On the existence of branched coverings between surfaces with prescribed branch data

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August 23, 2005

Abstract

Given a branched covering between closed connected surfaces, one can easily establish some relations between the Euler characteristic and orientability of the involved surfaces, the degree of the covering, the number of branching points, and the local degrees at these points. These relations can therefore be regarded as necessary conditions for the existence of the covering. A classical problem dating back to Hurwitz asks whether these necessary conditions are actually also sufficient. Thanks to the work of many authors, the problem is now completely solved (in the affirmative) when the base surface has non-positive Euler characteristic. The cases where the base surface is the sphere or the projective plane remain elusive, but the latter reduces to the former, and many partial results are known. In this paper we prove several new existence and non-existence results.

MSC (2000): 57M12.

1 Introduction and statements

A *branched covering* between surfaces is a map $f:\widetilde{\Sigma}\to\Sigma$, where $\widetilde{\Sigma}$ and Σ are closed connected surfaces and f is locally modelled on maps of the form $\mathbb{C}\ni z\mapsto z^k\in\mathbb{C}$ for some $k\geqslant 1$. The integer k is called the *local degree* at the point of $\widetilde{\Sigma}$ corresponding to 0 in the source $\mathbb{C}$. If $k>1$ then the point of Σ

*Supported by the INTAS YS fellowship 03-55-1423

**Supported by the INTAS project "CalcoMet-GT" 03-51-3663

corresponding to 0 in the target $\mathbb{C}$ is called a *branching point*. The branching points are isolated, hence there are finitely many, say n, of them. Removing the branching points in Σ and all their pre-images in $\widetilde{\Sigma}$, the restriction of f gives a genuine covering, whose degree we will denote by d. If the i-th branching point on Σ has m_i pre-images, the local degrees $(d_{ij})_{j=1,\ldots,m_i}$ at these points give a partition of d, namely $d_{ij} \geqslant 1$ and $\sum_{j=1}^{m_i} d_{ij} = d$. In the sequel we will always assume that in a partition $(d_1, \ldots, d_m)$ of d we have $d_1 \geqslant \ldots \geqslant d_m$, which allows to regard the partition as an array of integers rather than an unordered set with repetitions.

Branch data Suppose we are given closed connected surfaces $\widetilde{\Sigma}$ and Σ, integers $n \geqslant 0$ and $d \geqslant 2$, and for $i = 1, \ldots, n$ a partition $(d_{ij})_{j=1,\ldots,m_i}$ of d. The 5-tuple $\left(\widetilde{\Sigma}, \Sigma, n, d, (d_{ij})\right)$ will be called the *branch datum* of a candidate branched covering. To such a datum we will always associate the integer $\widetilde{n}$ defined as $m_1 + \ldots + m_n$.

Compatibility We define a branch datum to be *compatible* if the following conditions hold:

1. $\chi(\widetilde{\Sigma}) - \widetilde{n} = d \cdot (\chi(\Sigma) - n)$;
2. $n \cdot d - \widetilde{n}$ is even;
3. If Σ is orientable then $\widetilde{\Sigma}$ is also orientable;
4. If Σ is non-orientable and d is odd then $\widetilde{\Sigma}$ is also non-orientable;
5. If Σ is non-orientable but $\widetilde{\Sigma}$ is orientable then each partition $(d_{ij})_{j=1,\ldots,m_i}$ of d refines the partition $(d/2, d/2)$.

The meaning of Condition 5 is that $(d_{ij})_{j=1,\ldots,m_i}$ is obtained by juxtaposing two partitions of $d/2$ and reordering. Note that d is even by Condition 4.

The problem It is not too difficult to show that if a branched covering $\widetilde{\Sigma} \to \Sigma$ exists then the corresponding branch datum, with $n, d, (d_{ij}), \widetilde{n}$ defined as above, is compatible. Only Conditions 2 for non-orientable Σ and 5 are non-evident, and a short proof is included for the sake of completeness in Section 2. We will call *Hurwitz existence problem* the question of which compatible branch data are actually realized by some branched covering. In the sequel we will always consider the branch datum $\big(\widetilde{\Sigma}, \Sigma, n, d, (d_{ij})\big)$, and the corresponding $\widetilde{n}$, to be fixed. We will also mostly assume that each partition $(d_{ij})_{j=1,\ldots,m_i}$ is different from $(1,\ldots,1)$, for in this case we could just reduce n.

Remark 1.1. A perhaps more traditional viewpoint is to consider only $\Sigma, n, d, (d_{ij})$ to be given, and then determine the corresponding $\widetilde{\Sigma}$ using Conditions 1, 3, and 4. In this context, these conditions are replaced by the requirement that $\widetilde{n} + d\cdot(\chi(\Sigma) - n)$ should be at most 2. However, for non-orientable Σ and even d, two possibilities exist for $\widetilde{\Sigma}$, so we prefer to stick to our datum which also includes $\widetilde{\Sigma}$.

Known results for $\Sigma \neq \mathbb{S}$ In this paragraph we review several results from the literature, whose overall content is that, to get a complete solution of the Hurwitz existence problem, it would be sufficient to settle the case where the base surface Σ is the sphere $\mathbb{S}$. The first theorem we cite is attributed to Shephardson in [5, page 125] and explicitly proved in [11, Theorem 4] and [4, Proposition 3.3]:

Theorem 1.2. *A compatible branch datum with Σ orientable and $\chi(\Sigma) \leqslant 0$ is realizable.*

The next result is proved in [5, Theorem 3.4] and [4, Proposition 3.3]:

Theorem 1.3. *A compatible branch datum with Σ and $\widetilde{\Sigma}$ non-orientable and $\chi(\Sigma) \leqslant 0$ is realizable.*

We then quote the following elementary fact, stated in [4, Proposition 2.7], and its consequence [4, Proposition 3.4]:

Proposition 1.4. *A compatible branch datum with Σ non-orientable and $\widetilde{\Sigma}$ orientable is realizable if and only if it is possible to decompose for all i the partition $(d_{ij})_{j=1,\ldots,m_i}$ of d into partitions $(d'_{ij})_{j=1}^{m'_i}$ and $(d''_{ij})_{j=1}^{m''_i}$ of $d/2$ in such a way that the branch datum*

$$\big(\widetilde{\Sigma}, \Sigma', 2n, d/2, (d'_{ij}), (d''_{ij})\big)$$

is realizable, where Σ' is the orientable double covering of Σ.

Corollary 1.5. *A compatible branch datum with Σ non-orientable, $\widetilde{\Sigma}$ orientable and $\chi(\Sigma) \leqslant 0$ is realizable.*

The next result is due to Edmonds, Kulkarny and Stong [4, Theorem 5.1]. We denote by $\mathbb{P}$ the projective plane.

Theorem 1.6. *A compatible branch datum with $\Sigma = \mathbb{P}$ and non-orientable $\widetilde{\Sigma}$ is realizable.*

These theorems imply that only the following instances of the Hurwitz existence problem remain open:

- $\Sigma = \mathbb{S}$;
- $\Sigma = \mathbb{P}$ and $\widetilde{\Sigma}$ orientable.

However, Proposition 1.4 reduces the latter instance to the former one, to which we will therefore confine ourselves henceforth.

Known results for $\Sigma = \mathbb{S}$ When the base surface is the sphere $\mathbb{S}$, not every compatible branch datum is realizable. The easiest example is given in degree $d = 4$ with $n = 3$ branch points by the partitions $(3, 1), (2, 2), (2, 2)$, which implies that $\widetilde{\Sigma}$ is $\mathbb{S}$ too. A proof follows from Proposition 1.10 below, established in Section 3. In the rest of our paper the base surface Σ will always be the sphere $\mathbb{S}$, and we will say that a branch datum is *exceptional* if it is compatible but not realizable.

We will now review the main existence and non-existence theorems proved in the literature. However, we will not attempt to give a comprehensive list of the abstract statements. Instead, we will concentrate on the results which can be applied in a more direct fashion. The following was established in [4, Proposition 5.7].

Theorem 1.7. *For all non-prime d there exist exceptional branch data of degree d with $n = 3$ and $\widetilde{\Sigma} = \mathbb{S}$.*

Turning to existence, the most general known result appears to be the following:

Theorem 1.8. *A compatible branch datum is realizable if one of the partitions (d_{ij}) is given by (d) only.*

This fact was first stated by Thom [19, Theorem 1] for $\widetilde{\Sigma}$ also equal to $\mathbb{S}$, was reproved in [12, Theorem 2] and [2, Theorem 6] in the same context, and was generalized to arbitrary $\widetilde{\Sigma}$ in [4, Proposition 5.2]. A variation on this result is given in [4, Proposition 5.3], where the realizable branch data with one partition of the form $(d-1,1)$ are classified. In [4, page 775] a similar classification is announced for data with $n=3$ and one partition of the form $(m,1,\ldots,1)$.

Another major contribution of Edmonds, Kulkarni and Stong is the following [4, Theorem 5.4]:

Theorem 1.9. *A branch datum with $d\neq 4$ and $n\cdot d-\widetilde{n}\geqslant 3(d-1)$ is realizable. The exceptional data with $d=4$ are precisely those with partitions $(2,2),\ldots,(2,2),(3,1)$.*

Since no partition of d has the form $(1,\ldots,1)$, one easily sees that $n\cdot d-\widetilde{n}\geqslant n$, so a consequence of this result is that for fixed $d\neq 4$ the number of exceptional branch data of degree d is finite [4, Corollary 4.4].

Gersten [8, page 762] showed that for $d=9$ and $n=3$ the compatible data with partitions $(3,3,3),(3,3,3),(2,2,2,2,1)$ and $(4,4,1),(3,3,3),(2,2,2,2,1)$ are exceptional. More sporadic exceptional data were described in [4, Table 1], but we will not list them explicitly here. The next result is established in [4, Corollary 6.4]:

Proposition 1.10. *For $\widetilde{\Sigma}=\mathbb{S}$, $n=3$ and even d a branch datum with partitions $(x,d-x),(2,\ldots,2),(2,\ldots,2)$ is realizable if and only if $x=d/2$.*

The following results are due to Baránski [2, Proposition 10,Theorem 12, Corollary 15]. We note that the first one extends Theorem 1.8 in the special case where $\widetilde{\Sigma}=\mathbb{S}$, while the second one implies that for fixed d the number of exceptional branch data of degree d with $\widetilde{\Sigma}=\mathbb{S}$ is finite (which was already known after [4], for arbitrary $\widetilde{\Sigma}$ and $d\neq 4$).

Proposition 1.11. *A branch datum with $\widetilde{\Sigma}=\mathbb{S}$ is realizable if there exists r such that $m_1+\ldots+m_r=(r-1)\cdot d+1$.*

Proposition 1.12. *A compatible branch datum with $\widetilde{\Sigma}=\mathbb{S}$ and $n\geqslant d$ is realizable.*

Proposition 1.13. *A compatible branch datum with $\widetilde{\Sigma}=\mathbb{S}$, $d_{ij}\leqslant 2$ for all i,j, and $m_i\geqslant d-\sqrt{d/2}$ for all i is realizable.*

Considering Theorem 1.7 and the fact that all known exceptional branch data occur with non-prime degree d, one is naturally led to conjecture that for prime d the Hurwitz existence problem always has a positive solution. It is claimed in [4, page 787] that establishing this conjecture in the special case $n = 3$ would imply the general case.

Related known results and techniques The proofs of Theorems 1.2 and 1.3 are based on a reformulation of the main problem in terms of representations of the fundamental group of the n-punctured surface Σ into the symmetric group $\mathfrak{S}_d$. This technique goes back to Hurwitz himself [10, Section I.1] and was later revisited and refined by Ezell [5, Theorem 2.1], Singerman [18, Theorem 1], where a more complicated situation (which includes cusped surfaces) is considered, and Husemoller [11, Theorems 3 and 5], where surfaces with boundary are also accepted. The permutation technique will be carefully reviewed and reinterpreted below in Section 2. It is however worth remarking here that the main algebraic results on permutations leading to Theorems 1.2 and 1.3 were also established in an abstract context in [3].

The technique used by Gersten in [8] to prove the non-existence results mentioned above also has a group-theoretic nature and bypasses the permutation approach. This technique will not be reviewed in this paper.

Existence problems of branched coverings with additional constraints on the candidate covering map were faced in [6, 7], but the results of these papers do not seem to have any direct application to the basic Hurwitz existence problem as we have stated it.

In the special case where $\Sigma = \mathbb{S}$ and $n = 3$, which can be treated using the technique of "dessins d'enfants" (see Section 3) it can be shown [21, Proposition 3] that the Hurwitz existence problem has an equivalent formulation in terms of algebraic maps between algebraic curves. The problem also has a complete solution in this context [21, Theorem 2], but the necessary and sufficient condition for existence is an abstract algebraic one which it does not seem to be possible to check in practice.

Non-realizability of the sporadic exceptional data found by Edmonds, Kulkarni and Stong in [4] was proved using a rather subtle further necessary condition of arithmetic nature [4, Theorem 6.3].

The results of Baránski stated above are based on a geometric criterion [2, Lemma 5] for the realizability of a branched cover of $\mathbb{S}$ with $\widetilde{\Sigma}$ also equal to

$\mathbb{S}$, that we will extend to arbitrary $\widetilde{\Sigma}$ in Section 4.

Before turning to a description of our original contributions to the problem, we also mention the question of counting the number of (suitably defined) equivalence classes of branched coverings realizing a given branch datum. This counting problem in fact goes back to Hurwitz as well, for whom it was a primary motivation and who solved it in [10, pages 16, 22] for the case $n = 2d-2$ and all partitions of the form $(2, 1, \ldots, 1)$, with $\widetilde{\Sigma} = \mathbb{S}$. Complete formulas for the most general by Mednykh [15, page 138], [16, Theorem C]. Of course, a branch datum is realizable if and only if the corresponding counting formula returns a positive value, so these formulae give an implicit solution to the Hurwitz existence problem. But again the actual computation appears to be hopeless. A variation of Mednykh's formulae, where some explicit computation is possible, was recently derived in [17] for special types of branched coverings.

