessible vertex, while the G-arc e on D_4 will be used later in the proof. The partitions $(d_{ij})_{i=2,3}^{j=1,\ldots,m_i}$ are $(4,1),(3,1,1)$ for D_1, $(4,1),(2,2,1)$ for D_2, $(3,2),(3,1,1)$ for D_3, and $(3,2),(2,2,1)$ for D_4.

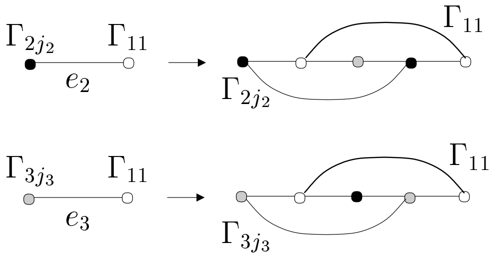

Figure 12: The move μ_1.

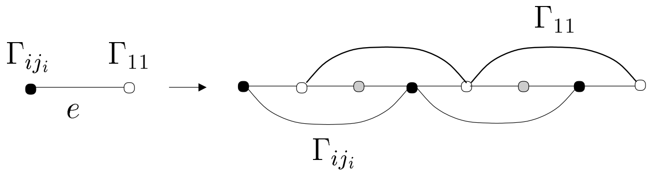

Figure 13: The move μ_2.

diagrams $D_1, \ldots, D_4$ of Fig. 11, which realize all the relevant data for $d = 5$, and successively applying certain moves to these diagrams so to realize all the relevant data for arbitrary d. However, we will have to deal with special sorts of data by separate inductions.

We will employ three moves μ_1, μ_2, μ_3, that we now describe:

μ_1: Choose G-arcs e_2 and e_3 such that each e_i has one end in Γ_{11} and the other end in some Γ_{ij_i}, and perform the modifications of Fig. 12;

μ_2: Choose a G-arc e with one end in Γ_{11} and one in some Γ_{ij_i}, and perform the modification of Fig. 13;

μ_3: Choose a G-arc e with one end in Γ_{11} and one in some Γ_{ij_i}, and perform the modification of Fig. 14. Since we will sometimes use a systematic iteration of this move, a new G-arc is specified and also labelled e in the new diagram.

It is evident that all three moves transform a diagram of degree d to a diagram of degree $d + 2$ which still satisfies the conditions of Theorem 2.5. We call a diagram *constructible* if it is obtained from one of the diagrams

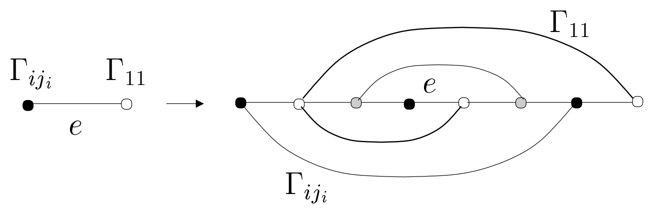

Figure 14: The move μ_3.

$D_1, \ldots, D_4$ by successive application of moves μ_*. We will prove that each relevant branch datum is realized by a constructible diagram.

Let us note now that the effect of the μ_*'s on the partitions (d_{2j}) and (d_{3j}) is as follows:

- $\widehat{\mu}_1$: Choose j_2, j_3, replace d_{2j_2} and d_{3j_3} by $d_{2j_2}+1$ and $d_{3j_3}+1$ respectively, add a 1 at the end of both partitions, and reorder (if necessary: recall that our partitions are arranged in non-increasing order);

- $\widehat{\mu}_2$: Choose $i \in \{2,3\}$ and j_i, let $\{2,3\} = \{i, i'\}$, replace d_{ij_i} by $d_{ij_i}+2$ and reorder, and add two 1's at the end of $(d_{i'j})$;

- $\widehat{\mu}_3$: Choose $i \in \{2,3\}$ and j_i, let $\{2,3\} = \{i, i'\}$, replace d_{ij_i} by $d_{ij_i}+1$, add a 1 at the end, add a 2 at the end of $(d_{i'j})$, and reorder.

Note also that the $\widehat{\mu}_*$'s can be viewed as moves transforming a relevant branch datum of degree d into one of degree $d+2$. However, *a priori* not every branch move $\widehat{\mu}_*$ on the branch datum realized by a constructible diagram is induced by the corresponding geometric move μ_* on that diagram. When this happens, we will say that the move $\widehat{\mu}_*$ itself is *constructible*. Of course a move $\widehat{\mu}_*$ applied to indices j_2, j_3 (for $\widehat{\mu}_1$) or j_i (for $\widehat{\mu}_2$ or $\widehat{\mu}_3$) is constructible if and only if the corresponding trees Γ_{2j_2} and Γ_{3j_3} or Γ_{ij_i} are accessible. Lemma 4.1 then implies that *a branch move $\widehat{\mu}_*$ is constructible if applied to indices such that the corresponding d_{2j_2} and d_{3j_3} or d_{ij_i} are larger than 1.* We also note that a move μ_* never creates inaccessible trees. Therefore *a branch move $\widehat{\mu}_*$ is constructible if it is applied to a diagram constructed starting from D_4 or to tree(s) created by previous geometric moves.* These facts will be used repeatedly below.

We now state and prove a series of claims that will lead to the conclusion. We start with an easy arithmetic one.

Claim 1. *Either* $(d_{ij})_{i=2,3}^{j=1,\ldots,m_i}$ *includes two* 1*'s or* $(d_{2j})_{j=1}^{m_2} = (3,2,\ldots,2)$ *and* $(d_{3j})_{j=1}^{m_3} = (2,\ldots,2,1)$*, up to permutation.* Assume first there are no 1's. Since d is odd we deduce that $d_{21}, d_{31} \geqslant 3$. Therefore

$$\left(d \geqslant 3 + 2(m_i - 1) \Rightarrow m_i \leqslant \frac{d-1}{2},\ i = 2,3\right) \Rightarrow m_2 + m_3 \leqslant d - 1,$$

which gives a contradiction. Suppose then that $d_{3m_3} = 1$ and there are no other 1's. Then again $m_2 \leqslant \frac{d-1}{2}$, and $m_3 \leqslant \frac{d+1}{2}$ by a similar argument. So both inequalities are equalities and the conclusion readily follows.