New results: exceptions The following results improve Theorem 1.7 in that they give more general exceptional data for non-prime degree d, $n = 3$, and $\widetilde{\Sigma} = \Sigma = \mathbb{S}$, and they show that such exceptions occur also for $n = 4$ and $\widetilde{\Sigma} \neq \mathbb{S}$. The first statement establishes a partial case of the result announced in [4, page 775]. To the best of our knowledge, none of the other exceptional series described in our statements was fully covered by previous papers.

Theorem 1.14. *Suppose that $d = kh$ with $k, h \geqslant 2$. Let $(h_j)_{j=1,\ldots,p}$ be a partition of h with $p \geqslant 2$. Then the branch datum*

$$\big(\mathbb{S}, \mathbb{S}, 3, d, (k, \ldots, k), (kh_1, \ldots, kh_p), (h + p - 1, 1, \ldots, 1)\big)$$

is exceptional.

Proposition 1.15. *Let $d \geqslant 8$ be even and consider compatible branch data of the form $\big(\widetilde{\Sigma}, \mathbb{S}, 3, d, (2, \ldots, 2), (5, 3, 2, \ldots, 2), (d_{3j})_{j=1,\ldots,m_3}\big)$.*

- *If $\widetilde{\Sigma} = \mathbb{T}$, whence $m_3 = 2$, the datum is realizable if and only if $(d_{31}, d_{32}) \neq (d/2, d/2)$;*
- *If $\widetilde{\Sigma} = \mathbb{S}$, whence $m_3 = 4$, the datum is realizable if and only if $(d_{3j})_{j=1,\ldots,4}$ does not have the form $(k, k, d/2 - k, d/2 - k)$ for some $k > 0$, or $(d/2, d/6, d/6, d/6)$ for d a multiple of 6.*

Proposition 1.16. *For any even $d \geqslant 6$ the following branch datum is exceptional:* $\big(\mathbb{S},\mathbb{S},4,d,(2,\ldots,2),(2,\ldots,2),(2,1,\ldots,1),(d-2,1,1)\big)$.

Proposition 1.17. *Let $\big(\mathbb{S},\mathbb{S},3,d,(d_{ij})\big)$ be a compatible branch datum with even d and $(d_{1j})=(2,\ldots,2)$. If $(d_{2j})=(3,3,2,\ldots,2)$ or $(d_{2j})=(3,2,\ldots,2,1)$ then the datum is realizable if and only if $d_{31}\neq d/2$.*

Theorem 1.18. *Suppose d and all d_{ij} for $i=1,2$ are even. If the branch datum $\big(\mathbb{S},\mathbb{S},3,d,(d_{ij})\big)$ is realizable then (d_{3j}) refines the partition $(d/2,d/2)$.*

This result implies for instance that the following is an exceptional datum: $\big(\mathbb{S},\mathbb{S},3,12,(4,2,2,2,2),(4,2,2,2,2),(7,3,1,1)\big)$. In Section 3 we will also describe other exceptional branch data with $d=12$, $n=3$ and $\Sigma=\mathbb{S}$.

New results: existence Our contributions in the positive sense to the solution of the Hurwitz existence problem are as follows:

Theorem 1.19. *Let $\big(\widetilde{\Sigma},\mathbb{S},3,d,(d_{ij})\big)$ be a compatible branch datum. Let $p\geqslant 3$ be odd and suppose that all d_{ij} are divisible by p. Then the datum is realizable.*

Theorem 1.20. *If d is odd then every compatible branch datum of the form $\big(\mathbb{S},\mathbb{S},3,d,(d-2,2),(d_{2j}),(d_{3j})\big)$ is realizable.*

Theorem 1.21. *With the single exception of $\big(\mathbb{T},\mathbb{S},3,6,(4,2),(3,3),(3,3)\big)$, every compatible branch datum of the form $\big(\mathbb{T},\mathbb{S},3,d,(d-2,2),(d_{2j}),(d_{3j})\big)$ is realizable.*

Techniques and organization of the paper Various equivalent ways of formulating the Hurwitz existence problem, and techniques to attack it, were developed over the time. Since we will make use of all these approaches, we will review them in Section 2, also trying to make the relations between them completely transparent.

Our results on non-realizability of compatible branch data are based on two main tools. The first is a suitable adjustment of Grothendieck's approach through "dessins d'enfants" to the study of algebraic maps between Riemann surfaces. This is described in Section 3, where the approach is used to establish Theorem 1.14 and Propositions 1.15, 1.16 and 1.17. The second tool, also described in Section 3, is based on the idea of expressing a covering as a composition of two non-trivial ones, and the corresponding idea of finding "block decompositions" of permutations. This tool leads to Theorem 1.18.

On the existence side, our main instrument is a geometric criterion due to Baránski [2, Lemma 5] for the case $\widetilde{\Sigma} = \mathbb{S}$, and extended to arbitrary $\widetilde{\Sigma}$ in Section 4. This criterion is used in Section 5 to prove our existence results Theorems 1.20 and 1.21, while Theorem 1.19, which is established in the same section, is again based on the idea of expressing a covering as the composition of two other ones.

ACKNOWLEDGEMENTS. We are grateful to Laurent Bartholdi for his instructions on how to use the software GAP to run a computer experiment which listed all the exceptional data with $n = 3$, $d \leqslant 10$ and with $n = 4, 5, 6$ for smaller degrees. This listing suggested to us some of the infinite exceptional series later proved to be so by theoretical means. We also thank Sergei Matveev, Laura Mazzoni, Alexander Mednykh, and Joan Porti for helpful conversations.

2 Alternative viewpoints on the problem

We start by describing equivalent formulations of the Hurwitz existence problem in terms of ordinary coverings of surfaces with boundary and in terms of 2-orbifolds. In some cases, one of these reformulations will be easier to handle than the original viewpoint of branched coverings.

Surfaces with boundary Let Σ_n denote the surface obtained from Σ by removing n open discs with disjoint closures. A branched covering realizing a datum $\left(\widetilde{\Sigma}, \Sigma, n, d, (d_{ij})_{i=1,\ldots,n}^{j=1,\ldots,m_i}\right)$ induces a genuine covering $\widetilde{\Sigma}_{\widetilde{n}} \to \Sigma_n$ in which the i-th component of $\partial\Sigma_n$ is covered by m_i components of $\partial\widetilde{\Sigma}_{\widetilde{n}}$, and the degrees of the restrictions to these components are $(d_{ij})_{j=1,\ldots,m_i}$. Conversely, any such genuine covering $\widetilde{\Sigma}_{\widetilde{n}} \to \Sigma_n$ induces a realization of the branch datum. We will therefore consider also such a covering a realization of the datum.

2-orbifolds Let $(\Sigma; p_1, \ldots, p_n)$ denote the (closed locally orientable) 2-orbifold with underlying surface Σ and cone points of orders $p_1, \ldots, p_n$. Choosing p_i to be divisible by d_{ij} for $j = 1, \ldots, m_i$, a branched covering realizing a datum $\big(\widetilde{\Sigma},\Sigma,n,d,(d_{ij})_{i=1,\ldots,n}^{j=1,\ldots,m_i}\big)$ induces an orbifold-covering [20]

$$\left(\widetilde{\Sigma};\frac{p_1}{d_{11}},\ldots,\frac{p_1}{d_{1m_1}},\ldots,\frac{p_n}{d_{n1}},\ldots,\frac{p_n}{d_{nm_n}}\right)\to(\Sigma;p_1,\ldots,p_n).$$

And again, with details that we can safely leave to the reader, a covering between 2-orbifolds induces a branched covering between surfaces.

Covering from permutations We will denote by $\mathbb{S}$, $\mathbb{T}$ and $\mathbb{P}$ the sphere, the torus and the projective plane respectively, whence by $g\mathbb{T}=\mathbb{T}\#\ldots\#\mathbb{T}$ (g times) and $g\mathbb{P}=\mathbb{P}\#\ldots\#\mathbb{P}$ (g times) the orientable and non-orientable surfaces of genus $g\geqslant 1$. The following group presentations are well-known:

$$\begin{aligned}\pi_1\big((g\mathbb{T})_n\big) &= \big\langle a_1,b_1,\ldots,a_g,b_g,e_1,\ldots,e_n\big|\ [a_1,b_1]\cdots[a_g,b_g]\cdot e_1\cdots e_n\big\rangle\\ \pi_1\big((g\mathbb{P})_n\big) &= \big\langle c_1,\ldots,c_g,e_1,\ldots,e_n\big|\ c_1^2\cdots c_g^2\cdot e_1\cdots e_n\big\rangle.\end{aligned}$$

In both cases the e_i's are represented by the boundary circles. In the orientable case a_k,b_k is a meridian-longitude pair on the k-th copy of $\mathbb{T}$. In the non-orientable case c_k is the only non-trivial loop on the k-th copy of $\mathbb{P}$.

The first alternative viewpoint on the Hurwitz existence problem described above allows to establish the following fact, originally due to Hurwitz, Husemoller, Ezell, and Singerman. We quickly review the geometric argument underlying the proof because we will need it below.

Theorem 2.1. *A compatible branch datum $\big(\widetilde{\Sigma},\Sigma,n,d,(d_{ij})\big)$ is realizable if and only if there exists a representation $\theta:\pi_1(\Sigma_n)\to\mathfrak{S}_d$ such that:*

1. *$Im(\theta)$ acts transitively on $\{1,\ldots,d\}$;*
2. *$\theta(e_i)$ has precisely m_i cycles of lengths $d_{i1},\ldots,d_{im_i}$;*
3. *For non-orientable Σ and orientable $\widetilde{\Sigma}$, each permutation $\theta(c_k)$ consists of cycles of even length only.*

Proof. Our proof only works for $n>0$, *i.e.* it does not for genuine coverings. Recall first [14, Chapter 2] that Σ_n can be obtained as shown in Fig. 1 from a disc Δ by identifying each arc $\epsilon_{i,-}$ to the arc $\epsilon_{i,+}$ and, depending on orientability, either each $\alpha_{k,-}$ to $\alpha_{k,+}$ and $\beta_{k,-}$ to $\beta_{k,+}$, or each $\gamma_{k,-}$ to $\gamma_{k,+}$. The figure also shows loops corresponding to the generators of $\pi_1(\Sigma_n)$ used in the above presentations.

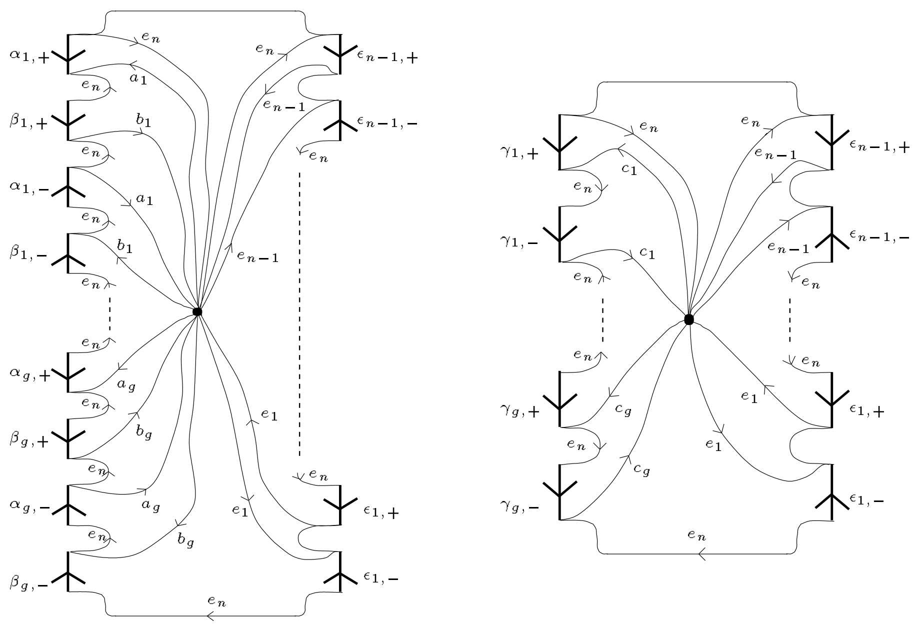


Figure 1: Construction of punctured surfaces.

The degree-d coverings of Σ_n are now obtained as follows. We first take the disjoint union of d copies $(\Delta^{(h)})_{h=1}^d$ of Δ, with the corresponding arcs $\epsilon_{i,\pm}^{(h)}$ etc. Then we glue each $\epsilon_{i,-}^{(h)}$ to some $\epsilon_{i,+}^{(\theta(e_i)(h))}$, where $\theta(e_i) \in \mathfrak{S}_d$, and similarly for the other arcs, using permutations $\theta(a_k)$ and $\theta(b_k)$, or $\theta(c_k)$. The corresponding covering is induced by the identification of each $\Delta^{(h)}$ with Δ, and of course the covering is connected if and only if the subgroup of $\mathfrak{S}_d$ generated by $\theta(e_1), \ldots, \theta(e_{n-1})$ and either the $\theta(a_k)$'s and $\theta(b_k)$'s or the $\theta(c_k)$'s acts transitively on $\{1, \ldots, d\}$. It is also obvious that for $i = 1, \ldots, n-1$ the way the i-th component of $\partial\Sigma_n$ is covered depends on the cyclic structure of $\theta(e_i)$ as described in the statement.

Considering the form of the presentation of $\pi_1(\Sigma_n)$ given above, we see that we can define $\theta(e_n) \in \mathfrak{S}_d$ in a unique fashion so to get a representation θ. To conclude we must show that the way the n-th component of $\partial\Sigma_n$ is covered depends on the cyclic structure of $\theta(e_n)$, and that the covering over non-orientable Σ_n is orientable if and only if each $\theta(c_k)$ contains cycles of even length only. Both assertions are easy and left as an exercise. $\square$

The parity condition We can now explain why Conditions 2 and 5 of the definition of compatible branch datum are necessary for realizability. The latter condition is actually easily deduced from the fact that a covering $\widetilde{\Sigma} \to \Sigma$ with orientable $\widetilde{\Sigma}$ and non-orientable Σ factors through the orientable double covering $\Sigma' \to \Sigma$ and some covering $\widetilde{\Sigma} \to \Sigma'$ of degree $d/2$. We can now prove necessity of the former condition using Theorem 2.1:

Lemma 2.2. *If a datum is realizable then $n \cdot d - \widetilde{n}$ is even.*

Proof. The conclusion is evident when Σ is orientable, because $\widetilde{\Sigma}$ also is, so $\chi(\Sigma)$ and $\chi(\widetilde{\Sigma})$ are even, and we have Condition 1. Assume then that Σ is non-orientable and consider a representation θ realizing the datum as in Theorem 2.1. Notice that $\theta(e_1) \cdots \theta(e_n)$ is an even permutation, since its inverse is the product of either commutators or squares. A permutation is even if and only if it contains an even number of cycles of even length, *i.e* if the sum of a contribution $\ell - 1$ for each cycle of length ℓ is even. Now the lengths of the cycles of $\theta(e_i)$ are the d_{ij}'s, whence

$$\sum_{i=1}^{n} \sum_{j=1}^{m_i} (d_{ij} - 1) = n \cdot d - \widetilde{n}$$

is even. □

Covering subgroup Consider a covering $f : \widetilde{\Sigma}_{\widetilde{n}} \to \Sigma_n$ realizing a branch datum via a representation θ as in Theorem 2.1, with notation as in its statement and proof, particularly that of Fig. 1. The following result describes the image in $\pi_1(\Sigma_n)$ of $\pi_1(\widetilde{\Sigma}_{\widetilde{n}})$.

Lemma 2.3. *If the base-point for $\pi_1(\widetilde{\Sigma}_{\widetilde{n}})$ is taken at the centre of $\Delta^{(h)}$ then $\theta(f_*(\pi_1(\widetilde{\Sigma}_{\widetilde{n}})))$ is the stabilizer of h in the image of θ.*

Proof. Denote by $\widetilde{x}_h$ this base-point. For $\varphi \in \pi_1(\Sigma_n)$ we can determine $\theta(\varphi)(h)$ as the index h' such that the lifting of φ which starts at $\widetilde{x}_h$ ends at $\widetilde{x}_{h'}$. This property is evident for the elements of the generating set of $\pi_1(\Sigma_n)$ obtained by omitting e_n, and it follows for the other elements. If φ belongs to the image of f_*, then the lifting of φ is a loop at $\widetilde{x}_h$, so $\theta(\varphi)(h) = h$. We have shown the inclusion $\theta(f_*((\pi_1(\widetilde{\Sigma}_{\widetilde{n}}))) \subseteq \mathrm{Stab}(h)$, and the opposite inclusion follows from the fact that both subgroups have index d in the image of θ. □

Before proceeding we note that, for an orbifold $(\Sigma; p_1, \ldots, p_n)$, the orbifold fundamental group [20] is the quotient of the corresponding $\pi_1(\Sigma_n)$, with presentation as above, with respect to the normal subgroup generated by $e_1^{p_1}, \ldots, e_n^{p_n}$. Now recall that, just as for ordinary coverings between well-behaved spaces, an orbifold covering $\widetilde{X} \to X$ is associated to a subgroup H of the orbifold fundamental group of X, where $\widetilde{X}$ is the quotient under H of the orbifold universal covering of X. Note that H is well-defined up to conjugation.

The next result, which follows quite easily from the previous lemma, was proved in the orientable case by Singerman, and we believe it clarifies quite well the relation between the approach through orbifolds and that through surfaces with boundary.

Proposition 2.4. *Consider an orbifold covering*

$$\left(\widetilde{\Sigma}; \frac{p_1}{d_{11}}, \ldots, \frac{p_1}{d_{1m_1}}, \ldots, \frac{p_n}{d_{n1}}, \ldots, \frac{p_n}{d_{nm_n}}\right) \to (\Sigma; p_1, \ldots, p_n)$$

arising from a realization as described in Theorem 2.1 of a certain branch datum. Then the subgroup of the orbifold fundamental group of $(\Sigma; p_1, \ldots, p_n)$ to which this covering is associated is the image in this group of $\theta^{-1}(\mathrm{Stab}(h))$.

3 Non-existence results

It turns out that in the particularly important case of coverings over the sphere $\mathbb{S}$ with $n = 3$ branching points there are many examples of exceptional (*i.e.* compatible but non-realizable) data, and in this section we present some of them, together with a few examples for $n \geqslant 4$. To prove exceptionality we employ dessins d'enfants and decomposition of coverings, as anticipated in Section 1.

Dessins d'enfants The first of our methods for establishing non-realizability of branch data is the correspondence between branched coverings and the so-called *dessins d'enfants* of Grothendieck [9]. To give the definition and describe the correspondence we recall that a graph (a finite 1-complex) is called *bipartite* if its set of vertices is split as $V_1 \sqcup V_2$ and each edge has an endpoint in V_1 and one in V_2.

Definition 3.1. A *dessin d'enfant* on the surface $\widetilde{\Sigma}$ is a bipartite graph $D \subset \widetilde{\Sigma}$ such that $\widetilde{\Sigma} \setminus D$ consists of open discs. The *length* of one of these discs is the number of edges of D along which its boundary passes (with multiplicity).

Proposition 3.2. *The realizations of a branch datum $\left(\widetilde{\Sigma}, \mathbb{S}, 3, d, (d_{ij})\right)$ correspond to the dessins d'enfants $D \subset \widetilde{\Sigma}$ with set of vertices split as $V_1 \sqcup V_2$ such that for $i = 1, 2$ the vertices in V_i have valences $(d_{ij})_{j=1,\ldots,m_i}$, and the discs in $\widetilde{\Sigma} \setminus D$ have lengths $(2d_{3j})_{j=1,\ldots,m_3}$.*

Proof. Suppose a realization $f : \widetilde{\Sigma} \to \mathbb{S}$ exists, let the branching points be p_1, p_2, p_3, choose an arc α joining p_1 to p_2 and avoiding p_3, and define D as $f^{-1}(\alpha)$. Setting $V_i = f^{-1}(p_i)$ for $i = 1, 2$, it is clear that D is a bipartite graph with partition of vertices $V_1 \sqcup V_2$ with the required valences. Now $\mathbb{S} \setminus \alpha$ is an open disc and the restriction of f to any component of $\widetilde{\Sigma} \setminus D$ is a covering onto this disc with a single branching point. Such a covering is always modelled on the covering $z \mapsto z^k$ of the open unit disc onto itself, so the components of $\widetilde{\Sigma} \setminus D$ are open discs. More precisely, there is one such disc for each element of $f^{-1}(p_3)$, and it is easy to see that the j-th one has length $2d_{3j}$.

Reversing this construction is a routine matter that we can leave to the reader. □

From dessins to permutations and back We recall that the conjugacy classes in $\mathfrak{S}_d$ are given precisely by the partitions of d, with the class of a permutation being the array of lengths of its cycles. The following is a consequence of Theorem 2.1, in which we emphasize the constructive nature of the theorem:

Corollary 3.3. *The realizations of a branch datum $\left(\widetilde{\Sigma}, \mathbb{S}, 3, d, (d_{ij})\right)$ correspond to the choices of $\tau_1 = \theta(e_1)$ and $\tau_2 = \theta(e_2)$ in $\mathfrak{S}_d$ such that $\langle \tau_1, \tau_2 \rangle$ is transitive and, setting $\tau_3 = \tau_2^{-1} \cdot \tau_1^{-1}$, for $i = 1, 2, 3$ the conjugacy class of τ_i is given by $(d_{ij})_{j=1,\ldots,m_i}$.*

Combining Proposition 3.2 and Corollary 3.3 we deduce a correspondence between (suitable) dessins d'enfants in $\widetilde{\Sigma}$ and (suitable) choices of $\tau_1, \tau_2 \in \mathfrak{S}_d$. Since we will use it in the sequel, we spell out this correspondence explicitly in the next two propositions. The proofs are easy and hence omitted. The first statement uses the easy fact that there are d edges in a dessin d'enfant corresponding to a branch datum of degree d.

Proposition 3.4. *Given a dessin d'enfant $D \subset \widetilde{\Sigma}$, with vertices split as $V_1 \sqcup V_2$, corresponding to a realization of branch datum $\left(\widetilde{\Sigma}, \mathbb{S}, 3, d, (d_{ij})\right)$, a choice of τ_1, τ_2 corresponding to the same realization is constructed by assigning labels from 1 to d to the edges of D and, for $i = 1, 2$ and $k \in \{1, \ldots, d\}$, by selecting the vertex of V_i to which the edge with label k is incident, and defining $\tau_i(k)$ as the label following k in positive order around that vertex.*

To describe the opposite correspondence, given $\tau_1, \tau_2 \in \mathfrak{S}_d$, we construct a graph $D(\tau_1, \tau_2)$ with vertices $V_1 \sqcup V_2$, where V_i is the set of cycles of τ_i, and for $k = 1, \ldots, d$ an edge labelled k joins the cycles of τ_1 and τ_2 which contain k. Note that $D(\tau_1, \tau_2)$ is connected if and only if $\langle \tau_1, \tau_2 \rangle$ is transitive. Then we consider in $D(\tau_1, \tau_2)$ the loops constructed as follows. We start from a vertex of V_2, we run along some edge k until we reach a vertex of V_1, then we run along the edge $\tau_1^{-1}(k)$ until we reach a vertex of V_2, we run along the edge $(\tau_2^{-1} \cdot \tau_1^{-1})(k)$, and we proceed similarly until we reach the original vertex in V_2 again.

Proposition 3.5. *Given $\tau_1, \tau_2 \in \mathfrak{S}_d$ corresponding to a realization of a branch datum $\left(\widetilde{\Sigma}, \mathbb{S}, 3, d, (d_{ij})\right)$, the space obtained from $D(\tau_1, \tau_2)$ by attaching discs to the loops just described is $\widetilde{\Sigma}$, and $D(\tau_1, \tau_2) \subset \widetilde{\Sigma}$ is a dessin d'enfant corresponding to the same realization of the branch datum.*

A sample application To investigate the realizability of a branch datum $\left(\widetilde{\Sigma}, \mathbb{S}, 3, d, (d_{ij})\right)$ using the permutation approach of Corollary 3.3, one should fix a certain τ_1 with cycle lengths (d_{1j}) and then let τ_2 vary in the conjugacy class (d_{2j}), checking that $\langle \tau_1, \tau_2 \rangle$ is transitive and that $\tau_1 \cdot \tau_2$ has cycle lengths (d_{3j}). Since conjugacy classes are huge, this method is only feasible for very small d, and we exploited it using the software GAP for $d \leqslant 10$. On the other hand the realizability criterion through dessins d'enfants, stated in Proposition 3.2, has the advantage of usually requiring the consideration of a much smaller number of cases. The geometric nature of the criterion often also makes it very easy to apply it. As a first example, we give a very simple proof of a result stated in Section 1 originally established in [4].

Proof of 1.10. If the datum is realizable then the dessin d'enfant associated to the first two branching points is just a circle embedded in $\mathbb{S}$. Such a dessin decomposes $\mathbb{S}$ into two discs of the same length, so $x = d - x$, whence $x = d/2$. The same argument proves also the opposite implication. $\square$

Exceptional data with non-prime degree We now establish the generalization of [4, Proposition 5.7] stated in Section 1. Its proof only uses the combinatorics of $D(\tau_1,\tau_2)$, without referring to its embedding in $\mathbb{S}$.

Proof of 1.14. First note that the number of 1's in the third partition of d is $(k-1)h-p+1$. Take arbitrary $\tau_1,\tau_2\in\mathfrak{S}_d$ with cyclic structures $(k,\ldots,k)$ and $(kh_1,\ldots,kh_p)$ such that $\langle\tau_1,\tau_2\rangle$ is transitive, and consider the associated graph $D(\tau_1,\tau_2)$. If τ_1, τ_2 realize the branch datum, $\tau_1\cdot\tau_2$ has $(k-1)h-p+1$ fixed points. Remark that to each fixed point of $\tau_1\cdot\tau_2$ there corresponds in $D(\tau_1,\tau_2)$ a pair of edges having the same ends. More precisely, let us define a multi-edge of $D(\tau_1,\tau_2)$ as the set of all edges having two given vertices as ends. Then one sees that a multi-edge φ gives rise to at most $\#(\varphi)$ fixed points of $\tau_1\cdot\tau_2$, and all fixed points arise like this. The case of $\#(\varphi)$ fixed points actually occurs only if τ_1 and τ_2 contain cycles which are the inverse of each other, but this is easily recognized to be incompatible with our assumptions. So a multi-edge φ contributes with at most $\#(\varphi)-1$ fixed points. We conclude that the total number of fixed points is at most the sum of all the multiplicities of the multi-edges, which is equal to kh, minus the number of multi-edges.

Let us now estimate the number of multi-edges. We first note that the vertices split as $V_1\sqcup V_2$, where V_1 consists of h vertices of valence k and V_2 consists of p vertices of valences $kh_1,\ldots,kh_p$. This implies that at least h_j multi-edges are incident to the j-th vertex of V_2. However, if there are exactly h_j multi-edges, connectedness of $D(\tau_1,\tau_2)$ easily implies that $p=1$, which was excluded. So there are at least h_j+1 multi-edges. Therefore there are in total at least $\sum_{j=1}^{p}(h_j+1)=h+p$ multi-edges. From the above we deduce that $\tau_1\cdot\tau_2$ has at most $kh-(h+p)=(k-1)h-p$ fixed points. This implies that τ_1,τ_2 cannot realize the branch datum. $\square$

Graph fattening and applications To apply Proposition 3.2 it is sometimes useful to switch the viewpoint: instead of trying to embed a dessin D in the surface $\widetilde{\Sigma}$, we try to thicken a given graph D to a surface with boundary so to get $\widetilde{\Sigma}$ by capping off the boundary circles. An application of this method is given by the following proof of one of the results stated in Section 1.