We are now ready to proceed with the main part of the proof, where we show that every relevant branch datum, *i.e.* one of the form

$$\left(\mathbb{S}, \mathbb{S}, 3, d, (d-2,2), (d_{2j})_{j=1}^{m_2}, (d_{3j})_{j=1}^{m_3}\right) \tag{1}$$

with odd $d \geqslant 7$ and $m_2, m_3 \geqslant 2$, is realized by some constructible diagram. The proof is separate for branch data that involve a partition of the form $(3,\ldots,3,2,\ldots,2)$, *i.e.* those of form

$$\left(\mathbb{S}, \mathbb{S}, 3, d, (d-2,2), (\underbrace{3,\ldots,3}_{k}, 2, \ldots, 2), (d_{3j})_{j=1}^{m_3}\right). \tag{2}$$

Claim 2. *If* $(d_{2j}) = (3,\ldots,3,2\ldots,2)$ *and* $(d_{3j}) = (2,\ldots,2,1,\ldots,1)$ *then the datum can be realized by an accessible constructible diagram.* Since the total number of entries in the two partitions is d, the number k of 3's in the first partition is equal to the number of 1's in the second one, and k is odd. We prove the claim by induction on k. The diagram obtained from D_4 by $(d-5)/2$ successive applications of the move μ_3 starting with the G-arc e indicated in Fig. 11 gives the base $k=1$ of the induction.

For the inductive step, notice that to get the datum (2) at level k in degree d with the relevant (d_{3j}) we can apply a move $\widehat{\mu}_1$ (with $j_2 = k-1$ and $j_3 = m_3 - 3$) followed by a move $\widehat{\mu}_2$ (with $i = 2$ and $j_2 = m_2$) to the same datum at level $k-2$ in degree $d-4$, and the inductive assumption easily implies the conclusion.

Claim 3. *If* $(d_{2j}) = (3,2\ldots,2)$ *then the datum can be realized by a constructible diagram.* By induction on $d \geqslant 5$. For $d=5$ the conclusion is given by diagrams D_3 and D_4. For the inductive step, suppose first that $d_{31} \leqslant 2$. It follows that $(d_{3j}) = (2,\ldots,2,1)$, and this case was settled in Claim 2. Then we assume $d_{31} \geqslant 3$. So the datum (2) in degree d is obtained using a move $\widehat{\mu}_3$ (with $i = 3$ and $j_3 = 1$) from the datum with partitions

$$(3, 2, \ldots, 2), (d_{31} - 1, d_{32}, \ldots, d_{3(m_3-1)})$$

in degree $d - 2$. This datum is constructible by the inductive assumption, and the move is constructible because $d_{31} - 1 \geqslant 2$, whence the conclusion.

Claim 4. *If* $(d_{2j}) = (3, \ldots, 3, 2 \ldots, 2)$ *then the datum can be realized by a constructible diagram.* By induction on the odd number k of 3's in (d_{2j}). The base $k = 1$ is given by Claim 3. Suppose $k \geqslant 3$. Then it is easy to see that (d_{3j}) has at least k entries equal to 1. Moreover by Claim 2 we can assume $d_{31} \geqslant 3$. Then the datum (2) at level k in degree d is obtained from the datum with partitions

$$(3, \ldots, 3, 2, \ldots, 2), (d_{31} - 1, d_{32}, \ldots, d_{3(m_3-3)})$$

at level $k - 2$ in degree $d - 4$ by a move $\widehat{\mu}_1$ (with $j_2 = k - 1$ and $j_3 = 1$) and a move $\widehat{\mu}_2$ (with $i = 2$ and $j_2 = m_2$) . The conclusion now follows from the inductive assumption and from the remark that both moves are constructible, because $\widehat{\mu}_1$ is applied to entries which are at least 2, and $\widehat{\mu}_2$ is applied to an entry 1 created by $\widehat{\mu}_1$.

To conclude the proof, consider a branch datum of form (1), and let us prove by induction on d that it is realized by a constructible diagram. The base step follows from Fig. 11. To proceed, assume that no partition is $(3, \ldots, 3, 2 \ldots, 2)$, which is not restrictive by Claim 4. Claim 1 then implies that there are at least two 1's in $(d_{ij})_{i=2,3}^{j=1,\ldots,m_i}$. If only (d_{2j}) contains 1's then $d_{31} \geqslant 4$ by the assumption just made, so the datum is obtained from a move $\widehat{\mu}_2$ (with $i = 3$ and $j_3 = 1$) from

$$(d_{21}, \ldots, d_{2(m_2-2)}), (d_{31} - 2, \ldots, d_{3m_3})$$

and the move is constructible because $d_{31} - 2 \geqslant 2$, whence the conclusion.

Let us then assume that both (d_{2j}) and (d_{3j}) contain some 1. If one of these partitions, say (d_{2j}), is $(2, \ldots, 2, 1)$, then we see that $d_{31} \geqslant 3$, and the datum is obtained from

$$(2, \ldots, 2, 1), (d_{31} - 1, \ldots, d_{3(m_3-1)})$$

by a move $\widehat{\mu}_3$ (with $i = 3$ and $j_3 = 1$), which is constructible because $d_{31} - 1 \geqslant 2$. If neither (d_{2j}) nor (d_{3j}) is $(2, \ldots, 2, 1)$, then the datum is obtained from

$$(d_{21} - 1, \ldots, d_{2(m_2-1)}), (d_{31} - 1, \ldots, d_{3(m_3-1)})$$

by a move $\widehat{\mu}_1$ (with $j_2 = j_3 = 1$), and the move is constructible because either $d_{i1} - 1 \geqslant 2$ or there are two 1's to choose from, one of which must be accessible by Lemma 4.1. $\square$

Coverings by the sphere: general case Here we complete the proof of Theorem 1.1, using the results and techniques of the previous paragraph. We begin with the following:

Lemma 4.3. *If $k \geqslant 1$ then the partitions $(y_j)_{j=1}^m$ of $2k$ with $m \geqslant k$ that do not refine the partition (k,k) are precisely the following:*

- $(k+1, 1, \ldots, 1)$;
- $(2, \ldots, 2)$ *when k is odd.*

Proof. We proceed by induction on k, the basis $k = 1$ being obvious. For the inductive step, we assume as usual that $y_1 \geqslant \ldots \geqslant y_m$. If $y_1 > k$ then the partition has the first special form listed and it does not refine (k,k). If $y_m \geqslant 2$ then the partition has the second special form listed and it refines (k,k) precisely when k is even. We are left to show that if $y_1 \leqslant k$ and $y_m = 1$ then $(y_j)_{j=1}^m$ refines (k,k). The conclusion is obvious if $y_1 = 1$, so we exclude this case. Defining $x_1 = y_1 - 1$ and $x_j = y_j$ for $j = 2, \ldots, m-1$ we get a partition $(x_j)_{j=1}^{m-1}$ of $2(k-1)$ to which the inductive assumption applies. And it is actually very easy to check that (y_j) refines (k,k) both when (x_j) refines $(k-1, k-1)$ and when (x_j) has the second special form listed (it cannot have the first special form). $\square$

Proof of 1.1. By Proposition 4.2 it remains to consider only the case of even degree. So we must prove realizability of

$$\big(\mathbb{S}, \mathbb{S}, 3, 2k, (2k-2, 2), (d_{2j}), (d_{3j})\big) \tag{3}$$

with the two series of exceptions

$$\begin{gathered}\big(\mathbb{S}, \mathbb{S}, 3, 2k, (2k-2, 2), (2, \ldots, 2), (2, \ldots, 2)\big), \qquad k > 2 \\ \big(\mathbb{S}, \mathbb{S}, 3, 2k, (2k-2, 2), (2, \ldots, 2), (k+1, 1, \ldots, 1)\big).\end{gathered}$$

We first note that indeed the data of these forms are non-realizable. For the former series this is shown in [2, Corollary 6.4], while for the latter this is a consequence of [5, Theorem 1.5], because (d_{3j}) does not refine (k,k).

In the rest of the proof we exclude that (3) has one of the exceptional forms and we show it is realizable. We first do this assuming that all d_{2j} are even. Since $d_{2j} \geqslant 2$, we have $m_2 \leqslant k$ and hence $m_3 \geqslant k$, so Lemma 4.3 applies to (d_{3j}). And it allows us to conclude that (d_{3j}) does refine (k,k), because if it does not the Riemann-Hurwitz condition implies that $(d_{2j}) = (2,\ldots,2)$, so we are in one of the cases excluded.

Let us then reorder (d_{3j}) so that $\sum_{j=1}^{h} d_{3j} = \sum_{j=h+1}^{m_3} d_{3j} = k$ for some h and consider the branch datum

$$\big(\mathbb{S},\mathbb{S},4,k,(k-1,1),(d_{2j}/2),(d_{3j})_{j=1}^{h},(d_{3j})_{j=h+1}^{m_3}\big). \tag{4}$$

We claim that it does not match any of the two data listed in Proposition 1.6. If (4) matches the first datum of Proposition 1.6 then $k=4$ and $m_2+m_3=6$, whereas we know that $m_2+m_3=2k$. And (4) can only match the second datum of Proposition 1.6 if $(d_{3j})_{j=h+1}^{m_3}=(1,\ldots,1)$ up to permutation, whence $h=1$ by the same arguments used above, and the datum is not matched anyway. We deduce that (4) is realizable, and the existence of a covering corresponding to (3) follows by composition with a covering realizing $\big(\mathbb{S},\mathbb{S},3,2,(2),(2),(1,1)\big)$, which obviously exists.

The argument carried out so far leaves us to consider the case where both partitions (d_{2j}) and (d_{3j}) contain an odd entry. To deal with it we follow the same line of reasoning as in the proof of Proposition 4.2. To this end we introduce a move μ on diagrams. The move depends on the choice of a G-arc e_i joining Γ_{11} to some Γ_{ij_i}, and its effect is described, depending on whether $i=2$ or $i=3$, by either the top or the bottom part of Fig. 12. So μ_1 is just the combination of two moves μ applied to some e_2 and e_3.

Of course the effect of μ on the partitions (d_{2j}) and (d_{3j}) is the move $\widehat{\mu}$ which consists in replacing d_{ij_i} by $d_{ij_i}+1$ and reordering, and appending 1 to the other partition. Not every move $\widehat{\mu}$ comes from some μ, but it does if $d_{ij_i} \geqslant 2$, since in this case the tree Γ_{ij_i} is accessible by Lemma 4.1.

We now establish two claims that will readily lead to the conclusion. Recall that both (d_{2j}) and (d_{3j}) have an odd entry

Claim 1. *At least one of d_{2m_2} d_{3m_3} is equal to 1.* Suppose the contrary. For $i=2,3$ not all d_{ij} are even, so at least two of them are greater than 2 (recall that (d_{ij}) is a partition of $2k$), whence $m_i \leqslant k-1$. But we know that $m_2+m_3=2k$, a contradiction.

Claim 2. *Up to switching (d_{2j}) and (d_{3j}), we have $d_{21} \geqslant 3$ and $d_{3m_3}=1$.* By Claim 1 we can assume that $d_{3m_3}=1$. Suppose that $d_{2j} \leqslant 2$ for all j.

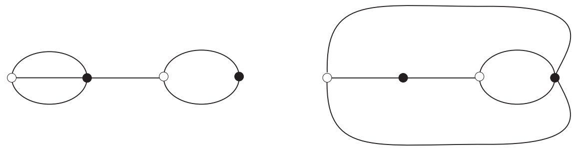

Figure 15: Two abstract bipartite graphs.