Proof of 1.15. We give a unified proof. A dessin d'enfant D corresponding to the first two partitions in the given branch datum, as an abstract graph, is homemorphic to one of the graphs X and Y shown in Fig. 2, with the

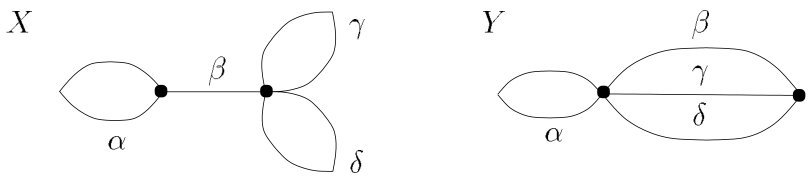


Figure 2: The two graphs with vertices of valences 5 and 3.

two visible vertices lying in V_2. To get D we must then insert on each edge of X or Y an odd number of vertices belonging to V_1 and V_2 alternatively. With a slight abuse of notation, suppose we add $2\alpha+1$ vertices on α, then $2\beta+1$ on β, and so on. Using the fact that V_1 has $d/2$ vertices, we see that $\alpha+\beta+\gamma+\delta=d/2-4$, and this is the only constraint on $\alpha,\beta,\gamma,\delta$.

Up to symmetry, the possible thickenings of X and Y to orientable surfaces with boundary are those described in Fig. 3, as explained in the caption. The associated closed surfaces $\widetilde{\Sigma}$ and the half-lengths of discs added are as follows:

$$
\begin{array}{ll}
X_1: & \mathbb{S},\ (\alpha+1,\gamma+1,\delta+1,\alpha+2\beta+\gamma+\delta+5)\\
X_2: & \mathbb{S},\ (\alpha+1,\gamma+1,\gamma+\delta+2,\alpha+2\beta+\delta+4)\\
X_3: & \mathbb{T},\ (\alpha+1,\alpha+2\beta+2\gamma+2\delta+7)\\
Y_1: & \mathbb{S},\ (\alpha+1,\beta+\gamma+2,\gamma+\delta+2,\alpha+\beta+\delta+3)\\
Y_2: & \mathbb{T},\ (\alpha+1,\alpha+2\beta+2\gamma+2\delta+7)\\
Y_3: & \mathbb{T},\ (\beta+\gamma+2,2\alpha+\beta+\gamma+2\delta+6)\\
Y_4: & \mathbb{T},\ (\alpha+\beta+\gamma+3,\alpha+\beta+\gamma+2\delta+5).
\end{array}
$$

We must then determine all the possible values which can be attained by these strings as $\alpha,\beta,\gamma,\delta$ vary among non-negative integers under the constraint $\alpha+\beta+\gamma+\delta=d/2-4$.

Let us now specialize the proof for $\widetilde{\Sigma}=\mathbb{T}$, corresponding to cases X_3, Y_2, Y_3, and Y_4. It is obvious that the string $(d/2,d/2)$ cannot be realized. If $1\leqslant k\leqslant d/2-3$ we can realize $(d/2+k,d/2-k)$ using Y_4 with $\alpha=d/2-3-k$, $\beta=\gamma=0$, $\delta=k-1$. Using X_3 and Y_3 we can obviously realize $(d-1,1)$ and $(d-2,2)$. This proves the first assertion.

Turning to $\widetilde{\Sigma}=\mathbb{S}$, let us first concentrate on case Y_1. We denote the unordered elements of $(d_{3j})_{j=1,\ldots,4}$ by $x,y,z,w=d-x-y-z$, and solve in

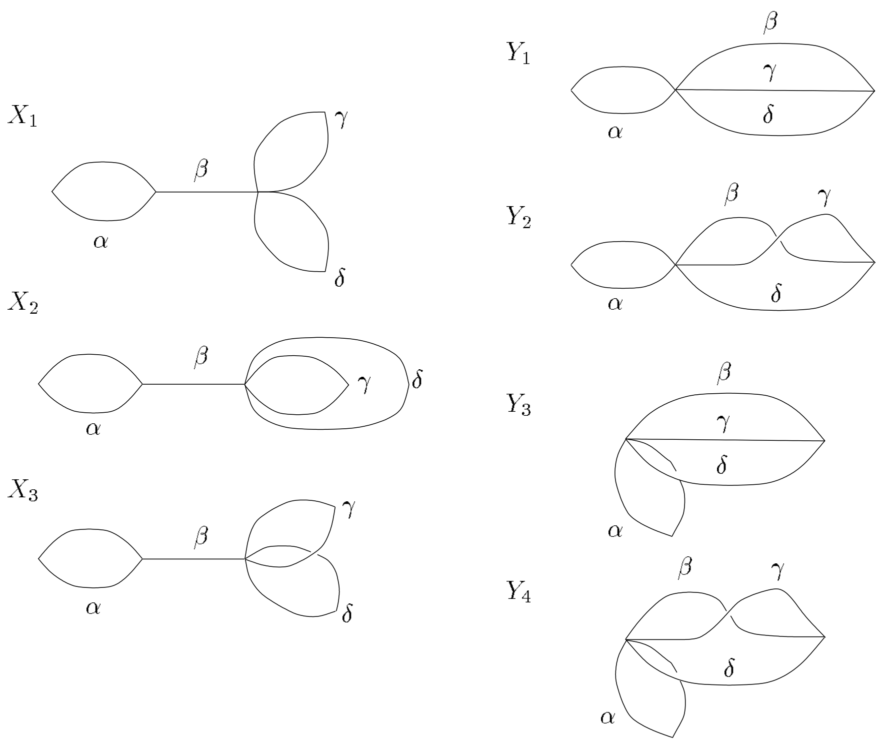


Figure 3: Thickenings of X and Y, always given by the immersion in the plane.

the unknowns $\alpha, \beta, \gamma, \delta$ the system

$$\left\{\begin{array}{lcl} x & = & \alpha+1 \\ y & = & \beta+\gamma+2 \\ z & = & \gamma+\delta+2 \\ w & = & \alpha+\beta+\delta+3 \end{array}\right. \qquad \Rightarrow \qquad \left\{\begin{array}{lcl} \alpha & = & x-1 \\ \beta & = & d/2-x-z-1 \\ \gamma & = & x+y+z-1-d/2 \\ \delta & = & d/2-x-y-1. \end{array}\right.$$

This solution is acceptable if and only if $\alpha, \beta, \gamma, \delta$ are all non-negative, namely if the following holds:

$$(*) \qquad \left\{\begin{array}{rcl} x+y & \leqslant & d/2-1 \\ x+z & \leqslant & d/2-1 \\ x+y+z & \geqslant & d/2+1. \end{array}\right.$$

We deduce that a branch datum with $\widetilde{\Sigma}=\mathbb{S}$ is realizable using case Y_1 if and only if it is possible to extract from $(d_{3j})_{j=1,\ldots,4}$ integers x, y, z satisfying $(*)$. Let us now denote by ℓ the largest of the d_{3j}'s and prove the following facts:
Claim 1. *If $\ell \geqslant d/2$ then the branch datum cannot be realized using Y_1.*
Claim 2. *If $\ell < d/2$ then the branch datum can be realized using Y_1 if and only if it does not have the form $(k, k, d/2-k, d/2-k)$.*

Claim 1 is easy: of course we cannot choose x, y, or z to be ℓ, otherwise one of the first two conditions in $(*)$ would be violated, whence $x+y+z = d-\ell \leqslant d-d/2 = d/2$, which contradicts the last condition in $(*)$. Turning to Claim 2, it is clear that from $(k, k, d/2-k, d/2-k)$ we cannot extract x, y, z satisfying $(*)$. To prove the converse, let us choose x, y, z, w in increasing order, *i.e.*

$$x \leqslant y \leqslant z \leqslant w = \ell = d-x-y-z < d/2$$

which implies the last condition in $(*)$. If the first condition in $(*)$ is violated then so is the second one, and we have

$$\begin{array}{l} x \leqslant y \leqslant z =: d/2-t \leqslant \ell =: d/2-s \\ x+z \geqslant d/2 \Rightarrow x \geqslant t \\ x+y = t+s \Rightarrow y \leqslant s \\ z \leqslant \ell \Rightarrow t \geqslant s. \end{array}$$

These facts imply that $x \leqslant y \leqslant s \leqslant t \leqslant x$. Calling k this common value, we deduce that (x, y, z, w) has the form $(k, k, d/2-k, d/2-k)$, and the claim is established.

To conclude the proof for $\widetilde{\Sigma} = \mathbb{S}$ it is now sufficient to establish the following:

Claim 3. *If $\ell \geqslant d/2$ then a branch datum can be realized using X_1 or X_2 if and only if $(d_{3j})_{j=1,\ldots,4}$ is not $(d/2, d/6, d/6, d/6)$.*

Claim 4. *No branch datum with $(d_{3j})_{j=1,\ldots,4}$ of the form $(k, k, d/2-k, d/2-k)$ can be realized using X_1 or X_2.*

Let us prove Claim 3. Again we denote the d_{3j}'s by x, y, z, w, and we choose them in increasing order, *i.e.*

$$x \leqslant y \leqslant z < w = \ell = d - x - y - z,$$

whence $x + y + z \leqslant d/2$. If we want to realize the datum using X_1 we have the following forced choice up to symmetry:

$$\left\{\begin{array}{rcl} x & = & \alpha + 1 \\ y & = & \gamma + 1 \\ z & = & \delta + 1 \\ w & = & \alpha + 2\beta + \gamma + \delta + 5 \end{array}\right. \qquad \Rightarrow \qquad \left\{\begin{array}{rcl} \alpha & = & x - 1 \\ \beta & = & d/2 - x - y - z - 1 \\ \gamma & = & y - 1 \\ \delta & = & z - 1. \end{array}\right.$$

This is an acceptable solution unless $x + y + z = d/2$. So we turn to case X_2 and try to realize the case $x \leqslant y \leqslant z < \ell = x + y + z$. We first solve

$$\left\{\begin{array}{rcl} x & = & \alpha + 1 \\ y & = & \gamma + 1 \\ z & = & \gamma + \delta + 2 \\ x + y + z & = & \alpha + 2\beta + \delta + 4 \end{array}\right. \qquad \Rightarrow \qquad \left\{\begin{array}{rcl} \alpha & = & x - 1 \\ \beta & = & y - 1 \\ \gamma & = & y - 1 \\ \delta & = & z - y - 1 \end{array}\right.$$

which is acceptable only for $z > y$. We are left to deal with the case $x \leqslant y = z < \ell = x + 2y$ and we try to use X_2 solving

$$\left\{\begin{array}{rcl} y & = & \alpha + 1 \\ x & = & \gamma + 1 \\ y & = & \gamma + \delta + 2 \\ x + 2y & = & \alpha + 2\beta + \delta + 4 \end{array}\right. \qquad \Rightarrow \qquad \left\{\begin{array}{rcl} \alpha & = & y - 1 \\ \beta & = & x - 1 \\ \gamma & = & x - 1 \\ \delta & = & y - x - 1 \end{array}\right.$$

which is acceptable only for $y > x$. We have thus realized the branch datum for all $(d_{3j})_{j=1,\ldots,4}$ except $(d/2, d/6, d/6, d/6)$, which of course cannot be realized.

Turning to Claim 4, we begin using X_2. Since $\gamma+1<\gamma+\delta+2$ and $\alpha+1<\alpha+2\beta+\delta+4$, there is only one attempt we can make:

$$\left\{\begin{array}{rcl} k & = & \alpha+1 \\ k & = & \gamma+1 \\ d/2-k & = & \gamma+\delta+2 \\ d/2-k & = & \alpha+2\beta+\delta+4 \end{array}\right. \qquad \Rightarrow \qquad \left\{\begin{array}{rcl} \alpha & = & k-1 \\ \beta & = & -1 \\ \gamma & = & k-1 \\ \delta & = & d/2-2k-1 \end{array}\right.$$

which is not acceptable. We then try X_1. Up to symmetry there is again one case only, which is of course impossible. $\square$

An argument similar to that just employed allows to prove Proposition 1.17.

Block decompositions We describe another technique that can be employed to prove exceptionality of branch data. The basic underlying remark is that certain patterns in a branch datum force the covering realizing the datum to be *decomposable*, namely the composition of two non-trivial coverings, and one can deduce strong restrictions on the whole datum from the information that all its realizations are decomposable.

We will now state an easy result characterizing decomposable coverings, apparently due to Ritt and cited (in a less detailed fashion) in [13, Theorem 1.7.6]. We first introduce some terminology and establish a lemma which are necessary for the statement.

Definition 3.6. If $1<k<d$ and $k|d$ we call *block decomposition* of *order* k for $\tau\in\mathfrak{S}_d$ a partition of $\{1,2,...,d\}$ into d/k subsets (the *blocks*) such that each block has k elements and τ induces a well-defined permutation $\widehat{\tau}$ of the blocks. We say that $G<\mathfrak{S}_d$ has a block decomposition of order k if there is a common such decomposition for all the elements of G.

Lemma 3.7. *Let τ have cyclic structure $(d_1,\ldots,d_m)$. Then τ admits a block decomposition of order k if and only if $(d_1,\ldots,d_m)$ can be partitioned into sets $D_1,\ldots,D_t$ and there exist integers $p_1,\ldots,p_t\geqslant 1$ such that for all j*

$$p_j|x \quad \forall x\in D_j \qquad \textit{and} \qquad \sum_{x\in D_j}\frac{x}{p_j}=k.$$

In this case $\widehat{\tau}$ has cyclic structure $(p_1,\ldots,p_t)$.