Since at least two of the d_{2j}'s are odd, we must have $d_{2m_2} = d_{2(m_2-1)} = 1$. In particular $m_2 \geqslant k+1$, hence $m_3 \leqslant k-1$, which implies that $d_{31} \geqslant 3$. Since $d_{2m_2} = 1$ we get the desired situation by switching the partitions.

To conclude the proof we remark that by Claim 2 the datum (3) can be obtained by the move $\widehat{\mu}$ from

$$\big(\mathbb{S}, \mathbb{S}, 3, 2k-1, (2k-3, 2), (d_{21}-1, d_{22}, \ldots, d_{2m_2}), (d_{31}, \ldots, d_{3(m_3-1)})\big),$$

and this datum is realizable by some diagram thanks to Proposition 4.2. In addition, the desired move is constructible because $d_{21} - 1 \geqslant 2$. $\square$

Coverings by the torus We now turn to the case $\widetilde{\Sigma} = \mathbb{T}$, where the result is even stronger.

Proof of 1.2. Note first that compatibility means $m_2 + m_3 = d - 2$. Let us first prove exceptionality of $\big(\mathbb{T}, \mathbb{S}, 3, 6, (4,2), (3,3), (3,3)\big)$ using dessins d'enfants, see Section 3. There are only two abstract bipartite graphs with two black vertices of valences 4 and 2 and two white vertices both of valence 3, see Fig. 15 (colors are used to represent the bipartition). It is now quite easy to analyze all the thickening of these graphs giving a torus minus some discs. The number of discs is automatically two, and one easily sees that one never gets two discs of length 6, which would be necessary to realize the partition $(3,3)$. For instance the thickenings given by taking regular neighbourhoods of the planar immersions of Fig. 16 give the partitions $(5,1)$ and $(4,2)$ respectively. Note that one could also use the second graph of Fig. 15 to get the partition $(5,1)$, but one could not the first graph of Fig. 15 to get the partition $(4,2)$.

Let us then prove realizability in general, excluding only the case of $\big(\mathbb{T}, \mathbb{S}, 3, 6, (4,2), (3,3), (3,3)\big)$. To do this we follow the same line of reasoning as in the proof of Theorem 1.1. Central to the proof is the move μ intro-

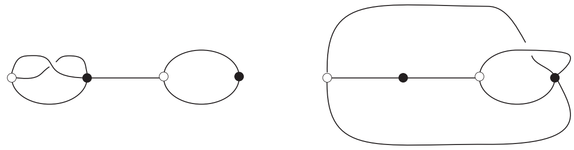

Figure 16: Two dessins d'enfants on the torus.

duced above. The following two facts concerning this move are repeatedly used below:

- A diagram realizing a datum with $d_{ij} \geqslant 2$ for all i, j is accessible;
- μ transforms accessible diagrams into accessible diagrams.

As in the previous proofs we proceed by establishing a series of claims.

CLAIM 1. *If d is odd and $d_{ij} \geqslant 2$ for all i, j then (d_{2j}) is $(3, 2, \ldots, 2)$ and (d_{3j}) is either $(5, 2, \ldots, 2)$ or $(4, 3, 2, \ldots, 2)$ or $(3, 3, 3, 2, \ldots, 2)$ up to permutation.* It follows from the assumptions that $m_2, m_3 \leqslant (d-1)/2$. Since their sum is $d-2$, the only option is that one be $(d-1)/2$ and the other $(d-3)/2$. The partitions of the statement of the claim are the only ones having these lengths and not containing 1's. A similar argument proves the next:

CLAIM 2. *If d is even and $d_{ij} \geqslant 2$ for all i, j then one of the following happens:*

- *Up to permutation, $(d_{2j}) = (2, \ldots, 2)$ and $(d_{3j}) \in \{(6, 2, 2, \ldots, 2),$ $(5, 3, 2, \ldots, 2), (4, 4, 2, \ldots, 2), (4, 3, 3, 2, \ldots, 2), (3, 3, 3, 3, 2, \ldots, 2)\}$;*
- $(d_{2j}), (d_{3j}) \in \{(4, 2, \ldots, 2), (3, 3, 2, \ldots, 2)\}$.

CLAIM 3. *If $d \geqslant 7$ and $d_{ij} \geqslant 2$ for all i, j then the datum is realizable.* This is proved using the dessins d'enfant of Section 3.

We present in Fig. 17 dessins d'enfants proving realizability of all the desired data except that with partitions $(3, 2, 2), (4, 3), (5, 2)$. For each dessin we have only drawn the minimal number of vertices it should have: an infinite series is obtained by adding pairs of vertices of opposite colors on any of the edges not incident to the length-4 disc that one easily sees in each picture. The branch datum with partitions $(3, 2, 2), (4, 3), (5, 2)$ is realized in Fig. 18.

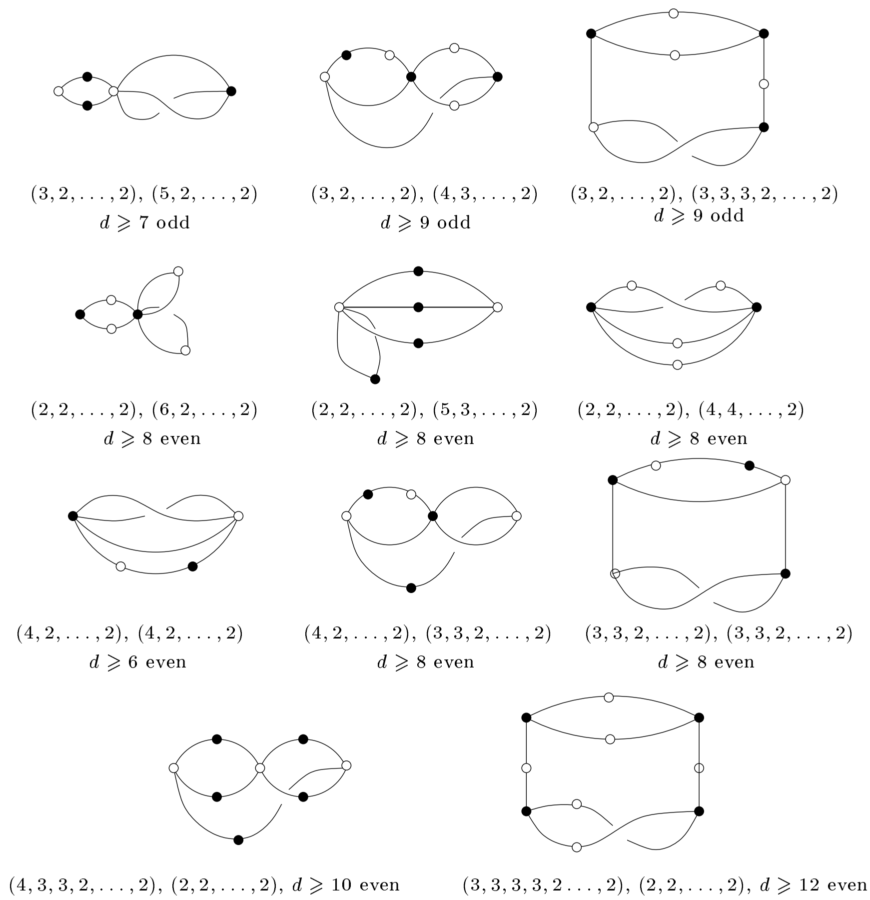

Figure 17: Dessins d'enfants on the torus for branch data not involving 1.

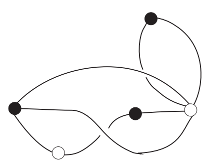

Figure 18: A dessin d'enfant for the branch datum with partitions $(3,2,2),(4,3),(5,2)$.

Claim 4. *If $d \leqslant 6$ then the datum is realizable by an accessible diagram.* The first relevant d is 4, where there is only the datum with partitions $(2,2),(4),(4)$. This datum is realizable by Theorem 1.5, and we conclude because $d_{ij} \geqslant 2$ for all i,j. For $d=5$ we have $(3,2),(3,2),(5)$ and $(3,2),(4,1),(5)$. Both are realizable by Theorem 1.5, the former has $d_{ij} \geqslant 2$ for all i,j, and the latter is obtained from $(2,2),(4),(4)$ by a move $\widehat{\mu}$, whence the conclusion. For $d=6$ we know experimentally that only the excluded datum is exceptional. The data with $d_{ij} \geqslant 2$ for all i,j are realizable by accessible diagrams, and those involving a 1 are obtained via a move $\widehat{\mu}$, and the conclusion follows from the case $d=5$.

Claim 5. *If $d \geqslant 7$ then the datum is realizable by an accessible diagram.* By induction. The base step $d=7$ and the inductive step are essentially identical. For both, either $d_{ij} \geqslant 2$ for all i,j, whence the conclusion from Claim 3, or there is some 1, so the datum is obtained via a move $\widehat{\mu}$ from a datum in degree $d-1$. For $d=7$ we use Claim 4 and the easy fact that the exceptional datum can always be avoided, while for $d \geqslant 8$ the conclusion is immediate from the inductive assumption. $\square$

Conjunction of diagrams In order to consider the case of genus higher than 1, we describe a procedure that we will use extensively.

Suppose we have two diagrams realizing branch data $\big(\widetilde{\Sigma},\mathbb{S},3,d,(d_{ij})\big)$ and $\big(\widetilde{\Sigma}',\mathbb{S},3,d',(d'_{ij})\big)$ respectively. Let α and α' be faces of opposite colors of these diagrams. As was shown in the proof of Theorem 2.5, α is incident to three trees $\Gamma_{1j_1},\Gamma_{2j_2},\Gamma_{3j_3}$ of distinct colors, and, similarly, α' to some $\Gamma'_{1j'_1},\Gamma'_{2j'_2},\Gamma'_{3j'_3}$. The following steps define the *conjunction of the diagrams along α and α'*:

- Contract the six trees Γ_{ij_i} and $\Gamma'_{ij'_i}$ to distinct points and remove the interiors of α and α'. This gives two compact surfaces, both bounded by a circle with three marked points bearing distinct colors 1,2,3.

- Glue the two surfaces along their boundaries, matching the colors of the marked points. This gives the closed surface $\widetilde{\Sigma}\#\widetilde{\Sigma}'$ with an embedded checkerboard graph and some trees.

- Perform the white and black moves of the proof of Theorem 2.5 until a diagram with minimal checkerboard graph is reached.

The resulting diagram of course satisfies the conditions of Theorem 2.5, and one easily sees that it realizes the datum $\big(\widetilde{\Sigma}\#\widetilde{\Sigma}', \mathbb{S}, 3, d+d'-1, (d''_{ij})\big)$, where for $i=1,2,3$ we get the partition (d''_{ij}) by appending (d'_{ij}) to (d_{ij}), removing the entries d_{ij_i} and $d'_{ij'_i}$, adding the entry $d_{ij_i}+d'_{ij'_i}-1$, and reordering. Note that conjunction induces a certain move on branch data. Constructibility of this move seems hard to investigate in general, but we will prove it when necessary.

Let us note that the coloring of faces and trees of one diagram can be changed arbitrarily, so neither the condition that α and α' have opposite colors nor the fact that the gluing be color-preserving is essential. We also remark that the move μ used above can be viewed as a conjunction with the diagram realizing the datum $\big(\mathbb{S}, \mathbb{S}, 3, 2, (2), (2), (1,1)\big)$.

The case of higher genus Here we consider the case when the covering surface has genus at least 2. In this case there are no obstructions to realizability.