Proof. Suppose the block decomposition exists, and take a cycle $(B_1,\ldots,B_p)$ of $\widehat{\tau}$, with $B_j=\{r_{j,1},\ldots,r_{j,k}\}$. Since $\tau(B_j)=B_{j+1}$ for $j=1,\ldots,p-1$, up to changing notation we have

$$\tau(r_{1,1})=r_{2,1}\qquad \tau(r_{2,1})=r_{3,1}\qquad \ldots\qquad \tau(r_{p-1,1})=r_{p,1}.$$

Now we either have $\tau(r_{p,1})=r_{1,1}$ or, up to changing notation, $\tau(r_{p,1})=r_{1,2}$. Proceeding similarly we see that τ contains a cycle

$$(r_{1,1},\ldots,r_{p,1},r_{1,2},\ldots,r_{p,2},\ldots,r_{1,b_1},\ldots,r_{p,b_1})$$

of order $x_1=p\cdot b_1$. Repeating the same argument from r_{1,b_1+1} and so on we find cycles of length $x_1,\ldots,x_s$ of τ such that $p|x_l$ and $\sum_{l=1}^{s}\frac{x_l}{p}=k$. The conclusion follows by considering the totality of the cycles of $\widehat{\tau}$, and the opposite implication is proved along the same lines. $\square$

Proposition 3.8. *Let $\theta:\pi_1(\Sigma_n)\to\mathfrak{S}_d$ be associated as in Theorem 2.1 to a branched covering realizing a branch datum $\big(\widetilde{\Sigma},\Sigma,n,d,(d_{ij})\big)$. Then the covering is decomposable if and only if Im(θ) has a block decomposition. Moreover, if Im(θ) has a block decomposition of order k with the decomposition of $\theta(e_i)$ corresponding to a partition of (d_{ij}) into $D_{i1},\ldots,D_{it_i}$ and integers $p_{i1},\ldots,p_{it_i}$ then the covering factors through coverings realizing branch data*

$$\left(\widetilde{\Sigma},\Sigma',t_1+\ldots+t_n,k,\left(\left(\frac{x}{p_{ij}}\right)_{x\in D_{ij}}\right)_{i=1,\ldots,n,\ j=1,\ldots,t_i}\right)$$
$$\left(\Sigma',\Sigma,n,d/k,\left((p_{ij})_{j=1,\ldots,t_i}\right)_{i=1,\ldots,n}\right).$$

Note that to apply this result for any given θ one could use [1]. The following is an easy geometric criterion which applies to our context. The proof immediately follows from Proposition 3.4, and a sample application is shown in Fig. 4.

Proposition 3.9. *Let a covering $\widetilde{\Sigma}\to\mathbb{S}$ be associated to a dessin d'enfant $D\subset\widetilde{\Sigma}$ as in Proposition 3.2, with vertices split as $V_1\sqcup V_2$. Then the covering is decomposable if and only if the edges of D can be colored using h colors so that $1<h<d$ and the simultaneous positive rotation around the vertices of V_1 induces a well-defined permutation of colors, and similarly for V_2.*

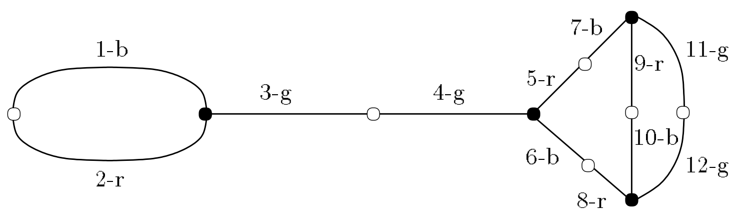


Figure 4: The dessin d'enfant on $\mathbb{S}$ associated as in Proposition 3.5 to the permutations $(1,2,3)(4,6,5)(7,9,11)(8,12,10)$ and $(1,2)(3,4)(5,7)(6,8)(9,10)(11,12)$, and an rgb coloring of the edges showing decomposability of the covering. The blocks of the corresponding block decomposition of $\mathrm{Im}(\theta)$ are $\{1,6,7,10\},\{2,5,8,9\},\{3,4,11,12\}$.

Very even data Here we apply the methods of the previous paragraphs to describe another infinite class of exceptional data. Theorem 1.18 is in fact an immediate consequence of the following result:

Proposition 3.10. *If d is even and all d_{ij} are also even for $i=1,2$ then any covering realizing a branch datum $\big(\mathbb{S},\mathbb{S},3,d,(d_{ij})\big)$ is a composition of coverings realizing data*

$$\big(\mathbb{S},\mathbb{S},4,d/2,(d_{1j}/2)_{j=1,\ldots,m_1},(d_{2j}/2)_{j=1,\ldots,m_2},(d_{3j})_{j=1,\ldots,h},(d_{3j})_{j=h,\ldots,m_3}\big),$$

up to changing the order of the d_{3j}'s, and $\big(\mathbb{S},\mathbb{S},3,2,(2),(2),(1,1)\big)$.

The proof requires a definition, that we will also employ in Section 4, and an easy lemma. We call *checkerboard graph* a finite 1-subcomplex of a surface whose complement consists of open discs each bearing a color black or white, so that each edge separates black from white.

Lemma 3.11. *A connected graph in $\mathbb{S}$ with all vertices of even valence is a checkerboard graph.*

Proof. By induction on the number v of vertices of the graph D. If $v=1$ then D is a wedge of circles, and the conclusion follows from the fact that embedded circles are separating on $\mathbb{S}$. If $v>1$ choose an edge of D having distinct ends, and let D' be obtained from D by contracting this edge to a vertex. Then the hypothesis applies to D', so there is a checkerboard coloring of $\mathbb{S}\setminus D'$. Now the regions of $\mathbb{S}\setminus D$ can be naturally identified to those of $\mathbb{S}\setminus D'$, and it is easy to see that the coloring of $\mathbb{S}\setminus D'$ also works for $\mathbb{S}\setminus D$. $\square$

Proof of 3.10. Let θ correspond to the realization of the datum, as in Proposition 3.3. By Proposition 3.8 it is enough to show that $\mathrm{Im}(\theta)$ has a block decomposition of order $d/2$, which we do by applying Proposition 3.9 to the dessin d'enfant $D \subset \mathbb{S}$ corresponding to the first two branch points. Recall that the vertices of D are split as $V_1 \sqcup V_2$, and by Lemma 3.11 the components of $\mathbb{S} \setminus D$ have a checkerboard coloring. We now color the edges of D black or white so that at each vertex of V_1 each corner of a black component has first a black and then a white edge in the positive order around the vertex. This gives a 2-coloring of the edges of D as prescribed by Proposition 3.9. Moreover τ_1 and τ_2 both transpose the colors, so the covering $\Sigma' \to \mathbb{S}$ given by Proposition 3.8 realizes a datum $\big(\mathbb{S}, \mathbb{S}, 3, 2, (2), (2), *\big)$, and the only such covering is as in the statement. $\square$

Blocks in degree 12 Again applying the decomposition technique of Proposition 3.8 we will now exhibit some exceptional data in degree 12. We begin with an easy consequence of Lemma 3.7 and one of Proposition 3.9.

Corollary 3.12. *If $\tau \in \mathfrak{S}_{12}$ has cyclic structure as in the first column of the next table, the orders of all possible block decompositions of τ and the cyclic structures of the corresponding $\widehat{\tau}$'s are as in the second and third column:*

(7,3,1,1),(7,2,2,1),(5,3,2,2)	—	—
(5,4,2,1),(5,3,3,1),(4,3,3,2)	6	(1,1)
(4,4,3,1)	4	(1,1,1) or (2,1)
(6,4,1,1)	6	(1,1)
	2	(3,2,1)
(6,2,2,2)	6	(1,1) or (2)
	3	(2,2)
	2	(3,1,1,1) or (3,2,1)

Corollary 3.13. *Suppose that $\tau_1, \tau_2 \in \mathfrak{S}_{12}$ have cyclic structures $(3, \ldots, 3)$ and $(2, \ldots, 2)$, that $\langle \tau_1, \tau_2 \rangle$ is transitive, and that $\tau_1 \cdot \tau_2$ is the product of 4 cycles. Then τ_1 and τ_2 have a common block decomposition. The possible orders of such decompositions and the cyclic structures of the corresponding $\widehat{\tau}_1$ and $\widehat{\tau}_2$ are as in the next table:*

4	(3)	(2,1)
3	(3,1)	(2,2)
2	(3,3)	(2,2,1,1)

Proposition 3.14. *If* $(d_{3j})_{j=1,\ldots,4}$ *is* $(7,3,1,1)$, $(7,2,2,1)$, $(6,4,1,1)$, $(6,2,2,2)$, $(5,4,2,1)$, $(5,3,3,1)$, $(5,3,2,2)$, *or* $(4,3,3,2)$ *then the compatible branch datum* $\big(\mathbb{S},\mathbb{S},3,12,(3,3,3,3),(2,2,2,2,2,2),(d_{3j})_{j=1,\ldots,4}\big)$ *is not realizable.*

Proof. If $\tau_1,\tau_2\in\mathfrak{S}_{12}$ give a realization of a such a datum according to Proposition 3.3 then they satisfy the assumptions of Corollary 3.13, so τ_1,τ_2 and $\tau_1\cdot\tau_2$ have a common block decomposition and Proposition 3.8 applies. Comparing the tables in Corollaries 3.12 and 3.13 one readily deduces that $(7,3,1,1)$, $(7,2,2,1)$, $(5,3,2,2)$, $(5,4,2,1)$, $(5,3,3,1)$, $(4,3,3,2)$ are not realizable, because there would be no common block decomposition. If $(6,4,1,1)$ were realizable then by Proposition 3.8 there would exist a covering realizing the datum $\big(\Sigma',\mathbb{S},3,6,(3,3),(2,2,1,1),(3,2,1)\big)$ but this datum is not compatible (Condition 2 is violated). For $(6,2,2,2)$ there would exist a covering realizing either the datum just written or one of the following:

$$\big(\Sigma',\mathbb{S},3,4,(3,1),(2,2),(2,2)\big),\quad \big(\Sigma',\mathbb{S},3,6,(3,3),(2,2,1,1),(3,1,1,1)\big).$$

The former of these data is compatible but not realizable by Proposition 1.10, while the latter is not compatible, because $\chi(\Sigma')$ would be $+4$. $\square$

Using dessins d'enfants as we did to prove Proposition 1.15, which in this case requires to consider 5 different abstract graphs, one can prove the following result, which shows that, at least in this case, the search for exceptions through block decompositions is rather effective:

Proposition 3.15. *The only exceptional branch datum of the form*

$$\big(\mathbb{S},\mathbb{S},3,12,(3,\ldots,3),(2,\ldots,2),(d_{3j})_{j=1,\ldots,4}\big)$$

not listed in Proposition 3.14 is that with $(d_{3j})_{j=1,\ldots,4}=(4,4,3,1)$.

Four branching points Using a computer and the approach through permutations of Corollary 3.3 we have determined all the exceptional data of the form $\big(\widetilde{\Sigma},\mathbb{S},4,d,(d_{ij})\big)$ for $d\leqslant 8$. As expected from the conjecture that there should be no exceptional data with prime d, we have not found any for $d=2,3,5,7$. For $d=4,6,8$ we have found the following:

$\big(\mathbb{T},\mathbb{S},4,4,(2,2),(2,2),(2,2),(3,1)\big)$

$\big(\mathbb{S},\mathbb{S},4,6,(2,2,2),(2,2,2),(2,2,2),(2,1,1,1,1)\big)$

$\big(\mathbb{S},\mathbb{S},4,6,(2,2,2),(2,2,2),(4,1,1),(2,1,1,1,1)\big)$
$\big(\mathbb{T},\mathbb{S},4,6,(2,2,2),(2,2,2),(2,2,2),(3,2,1)\big)$

$\big(\mathbb{S},\mathbb{S},4,8,(2,2,2,2),(2,2,2,2),(2,2,2,2),(3,1,1,1,1,1)\big)$
$\big(\mathbb{S},\mathbb{S},4,8,(2,2,2,2),(2,2,2,2),(6,1,1),(2,1,1,1,1,1,1)\big)$
$\big(\mathbb{S},\mathbb{S},4,8,(2,2,2,2),(2,2,2,2),(5,2,1),(2,1,1,1,1,1,1)\big)$
$\big(\mathbb{S},\mathbb{S},4,8,(2,2,2,2),(2,2,2,2),(5,1,1,1),(3,1,1,1,1,1)\big)$
$\big(\mathbb{S},\mathbb{S},4,8,(2,2,2,2),(2,2,2,2),(5,1,1,1),(2,2,1,1,1,1)\big)$
$\big(\mathbb{S},\mathbb{S},4,8,(2,2,2,2),(2,2,2,2),(3,3,2),(2,1,1,1,1,1,1)\big)$
$\big(\mathbb{S},\mathbb{S},4,8,(2,2,2,2),(4,2,2),(5,1,1,1),(2,1,1,1,1,1,1)\big)$
$\big(\mathbb{T},\mathbb{S},4,8,(2,2,2,2),(2,2,2,2),(2,2,2,2),(5,1,1,1)\big)$
$\big(\mathbb{T},\mathbb{S},4,8,(2,2,2,2),(2,2,2,2),(2,2,2,2),(3,2,2,1)\big)$
$\big(2\mathbb{T},\mathbb{S},4,8,(2,2,2,2),(2,2,2,2),(2,2,2,2),(5,3)\big)$

In fact, as in the case of $n=3$, there are systematic exceptions, as already stated in Section 1:

Proof of 1.16. Suppose a realization exists. Let us modify the technique based on dessins d'enfants by considering, on the base $\mathbb{S}$, instead of a segment, a triod with endpoints at the first three of the branching points and missing the fourth one. The pre-image of this triod is a graph $X\subset\mathbb{S}$ with d trivalent, $d-2$ univalent, and $d+1$ bivalent (fake) vertices, whose complement consists of discs of lengths $6(d-2),6,6$.

Let us color blue, red, and green the pre-images of the first, second, and third branching point, and black those of the vertex of the triod. Then X is obtained in one of the following ways:

I. Put $2d$ points on a circle and color them by repeating $d/2$ times the pattern blue-black-red-black. Join two black points by an arc with one green point in the middle. For each of the other black points, add a tail with a green loose end;

II. Take two circles as above with $4l$ and $2d-4l$ points. Join two black points of different circles by an arc with one green point in the middle. For each of the other black points, add a tail with a green loose end.

Let us show that if $d\geqslant 6$ then there cannot be two discs of length 6 in the complement of X. If X is constructed as in (I), before adding the arc and the tails we have two discs of length $2d\geqslant 12>6$. So we must create a disc of length 6 by adding the arc, after which the other two discs have lengths $2d \geqslant 12 > 6$ and $2d-4+2 \geqslant 10 > 6$. Addition of tails can now only increase the lengths, and we are done. If X is constructed as in (II), before adding the tails we have discs of lengths $4l$, $2d-4l$, and $2d+4 \geqslant 16 > 6$. Since $d \geqslant 6$, either $4l$ or $2d-4l$ is greater 6, and we conclude as in the other case. $\square$

More branching points Again using a computer we have investigated the case of coverings of degree at most 6 over $\mathbb{S}$ with 5 or 6 branching points. The outcome, which is somewhat surprising in the case of degree 6, was that only the following coverings, which are part of the series described in Theorem 1.9, are exceptional:
$\big(2\mathbb{T}, \mathbb{S}, 5, 4, (2,2), (2,2), (2,2), (2,2), (3,1)\big)$
$\big(3\mathbb{T}, \mathbb{S}, 6, 4, (2,2), (2,2), (2,2), (2,2), (2,2), (3,1)\big)$.

4 A geometric criterion for existence

In this section we prove that the Hurwitz existence problem with $\Sigma = \mathbb{S}$ has a geometric equivalent in terms of the existence of certain families of graphs on the putative covering surface $\widetilde{\Sigma}$. Our result extends that proved by Baránski [2] for $\widetilde{\Sigma} = \mathbb{S}$.

Minimal checkerboard graphs In the surface $\widetilde{\Sigma}$ we consider *checkerboard graphs*, as defined in Section 3, and we regard them up to homeomorphism of $\widetilde{\Sigma}$ and switching of colors. A checkerboard graph is called *minimal* if at every vertex all the black germs of discs are contained in the same global disc, and similarly for white.

Lemma 4.1. *If G is minimal then $\widetilde{\Sigma} \setminus G$ consists of one black and one white disc.*

Proof. If x_0 and x_1 are vertices of G joined by an edge, then there is a black disc incident to both x_0 and x_1, so there is only one global black disc incident to x_0 and x_1, and similarly for white. Now the conclusion follows from the remark that G is connected, because its complement consists of discs. $\square$

Minimal graphs from permutations We will now prove that for any g there are finitely many minimal checkerboard graphs on the closed orientable surface $g\mathbb{T}$ of genus g (including the case $g = 0$), and that they can be constructed algorithmically. Fix two copies B and W of the unit disc of $\mathbb{C}$, and for every integer $p > 1$ denote by $\mathcal{R}_p^B$ (respectively, $\mathcal{R}_p^W$) the set of p-th roots of unity on ∂B (respectively, ∂W).

Proposition 4.2. *1. Let $p > 1$ and $f \in \mathfrak{S}(\mathcal{R}_p^B)$. Define $\widetilde{f} \in \mathfrak{S}(\mathcal{R}_p^B)$ as $\widetilde{f}(z) = f(\mathrm{e}^{2\pi i/p} \cdot z)$. Suppose that:*

- *f consists of q cycles each having length at least 2;*
- *$\widetilde{f}$ is a full cycle;*
- *$p - q = 2g$.*

Then define $h : \partial W \to \partial B$ as

$$h(\mathrm{e}^{2\pi i(k+t)/p}) = \mathrm{e}^{2\pi i t/p} \cdot (\widetilde{f})^k(1), \qquad k = 0, \ldots, p-1, \quad 0 \leqslant t < 1,$$

and note that h is bijective. Define $f' \in \mathfrak{S}(\mathcal{R}_p^W)$ as $f' = h^{-1} \circ f \circ h$ and remark that h induces a homeomorphism $\overline{h} : \partial B/_f \to \partial W/_{f'}$. Then:

- *The gluing of $B/_f$ to $W/_{f'}$ along $\overline{h}$ gives $g\mathbb{T}$, with $g > 0$;*
- *The image in $g\mathbb{T}$ of ∂B, which is equal to the image of ∂W, is a minimal checkerboard graph on $g\mathbb{T}$;*

2. If $g > 0$, all minimal checkerboard graphs on $g\mathbb{T}$ arise as above for suitable $p > 1$ and $f \in \mathfrak{S}(\mathcal{R}_p^B)$.

Before turning to the proof of this result, we provide an alternative description of the statement and an example which show that the construction, despite its apparent complication, is actually straight-forward. Let the cycles of f have lengths $\ell_1, \ldots, \ell_q$. Give labels $a_j^{(i)}$ to the points of $\mathcal{R}_p^B$ for $i = 1, \ldots, q$ and $j = 0, \ldots, \ell_i - 1$, in such a way that $f(a_j^{(i)}) = a_{j+1}^{(i)}$ with indices modulo ℓ_i. An example is shown for $p = 7, q = 3, \ell_1 = \ell_2 = 2, \ell_3 = 3$ in Fig. 5-left, where for simplicity we have replaced $a^{(1)}, a^{(2)}, a^{(3)}$ by a, b, c. The definition of $\widetilde{f}$ is then as follows: from any point of $\mathcal{R}_p^B$, first go to the next point on ∂B in the counter-clockwise order, and then jump to the next point according to f. In our case, we get the full cycle $a_0 \mapsto b_0 \mapsto a_1 \mapsto b_1 \mapsto c_2 \mapsto c_1 \mapsto c_0$. If we label the points of $\mathcal{R}_p^W$ according to the labels in this cycle $\widetilde{f}$, as in Fig. 5-right, the definitions of f' and h are now immediately understood: again $f'(a_j^{(i)}) = a_{j+1}^{(i)}$, while $h : \partial W \to \partial B$ maps each open arc of $\partial W \setminus \mathcal{R}_p^W$ to the

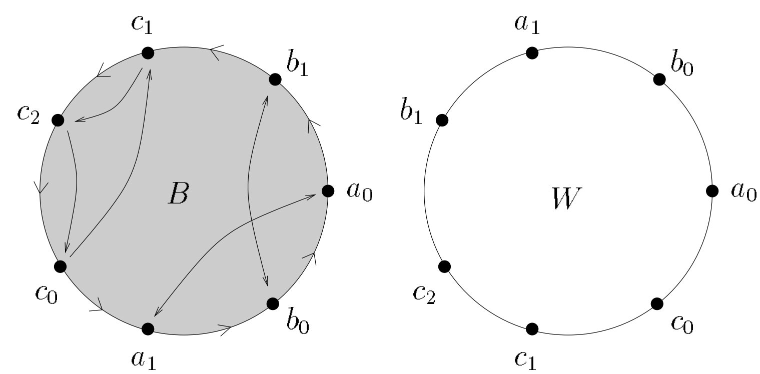


Figure 5: A permutation of the black 7-th roots of unity and the associated white permutation.

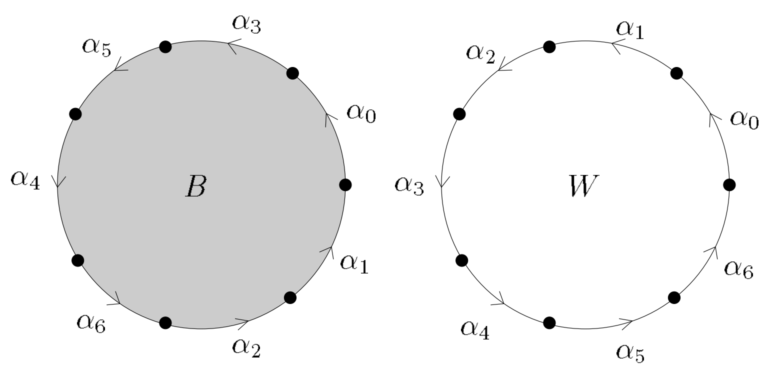


Figure 6: Example continued: identifications of arcs on ∂B and ∂W.

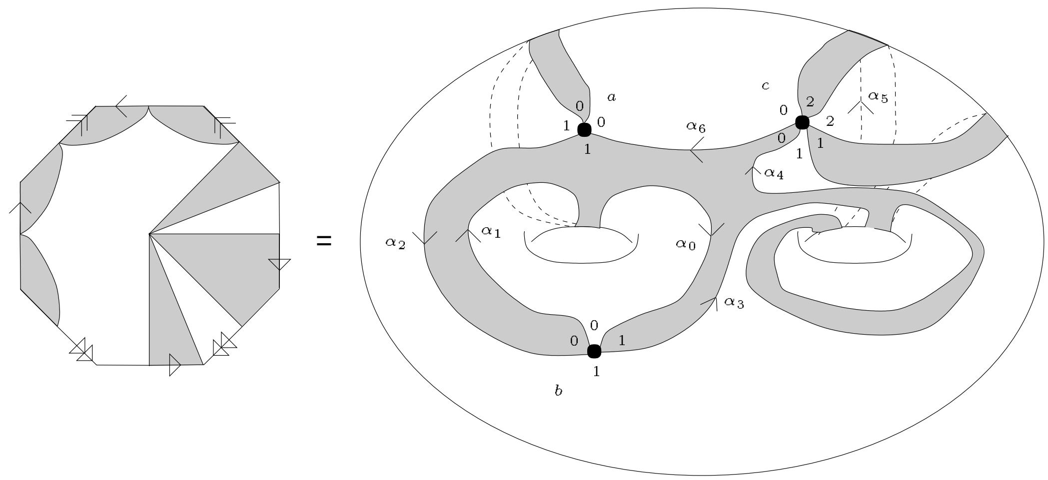


Figure 7: Example continued: a minimal checkerboard coloring of the genus-2 surface.

open arc of $\partial B \setminus \mathcal{R}_p^B$ having *first* endpoint with the same label. See Fig. 6 for the identifications of arcs in our example, and Fig. 7 for the resulting checkerboard coloring of the genus-2 surface.

Proof. Suppose a minimal checkerboard graph G is given. Since g is positive, G has vertices. Suppose there are q of them, and label them by $a^{(i)}$ for $i = 1, \ldots, q$. If $a^{(i)}$ has valence $2\ell_i$, label the germs of black and white discs around $a^{(i)}$ in positive order as

$$a_0^{(i)}, a_0^{(i)}, a_1^{(i)}, a_1^{(i)}, \ldots, a_{\ell_i-1}^{(i)}, a_{\ell_i-1}^{(i)}$$

starting from an arbitrary *white* germ. Recalling that there is only one white disc, label the edges of G as $\alpha_1, \ldots, \alpha_p$ while following the boundary of the white disc in the *negative* fashion. Now read the labels of the germs of discs and of the edges on the boundary of the abstract black and white discs B and W with respect to the obvious immersions from B and W to the surface. The maps f and f' of the definition are now those telling that the vertices $a_j^{(i)}$, for $j = 0, \ldots, \ell_i - 1$, become the single vertex $a^{(i)}$ in the surface, and describing the way the germs are cyclically arranged around $a^{(i)}$. Similarly, the map h says that each arc α_k on ∂W is glued to the corresponding α_k on ∂B. The cell decomposition of the surface associated to G consists of q

vertices, p edges, and 2 discs, whence $p-q=2g$, which implies that G arises from $f\in\mathfrak{S}(\mathcal{R}_p^B)$ as described in point 1.

Now suppose $f\in\mathfrak{S}(\mathcal{R}_p^B)$ is given. For each orbit $a_0^{(i)},\ldots,a_{\ell_i-1}^{(i)}$ of f we identify the points of the orbit to a single point $a^{(i)}$ and we give to the quotient space a planar structure near $a^{(i)}$ so that the germs of discs arising from the $a_j^{(i)}$'s are cyclically arranged in the order $j=0,\ldots,\ell_i-1$. With this choice a system of attaching loops for white discs is well-defined. The condition that $\widetilde{f}$ is a full cycle translates the fact that a single white disc is glued to the black disc, and the condition that $p-q=2g$ means that the result of the gluing is the orientable genus-g surface. $\square$

Corollary 4.3. *On $g\mathbb{T}$ there are finitely many minimal checkerboard graphs.*

Proof. With the notation of the previous proposition, we have that $q\leqslant p/2$ from the assumption that f has no fixed points. Whence $p\leqslant 4g$ from the condition $p-q=2g$. The conclusion follows. $\square$

Genus 0 **and** 1 The only minimal checkerboard graph on $\mathbb{S}$ is of course a plain circle, which splits $\mathbb{S}$ into an embedded black and an embedded white disc, as considered in [2]. This corresponds in Proposition 4.2 to the limit case $p=q=0$.

For genus 1, the argument proving Corollary 4.3 shows that $p\leqslant 4$. Moreover $p-q=2$. Now we note the following easy general fact:

Remark 4.4. Let $f\in\mathfrak{S}(\mathcal{R}_p^B)$ and suppose that f maps some point in $\mathcal{R}_p^B$ to the point preceding it in the counter-clockwise order on ∂B. Then the corresponding $\widetilde{f}$ as in Proposition 4.2 has a fixed point, so it is not a full cycle.

According to this result, only the p's and f's as in Fig. 8 can be relevant for $g=1$. Both these permutations f satisfy the condition that the corresponding $\widetilde{f}$ as in Proposition 4.2 is a full cycle. The associated minimal checkerboard graphs (and colorings) of the torus are shown in Fig. 9.

Existence of coverings We now prove our extension of Baránski's criterion for existence of a branched covering onto $\mathbb{S}$. We denote by $V(G)$ the set of vertices of a graph G and by $\mathrm{val}_G(w)$ the valence in G of a vertex w.

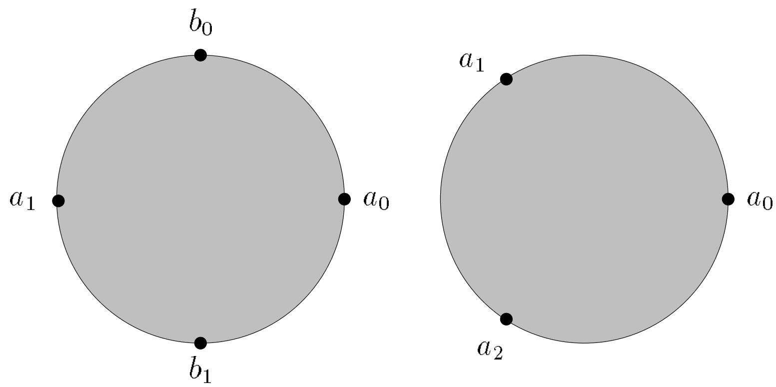


Figure 8: Permutations of roots of unity relevant for the torus.