Proof of 1.3. We will actually establish the following stronger statement: *any compatible datum of the form*

$$\big(g\mathbb{T}, \mathbb{S}, 3, d, (d-2,2), (d_{2j})_{j=1}^{m_2}, (d_{3j})_{j=1}^{m_3}\big), \qquad g \geqslant 2 \tag{5}$$

is realized by an accessible diagram. Note that compatibility means that $m_2+m_3=d-2g$, whence $d\geqslant 6$, and that the same statement was already proved above for $g=1$, with a single exception for $d=6$.

The proof is by induction on d simultaneously for all g. The base step is easy: for $d=6$, there is a single compatible datum as in (5), namely $\big(2\mathbb{T}, \mathbb{S}, 3, 6, (4,2), (6), (6)\big)$, which is realizable by Theorem 1.5. Then by Theorem 2.5 there is a diagram realizing the datum, and the diagram is automatically accessible by Lemma 4.1.

For the inductive step we need a preliminary fact. Recall that our partitions are arranged non-increasingly.

Lemma 4.4. *For a compatible datum as in (5), at least one of the following holds up to switching (d_{2j}) and (d_{3j}):*

1. $d_{3m_3}=1$;

2. *All d_{2j} and d_{3j} are at least 2, and $d_{3(m_3-1)}=d_{3m_3}=2$;*

3. $d_{2m_2} = d_{3m_3} = 2$*;*

4. There is an index j *such that* $d_{3j} \geqslant 3$ *and* $d_{21} > d_{3j}$*;*

5. All d_{2j} *and* d_{3j} *are equal to one and the same integer* k*.*

Proof. Assume that neither of the first three cases holds, *i.e.* that $d_{2m_2} \geqslant 3$, $d_{3m_3} \geqslant 2$, and $d_{3(m_3-1)} \geqslant 3$. If the fourth case also does not hold, we deduce that $d_{21} \leqslant d_{3(m_3-1)}$ and $d_{31} \leqslant d_{2m_2}$. Therefore

$$d_{21} \leqslant d_{3(m_3-1)} \leqslant d_{3(m_3-2)} \leqslant \ldots \leqslant d_{31} \leqslant d_{2m_2} \leqslant d_{2(m_2-1)} \leqslant \ldots d_{21},$$

which implies that $d_{21} = \ldots = d_{2m_2} = d_{31} = \ldots = d_{3(m_3-1)}$. Denoting this number by k, we see that $d = km_2 = k(m_3 - 1) + d_{3m_3}$. This implies that d_{3m_3} is divisible by k, and since $0 < d_{3m_3} \leqslant d_{3(m_3-1)} = k$, it is also equal to k. Thus, we have the fifth case. $\square$

We now proceed with the inductive step, supposing $d > 6$ and the realizability of (5) to be known for degrees smaller than d. We treat the different cases described in Lemma 4.4 separately, employing both the technique of dessins d'enfants and the criterion of Theorem 2.5. We also notice that by Lemma 4.1 a realizable datum that does not involve 1's is always realized by an accessible diagram, so in Cases 2-5 (where we always assume that Case 1 is not applicable) it will be sufficient to show that our datum is realizable. For this reason, recalling Theorem 1.5, in Cases 2-5 we will also assume $m_2, m_3 \geqslant 2$.

Case 1: $d_{3m_3} = 1$. In this case (5) is obtained by the move $\widehat{\mu}$ (described in the proof of Theorem 1.1) from the datum

$$\big(g\mathbb{T}, \mathbb{S}, 3, d-1, (d-2, 2), (d_{21} - 1, d_{22}, \ldots, d_{2m_2}), (d_{31}, \ldots, d_{3(m_3-1)})\big).$$

This move is constructible, since the latter datum can be realized by an accessible diagram by the inductive assumption. The arising diagram is also accessible, since the move μ preserves accessibility.

Case 2: $d_{2m_2} \geqslant 2$ *and* $d_{3m_3} = d_{3(m_3-1)} = 2$. We first note that we can assume $d_{21} > 2$, because if $d_{21} = 2$ then either $d_{31} > 2$, and we can switch the partitions, or $d_{ij} = 2$ for $i = 2, 3$ and all j, which is only possible for $g = 0$. Now we consider the compatible datum

$$\big(g\mathbb{T}, \mathbb{S}, 3, d-4, (d-4), (d_{21} + d_{22} - 4, d_{23}, \ldots, d_{2m_2}), (d_{31}, \ldots, d_{3(m_3-2)})\big).$$

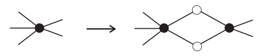

Figure 19: A move on dessins d'enfants.

By Theorem 1.5 it is realizable, so there exists in $g\mathbb{T}$ a dessin d'enfant D realizing it and such that $g\mathbb{T}\setminus D$ is a disc of length $2(d-4)$. We take the vertex of D corresponding to the entry $d_{21}+d_{22}-4$ and we perform at it the move shown in Fig. 19. Clearly, we can do so in such a way that the new black vertices have valences d_{21} and d_{22}. Hence we get a dessin d'enfant realizing (5).

CASE 3: $d_{2m_2}=d_{3m_3}=2$. Since $m_2,m_3\geqslant 2$ we easily see that $d\geqslant 8$, so the inductive assumption implies realizability of the compatible datum

$$\big(g\mathbb{T},\mathbb{S},3,d-2,(d-4,2),(d_{21},\ldots,d_{2(m_2-1)}),(d_{31},\ldots,d_{3(m_3-1)})\big).$$

Let D be a dessin d'enfant in $g\mathbb{T}$ realizing it and such that $g\mathbb{T}\setminus D$ consists of two discs of lengths $2(d-4)$ and 4. Since $2(d-4)>4$ there exists an edge e of D to which the first disc is doubly incident. We then place on e two vertices of valence 2 and give them colors so to get a bipartite graph. The result is of course a dessin d'enfant realizing (5).