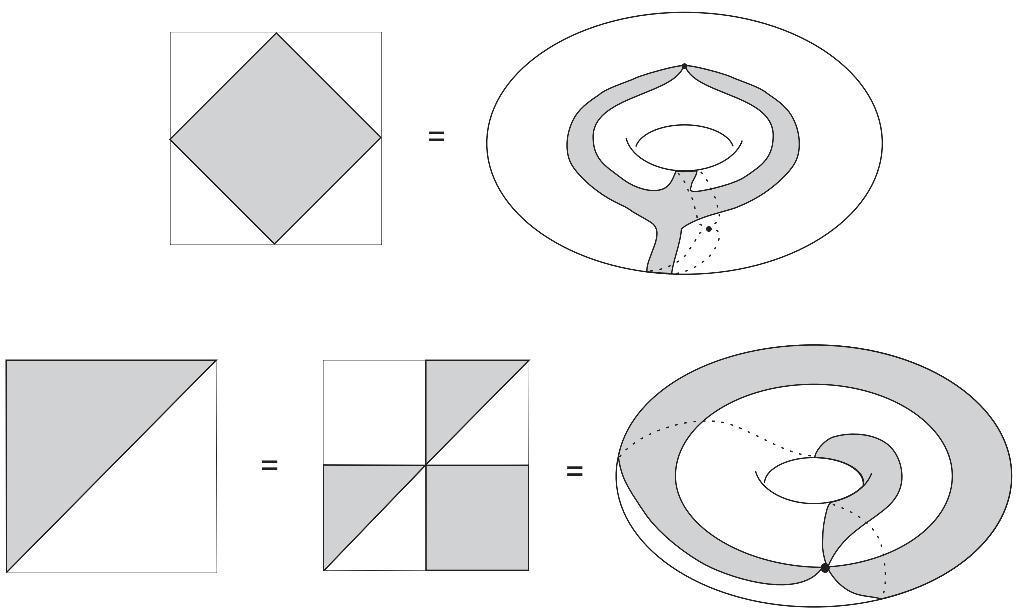


Figure 9: Minimal checkerboard graphs on the torus.

Theorem 4.5. *A branch datum with* $\Sigma = \mathbb{S}$ *and* $\widetilde{\Sigma} = g\mathbb{T}$ *is realizable if and only if there exists on* $g\mathbb{T}$ *a minimal checkerboard graph* G *associated to* $f \in \mathfrak{S}(\mathcal{R}_p^B)$ *as in Proposition 4.2, a set* S *of* $n \cdot d - p$ *points of* $G \setminus V(G)$*, a labelling in* $\{1, \ldots, n\}$ *of* $V(G) \cup S$ *and a collection* $\Gamma = \bigcup \Gamma_{ij}$ *of graphs on* $g\mathbb{T}$ *such that:*

- *The* Γ_{ij}*'s are pairwise disjoint trees;*
- *The set of vertices of* Γ *is equal to* $G \cap \Gamma$ *and is also equal to* $V(G) \cup S$*;*
- *Pulling back the labelling of the points through the projection* $\partial B \to G$ *(or, equivalently,* $\partial W \to G$*) we find* $1, \ldots, n, \ldots, 1, \ldots, n$*, with each string* $1, \ldots, n$ *appearing* d *times, in counter-clockwise order;*
- $\Gamma_{ij} \cap G$ *consists of points having label* i*;*
- *The number of edges of* Γ_{ij} *is given by*

$$d_{ij} - 1 - \sum_{w \in V(G) \cap \Gamma_{ij}} \left(\frac{1}{2} \mathrm{val}_G(w) - 1 \right).$$

Proof. Suppose the labelling of points and the trees Γ_{ij} exist. Let us assign to all the points of Γ_{ij} the label i, and let us call *i-portion* of a set X a connected component of the set of points of X having label i. We claim the following assertion:

> *The complement in* $g\mathbb{T}$ *of* $G \cup \Gamma$ *consists of* d *black and* d *white discs. On each black (respectively, white) disc the labelled portions of* Γ *cyclically found on the boundary in the positive (respectively, negative) order have labels* $1, \ldots, n$*.*

Assuming the assertion for a moment, let us explain how to construct the desired covering. Since the Γ_{ij}'s are disjoint trees, we can collapse them to points without changing the topology of $g\mathbb{T}$. After this collapse, G gives a graph G' on $g\mathbb{T}$ with labelled vertices and the complement consisting of d black and d white discs. Moreover on each black (respectively, white) disc the labels of the vertices cyclically found in the positive (respectively, negative) order on the boundary are $1, \ldots, n$. And each edge of G' separates discs of different colors, because this is true for G. The last two statements easily imply that the discs of $g\mathbb{T} \setminus G'$ have embedded closures.

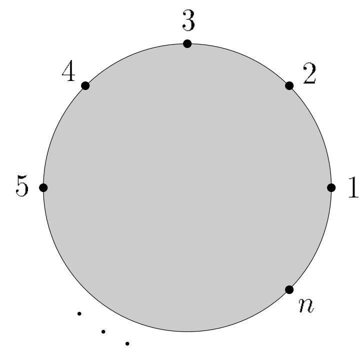


Figure 10: The sphere as a union of a black and a white disc.

Consider now the analogous decomposition of $\mathbb{S}$ as shown in Fig. 10, with one black and one white disc. We map the closure of each black (respectively, white) disc on $g\mathbb{T} \setminus G'$ to the closure of the black (respectively, white) disc on $\mathbb{S}$, matching the labels of the vertices. These maps can be arranged to coincide on the edges of G (which are shared by black and white discs). The result is of course a branched covering, and to conclude we only need to prove that the local degree at the point of $g\mathbb{T}$ arising from the collapse of Γ_{ij} is d_{ij}. This local degree is half of the valence of the vertex in G'. We now show that half of this valence is given by the number of edges of Γ_{ij} plus

$$1 + \sum_{w \in V(G) \cap \Gamma_{ij}} \left(\frac{1}{2} \mathrm{val}_G(w) - 1 \right)$$

which implies the conclusion by the assumption. The formula is of course correct when Γ_{ij} has no edges, because it must consist of a single point of $S \cup V(G)$. Adding an edge with endpoint w to an already existing Γ_{ij}, half of the valence in G' increases by $\frac{1}{2}\mathrm{val}_G(w) = 1 + \left(\frac{1}{2}\mathrm{val}_G(w) - 1\right)$. The conclusion follows.

Let us now prove the claimed assertion. Note first that, by the second assumption and the last one, the total number of edges in Γ is

$$\sum_{i=1}^{n} \sum_{j=1}^{m_i} (d_{ij} - 1) - \sum_{w \in V(G)} \left(\frac{1}{2} \mathrm{val}_G(w) - 1 \right).$$

The first sum gives $n \cdot d - \widetilde{n}$ while the second sum gives $p - q$. Since $n \cdot d - \widetilde{n} = 2d - 2 + 2g$ and $p - q = 2g$, the total number of edges is $2d - 2$. Before adding edges, *i.e.* in $g\mathbb{T}\setminus G$, there are two discs, and each time we add an edge we increase by one the number of discs, therefore the total number if discs in $g\mathbb{T}\setminus G'$ is indeed $2d$.

To proceed, let us call *G-arc* a connected component of $G\setminus(S\cup V(G))$ and let us assign the label i to all the points of a G-arc having ends i and $i+1$ in counter-clockwise order with respect to B. If U is a component of $g\mathbb{T}\setminus(G\cup\Gamma)$ and we are given distinct points of ∂U, we can speak of the two halves into which the points split ∂U, even if we do not know that the closure of U is embedded. We claim the following: *for every component U of $g\mathbb{T}\setminus(G\cup\Gamma)$, for all $i\neq i'$ and for every two distinct points of label i' in ∂U, at least one of the halves into which the points split ∂U contains a G-arc of label i. The other half is either entirely labelled with i' or it also contains a G-arc of label i.* We prove this claim again by imagining that Γ has been constructed adding one edge after the other. The claim is of course true before adding edges, *i.e.* in G. Assume the claim is true up to some stage. When we add the next edge inside a certain region U, we replace U by two regions U' and U'', and both $\partial U'$ and $\partial U''$ differ from ∂U as follows: a (possibly immersed) segment having ends of a certain label i' is replaced by an embedded segment entirely labelled by i'. One easily sees that the property stated in the claim is preserved by such a transformation (recall that Γ consists of trees).

The claim implies that each component of $g\mathbb{T}\setminus(G\cup\Gamma)$ contains at least one G-arc of each label. But there are $2d$ components, n labels and $n\cdot d$ arcs, and each G-arc is shared by 2 components, so each component of $g\mathbb{T}\setminus(G\cup\Gamma)$ contains precisely one G-arc for each label. For the same reasons, there are d black and d white components. The cyclic order of the labels of the G-arcs is preserved at each insertion of an edge, so it is $1,\ldots,n$ in positive (respectively, negative) order on the black (respectively, white) discs. To conclude the proof of the assertion we only need to note that a G-arc of label i and one of label $i+1$ are always separated by an i-portion of Γ. The assertion, and hence the existence of the covering, are eventually established.

Having proved the sufficiency of the conditions stated in the theorem, let us turn to the opposite implication. We then assume that the desired branched covering $g\mathbb{T}\to\mathbb{S}$ exists and we suppose, which of course we can, that the branching points on $\mathbb{S}$ are the labelled points in Fig. 10.

Let H be the pre-image in $g\mathbb{T}$ of the circle in Fig. 10 and $\Delta=(\Delta_{ij})$ be the collection of pre-images of the branching points. We give a label i to Δ_{ij} and a black or white color to each disc in $g\mathbb{T}\setminus H$, according to the color of

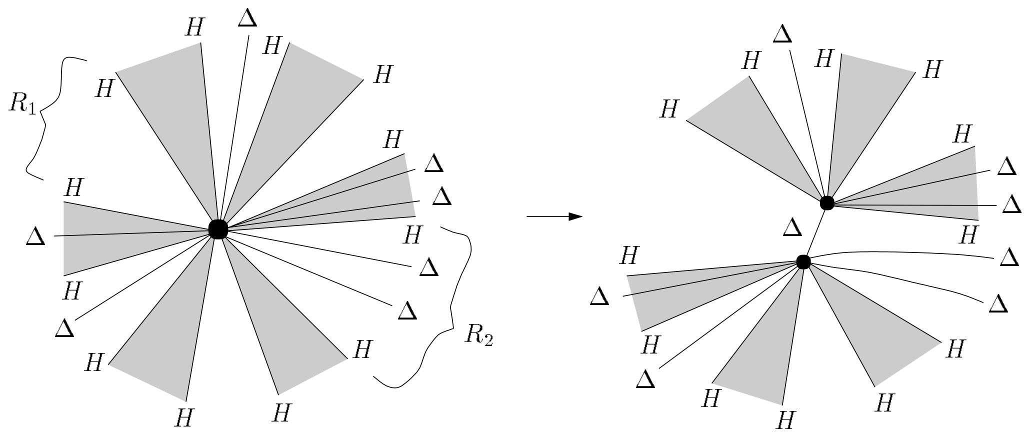


Figure 11: The white move.

its projection. We consider moves on the pair $(H, (\Delta_{ij}))$ which preserve the following properties:

- H is a checkerboard graph on $g\mathbb{T}$;
- The Δ_{ij}'s are mutually disjoint trees;
- The set of vertices of each Δ_{ij} is given by its intersection with H;
- The number of edges of Δ_{ij} plus a contribution of $\frac{1}{2}\mathrm{val}_H(v)-1$ for each vertex v of H contained in Δ_{ij} is $d_{ij}-1$.

Note that the conditions are of course met at the beginning. There are two moves that we employ, a black one and a white one. The white one applies if at some vertex of H there are two distinct white discs. The move itself is described in Fig. 11 where we assume that $R_1 \neq R_2$. Note that there is some arbitrariness in the move, related to the choice of the vertex where it is applied, to the white discs of $g\mathbb{T} \setminus H$ which are merged, and to the new position of the edges in Δ coming from the previous steps. The black move is completely analogous to the white one.

Let us now apply the moves as long as possible and call $(G, (\Gamma_{ij}))$ the final $(H, (\Delta_{ij}))$. By definition G is a minimal checkerboard graph. Let us define S to be the set of points of $G \cap \Gamma$ which are not vertices of $V(G)$, and let us assign the label i to a point of $S \cup V(G)$ if it belongs to some Γ_{ij}. It

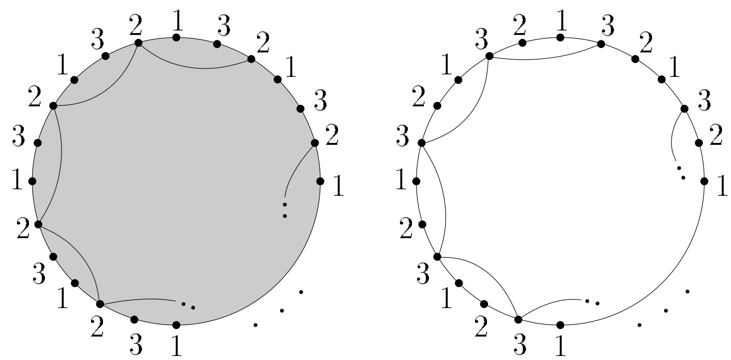


Figure 12: Trees showing that all minimal checkerboard graphs give rise to some covering.

is now a routine matter to repeat the above arguments in the reverse order to show that the assumptions of the theorem are verified. □

Useful minimal checkerboard graphs According to Theorem 4.5, only the minimal checkerboard graphs supporting the further structures described in the theorem are relevant to the Hurwitz existence problem, so *a priori* some graph may be dispensable. However we have the following:

Proposition 4.6. *If G is a minimal checkerboard graph for $g\mathbb{T}$ then there exists a branch datum with $n = 3$ and suitably large d which is realized according to Theorem 4.5 using the graph G.*

Proof. Let us give label 1 to all the vertices of G. Let S consist of two points (with labels 2 and 3) on each edge of G. Now consider in B and W the trees as shown in Fig. 12. These trees give a branched covering of type $(d),(d),(*)$. □

This result does not imply, however, that all minimal checkerboard graphs are necessary to construct the realizable branch data, because a branch datum could be constructed using different graphs. The next example shows that at least for the torus both graphs of Fig. 9 are necessary.