CASE 4: *$d_{2m_3}\geqslant 3$, $d_{3(m_3-1)}\geqslant 3$, $d_{3m_3}\geqslant 2$ and $d_{21}>d_{3j}\geqslant 3$ for some j.* This implies that either $d_{21}>d_{3m_3}\geqslant 3$ or $d_{21}>d_{3(m_3-1)}$ and $d_{3m_3}=2$. Let k be the smallest entry among d_{2j}, d_{3j} that is not equal to 2. Clearly, k can only be $d_{3(m_3-1)}$, d_{3m_3}, or d_{2m_2}. The proof is basically the same in all three cases. We then assume the first one occurs (which implies $d_{3m_3}=2$), but later we will mention the variations for the other two cases. The current assumption $m_2,m_3\geqslant 2$ easily implies that $k\leqslant d-3$.

CASE 4.1: *k is odd.* We consider the compatible data

$$\begin{array}{r}\big(\gamma\mathbb{T},\mathbb{S},3,d-k,(d-2-k,2),(d_{21}-k,d_{22},\ldots,d_{2m_2}),\\ (d_{31},\ldots,d_{3(m_3-2)},d_{3m_3})\big)\end{array}\tag{6}$$

with $\gamma=g-(k-1)/2=(d-k-(m_2+m_3)+1)/2$, and

$$\big((k-1)/2\cdot\mathbb{T},\mathbb{S},3,k+1,(k+1),(k+1),(k,1)\big).\tag{7}$$

We first show that if (6) is non-realizable then (5) is realizable. By the inductive assumption, Proposition 4.2 and Theorem 1.2 we know (6) can

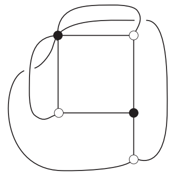

Figure 20: A dessin d'enfant realizing $\big(2\mathbb{T},\mathbb{S},3,9,(7,2),(6,3),(3,3,3)\big)$.

only be non-realizable if $\gamma=0$ or (6) is $\big(\mathbb{T},\mathbb{S},3,6,(4,2),(3,3),(3,3)\big)$. In the second case (5) has the form

$$\big(g\mathbb{T},\mathbb{S},3,k+6,(k+4,2),(k+3,3),(3,k,3)\big)$$

but by the definition of k we must have $k=3$, so (5) is

$$\big(2\mathbb{T},\mathbb{S},3,9,(7,2),(6,3),(3,3,3)\big),$$

which is realized in Fig. 20. Suppose $\gamma=0$. Since $m_2\leqslant d/k$ and $m_3\leqslant(d-2)/k+1$, we have

$$0=2\gamma\geqslant d-k-2(d-1)/k.$$

Knowing that $3\leqslant k\leqslant d-3$, a tiny bit of algebra shows that this can only happen for $d=7$ (and $k=3,4$), but we are assuming $m_2,m_3\geqslant 2$ and $g\geqslant 2$, so the compatibility condition $m_2+m_3=d-2g$ is impossible for $d=7$.

We can then assume (6) is realizable, so it is realized by a diagram which is automatically accessible. Within any such diagram we can select a face α incident to Γ''_{11} and Γ''_{21}, whence of type $(d-2-k,d_{21}-k,d_{3j})$ for some j. By Theorem 1.5 the datum (7) is also realizable by a diagram, and of course we can find in the diagram a face α' of type $(k+1,k+1,1)$. Performing the conjunction along α and α' of the diagrams realizing (6) and (7) we get a diagram realizing (5).

Case 4.2: *k is even.* We consider the compatible data

$$\big(\gamma\mathbb{T},\mathbb{S},3,d-k,(d-k),(d_{21}-k,d_{22},\ldots,d_{2m_2}),(d_{31},\ldots,d_{3(m_3-2)},d_{3m_3})\big)\quad(8)$$

with $\gamma=g-(k-2)/2=(d-k-(m_2+m_3)+2)/2$, and

$$\big((k-2)/2\cdot\mathbb{T},\mathbb{S},3,k+1,(k-1,2),(k+1),(k,1)\big).\quad(9)$$

Both these data are realizable by Theorem 1.5. Any diagram realizing (8) has a face α of type $(d-k, d_{21}-k, d_{3j})$ for some j. Moreover, since $k>2$, the inductive assumption implies that (9) can be realized by an accessible diagram, so we can choose in it a face α' incident to trees of degrees $k-1$ and 1. Therefore α' has type $(k-1, k+1, 1)$. Just as above, we do the conjunction along α and α' of the diagrams realizing (8) and (9), getting a diagram realizing (5).

Recall now that in Cases 4.1-2 we have assumed that k is $d_{3(m_3-1)}$, whereas it can also be d_{3m_3} or d_{2m_2}, but the arguments given readily extend to these cases. If k is d_{3m_3} we replace the second and third partitions of d in both (6) and (8) by

$$(d_{21}-k, d_{22}, \ldots, d_{2m_2}), \quad (d_{31}, \ldots, d_{3(m_3-1)}).$$

If k is d_{2m_2} we replace them by

$$(d_{21}, \ldots, d_{2(m_2-1)}), \quad (d_{31}-k, d_{32}, \ldots, d_{3m_3}).$$

Case 5: *all d_{2j} and d_{3j} are equal to some k.* We must prove realizability of a compatible datum of the form

$$\big(g\mathbb{T}, \mathbb{S}, 3, kh, (kh-2, 2), (k, \ldots, k), (k, \ldots, k)\big), \qquad g \geqslant 2,\ k \geqslant 3,\ h \geqslant 1. \tag{10}$$

Compatibility means that $kh = 2(g+h)$, so either k or h is even, and one easily sees that one of the following happens:

(a) $k=3$ and $h \geqslant 4$ is even;

(b) $k=4$ and $h \geqslant 2$;

(c) $k \geqslant 6$ is even and $h \geqslant 1$;

(d) $k \geqslant 5$ is odd and $h \geqslant 2$ is even.

In cases (a,b,c) we will prove realizability of (10) fixing k and proceeding by induction on h, with an induction step of length 2 in case (a). In case (d) we will give a direct (but very similar) argument. The base step of the induction in cases (a) and (b) is established in Fig. 21, and it is a consequence of Theorem 1.5 in case (c). The core of our arguments is the following:

Claim. *Suppose that (10) is realizable. Then*

$$\big(g\mathbb{T}, \mathbb{S}, 3, kh+2, (kh, 2), (k, \ldots, k, 2), (k, \ldots, k, 2)\big) \tag{11}$$

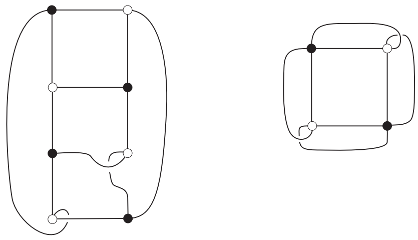

Figure 21: Dessins d'enfants realizing the data $\big(2\mathbb{T}, \mathbb{S}, 3, 12, (10,2), (3,3,3,3), (3,3,3,3)\big)$ and $\big(2\mathbb{T}, \mathbb{S}, 3, 8, (6,2), (4,4), (4,4)\big)$.

can be realized by a diagram having a face of type $(kh, 2, 2)$. Indeed, consider a dessin d'enfant $D \subset \widetilde{\Sigma}$ realizing (10) and such that all vertices of D have valence k. Then the complement of D consists of two discs B' and B'', of lengths $2(kh-2)$ and 4 respectively. Since $kh \geqslant 6$ we have $2(kh-2) > 4$, so there is an edge e of D to which B' is doubly incident. Then we place two vertices u and v of valence 2 on e and, as in the proof of Case 3 above, we get a dessin d'enfant $\widetilde{D}$ realizing (11).

It follows from the proof of Theorem 2.5 that a diagram realizing (11) can now be constructed as follows. Choose a point x' in the interior of B' and take a cone (in B') with center x' over the pull-back of the vertex set of $\widetilde{D}$ in the boundary of B'. Do the same for some interior point x'' of B''. We get a triangulation of $\widetilde{\Sigma}$ that we can color in a checkerboard fashion, and we notice that there is a triangle incident to both u and v (two of them, actually, a white one and a black one). Assign color 1 to x' and x'', and colors 2 and 3 to the members of the two different partitions of the vertex set of the bipartite graph $\widetilde{D}$. It is easy to see now that applying to this triangulation as long as possible the black and white moves described in the proof of Theorem 2.5 yields a diagram with a white and a black disc of type $(kh, 2, 2)$. The Claim is proved.

Inductive step in case (a). The inductive assumption is that some datum

$$\big(h/2 \cdot \mathbb{T}, \mathbb{S}, 3, 3h, (3h-2, 2), (3, \ldots, 3), (3, \ldots, 3)\big)$$

is realizable. By the Claim we can then realize

$$\big(h/2 \cdot \mathbb{T}, \mathbb{S}, 3, 3h+2, (3h, 2), (3, \ldots, 3, 2), (3, \ldots, 3, 2)\big) \tag{12}$$

via a diagram with a face α of type $(3h, 2, 2)$. Now, starting from the obvious dessin d'enfant realizing

$$\big(\mathbb{T}, \mathbb{S}, 3, 5, (5), (3, 2), (3, 2)\big) \tag{13}$$

and applying the triangulation trick used to prove the Claim, we see that this datum is realized by a diagram with a face α' of type $(5, 2, 2)$. Taking the conjunction of the diagrams realizing (12) and (13) along α and α' we get a realization of the datum

$$\big((h + 2)/2 \cdot \mathbb{T}, \mathbb{S}, 3, 3(h + 2), (3h + 4, 2), (3, \ldots, 3), (3, \ldots, 3)\big)$$

i.e. the desired conclusion.

Inductive step in cases (b,c). The inductive assumption is that some datum

$$\big(g\mathbb{T}, \mathbb{S}, 3, kh, (kh - 2, 2), (k, \ldots, k), (k, \ldots, k)\big)$$

with $g = (k - 2)h/2$ is realizable. By the Claim we can then realize

$$\big(g\mathbb{T}, \mathbb{S}, 3, kh + 2, (kh, 2), (3, \ldots, 3, 2), (3, \ldots, 3, 2)\big) \tag{14}$$

via a diagram with a face α of type $(kh, 2, 2)$. Now note that the datum

$$\big((k - 2)/2 \cdot \mathbb{T}, \mathbb{S}, 3, k - 1, (k - 1), (k - 1), (k - 1)\big) \tag{15}$$

is compatible, whence realizable by Theorem 1.5. Taking the conjunction of the diagram realizing (14) and any diagram realizing (15) along α and any face of the latter we get a realization of the datum

$$\big(\gamma\mathbb{T}, \mathbb{S}, 3, k(h + 1), (k(h + 1) - 2, 2), (k, \ldots, k), (k, \ldots, k)\big)$$

with $\gamma = (k - 2)(h + 1)/2$. This is what we had to prove.

Proof of realizability in case (d). We begin by noting that if $h \geqslant 2$ is even and $k \geqslant 5$ is odd then the datum

$$\big(\gamma\mathbb{T}, \mathbb{S}, 3, k(h - 1), (k(h - 1)), (k, \ldots, k), (k, \ldots, k)\big),$$

with $\gamma = ((k - 2)(h - 1) + 1)/2$, is compatible, whence realizable by Theorem 1.5. The very same argument used to prove the Claim then shows that the datum

$$\big(\gamma\mathbb{T}, \mathbb{S}, 3, k(h - 1) + 2, (k(h - 1) + 2), (k, \ldots, k, 2), (k, \ldots, k, 2)\big) \tag{16}$$

is realized by a diagram with a face α of type $(k(h-1)+2,2,2)$. Using the same techniques one can also easily show that the datum

$$\big((k-3)/2\cdot\mathbb{T},\mathbb{S},3,k-1,(k-3,2),(k-1),(k-1)\big) \tag{17}$$

is realized by a diagram with a face α' of type $(k-3,k-1,k-1)$. And again the conclusion follows because the conjunction of the diagrams realizing (16) and (17) along α and α' gives a realization of (10) under the assumptions of (d).

Having treated all five cases of Lemma 4.4 we have eventually shown Theorem 1.3. □

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