Example 4.7. The minimal checkerboard graph, labels and trees in Fig. 13-left give a covering realizing the datum $\big(\mathbb{T},\mathbb{S},4,2,(2),(2),(2),(2)\big)$, whose non-trivial automorphism is the hyperelliptic involution, which cannot be obtained using the graph in Fig. 9-right. The graph, labels and trees in Fig. 13-

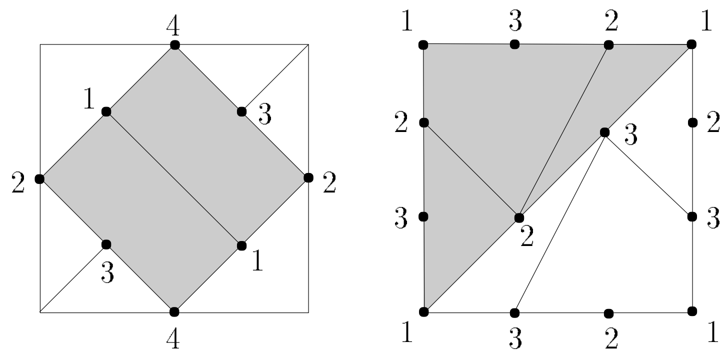


Figure 13: Realizations of coverings $\mathbb{T} \to \mathbb{S}$.

right give a covering realizing the datum $\big(\mathbb{T}, \mathbb{S}, 3, 3, (3), (3), (3)\big)$, which cannot be obtained using the graph in Fig. 9-left.

5 Existence results

In this section we establish the existence results stated in Section 1.

Existence by divisibility Our first argument exploits the idea of composing coverings, already used in Section 3 to prove non-realizability.

Proof of 1.19. Let Σ' be the orientable surface with $\chi(\Sigma') = 3 - p$. Then the branch datum $\big(\Sigma', \mathbb{S}, 3, p, (p), (p), (p)\big)$ is compatible, whence realizable by Theorem 1.8. It is now easy to see that the datum $\big(\widetilde{\Sigma}, \Sigma', 3, d/p, (d_{ij}/p)\big)$ is also compatible. Now $\chi(\Sigma') \leqslant 0$, so the datum is realizable by Theorem 1.2, and the desired covering $\widetilde{\Sigma} \to \mathbb{S}$ can then be constructed as the composition of the coverings $\widetilde{\Sigma} \to \Sigma'$ and $\Sigma' \to \mathbb{S}$. $\square$

Diagrams and accessible trees We now investigate branch data with $\widetilde{\Sigma} \in \{\mathbb{S}, \mathbb{T}\}$ involving the partition $(d-2, 2)$. Our motivation for this is that the conjugacy class in $\mathfrak{S}_d$ of $(d-2, 2)$ is among the largest ones, so this case is not suited for the computer investigation based on the permutation approach of Corollary 3.3.

We will employ the geometric existence criterion Theorem 4.5, of which we will employ the notation throughout. For the sake of brevity, a triple

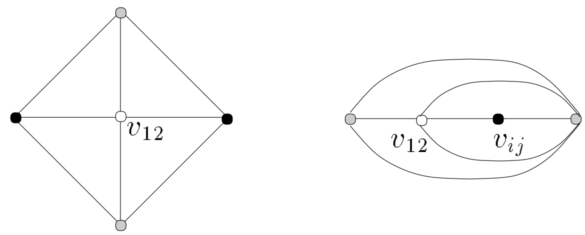


Figure 14: The triangles incident to v_{12}.

G, S, Γ as in the statement of this theorem will be called a *diagram* realizing the corresponding branch datum. For the rest of the paper we fix branch data as in the assumptions of Theorems 1.20 or 1.21, *i.e.* we will henceforth assume that $m_1 = 2$ and $(d_{11}, d_{12}) = (d-2, 2)$. Moreover, in all our diagrams the trees Γ_{11} and Γ_{12} will correspond to this partition of d. For $i = 2, 3$ we will say that a tree Γ_{ij} of a diagram is *accessible* if there is a G-arc with one endpoint in Γ_{ij} and the other endpoint in Γ_{11}. Recall that a G-arc is a connected component of $G \setminus (S \cup V(G))$. A diagram is called *accessible* if all Γ_{ij}'s with $i = 2, 3$ are accessible.

Lemma 5.1. *In a diagram realizing our branch datum there is at most one inaccessible tree, which must be a point.*

Proof. Recall the collapse used in the proof of Theorem 4.5, where each Γ_{ij} is shrunk to a point v_{ij} labelled $i \in \{1, 2, 3\}$, and G is transformed to some graph G'. The valence of v_{ij} is $2d_{ij}$, so v_{11} has valence $2d - 4$ and v_{12} has valence 4. The complement of G' consists of $2d$ triangles, each with embedded closure and vertices labelled $1, 2, 3$. So there are $2d - 4$ triangles incident to v_{11}, and, if Γ_{ij} is inaccessible, then v_{ij} does not belong to any of them. So it belongs to the interior of the union U of the other 4 triangles, which are all incident to v_{12}. The only two possibilities for U are as shown in Fig. 14, and the conclusion follows. □

Coverings by the sphere We now restrict to the case $\widetilde{\Sigma} = \mathbb{S}$.

Proof of 1.20. Note first that compatibility means that $m_2 + m_3 = d$, and recall that, in a diagram G, S, Γ realizing the datum, G is a plain circle, S consists of $3d$ points, and Γ_{ij} is a tree with $d_{ij} - 1$ edges. So Γ_{11} has $d - 3$ edges and Γ_{12} is just one edge.

By Theorem 1.8 the datum is realizable if $(d_{ij})_{j=1,\ldots,m_i} = (d)$ for some $i \in \{2,3\}$. We will then assume henceforth that this is not the case, which easily implies that $d \geqslant 5$ (recall that d is odd). To prove the theorem the general strategy will now be to proceed by induction on d, starting from the diagrams $D_1, \ldots, D_4$ of Fig. 15, which realize all the relevant data for $d = 5$, and successively applying certain moves to these diagrams so to realize all the relevant data for arbitrary d. However, we will have to deal with special sorts of data by separate inductions.

We will employ three moves μ_1, μ_2, μ_3, that we now describe:

μ_1: Choose G-arcs e_2 and e_3 such that each e_i has one end in Γ_{11} and the other end in some Γ_{ij_i}, and perform the modifications of Fig. 16;

μ_2: Choose a G-arc e with one end in Γ_{11} and one in some Γ_{ij_i}, and perform the modification of Fig. 17;

μ_3: Choose a G-arc e with one end in Γ_{11} and one in some Γ_{ij_i}, and perform the modification of Fig. 18. Since we will sometimes use a systematic iteration of this move, a new G-arc is specified and also labelled e in the new diagram.

It is evident that all three moves transform a diagram of degree d to a diagram of degree $d+2$ which still satisfies the conditions of Theorem 4.5. We call a diagram *constructible* if it is obtained from one of the diagrams $D_1, \ldots, D_4$ by successive application of moves μ_*. We will prove that each relevant branch datum is realized by a constructible diagram.

Let us note now that the effect of the μ_*'s on the partitions (d_{2j}) and (d_{3j}) is as follows:

$\widehat{\mu}_1$: Choose j_2, j_3, replace d_{2j_2} and d_{3j_3} by $d_{2j_2}+1$ and $d_{3j_3}+1$ respectively, add a 1 at the end of both partitions, and reorder (if necessary: recall that our partitions are arranged in non-increasing order);

$\widehat{\mu}_2$: Choose $i \in \{2,3\}$ and j_i, let $\{2,3\} = \{i, i'\}$, replace d_{ij_i} by $d_{ij_i}+2$ and reorder, and add two 1's at the end of $(d_{i'j})$;

$\widehat{\mu}_3$: Choose $i \in \{2,3\}$ and j_i, let $\{2,3\} = \{i, i'\}$, replace d_{ij_i} by $d_{ij_i}+1$, add a 1 at the end, add a 2 at the end of $(d_{i'j})$, and reorder.

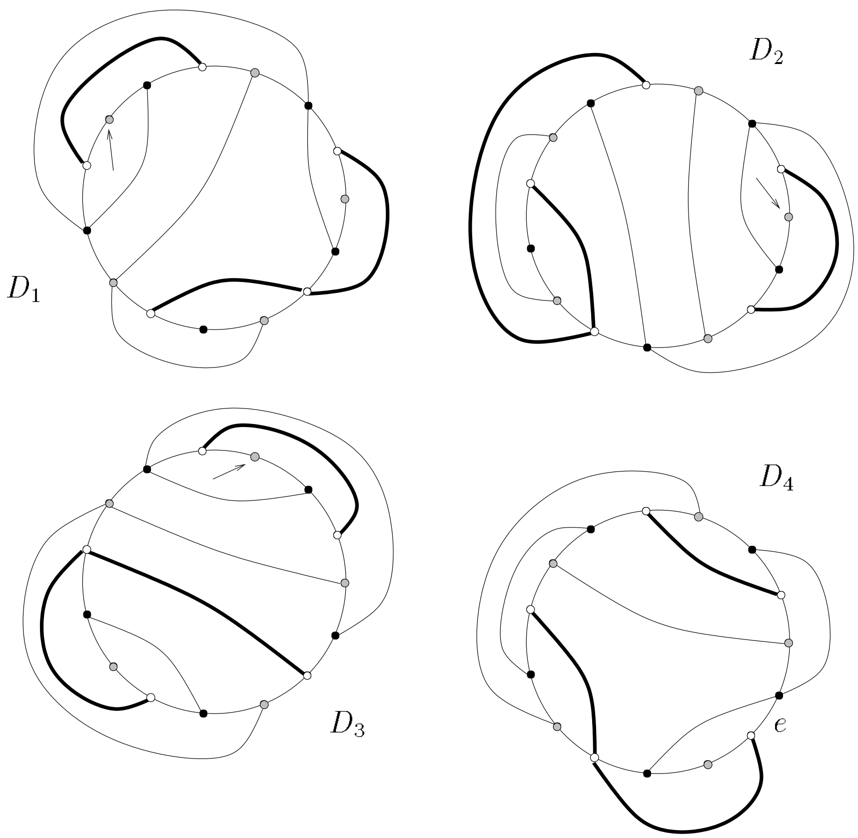


Figure 15: The diagrams $D_1, \ldots, D_4$, with Γ_{11} and Γ_{12} marked by thicker lines. In D_1, D_2, D_3 a small arrow indicates the only inaccessible vertex, while the G-arc e on D_4 will be used later in the proof. The partitions $(d_{ij})_{i=2,3}^{j=1,\ldots,m_i}$ are $(4,1),(3,1,1)$ for D_1, $(4,1),(2,2,1)$ for D_2, $(3,2),(3,1,1)$ for D_3, and $(3,2),(2,2,1)$ for D_4.

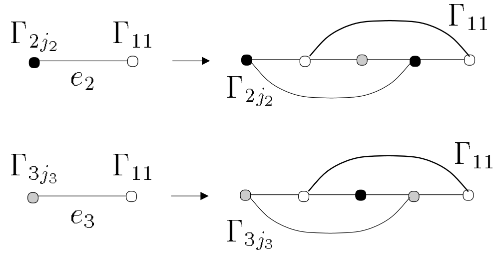


Figure 16: The move μ_1.

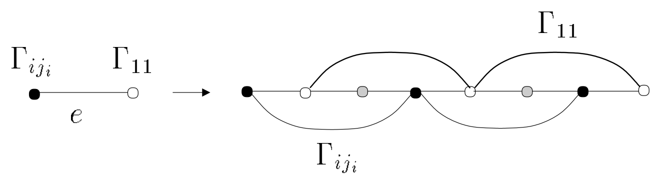


Figure 17: The move μ_2.

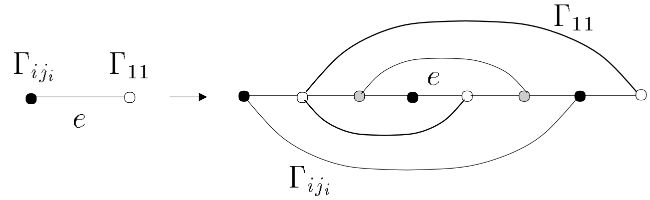


Figure 18: The move μ_3.

Note also that the $\widehat{\mu}_*$'s can be viewed as moves transforming a relevant branch datum of degree d into one of degree $d+2$. However, *a priori* not every branch move $\widehat{\mu}_*$ on the branch datum realized by a constructible diagram is induced by the corresponding geometric move μ_* on that diagram. When this happens, we will say that the move $\widehat{\mu}_*$ itself is *constructible*. Of course a move $\widehat{\mu}_*$ applied to indices j_2, j_3 (for $\widehat{\mu}_1$) or j_i (for $\widehat{\mu}_2$ or $\widehat{\mu}_3$) is constructible if and only if the corresponding trees Γ_{2j_2} and Γ_{3j_3} or Γ_{ij_i} are accessible. Lemma 5.1 then implies that *a branch move $\widehat{\mu}_*$ is constructible if applied to indices d_{2j_2}, d_{3j_3} or d_{ij_i} larger than* 1. We also note that a move μ_* never creates inaccessible trees. Therefore *a branch move $\widehat{\mu}_*$ is constructible if it is applied to a diagram constructed starting from D_4 or to tree(s) created be previous geometric moves.* These facts will be used repeatedly below.

We now state and prove a series of claims that will lead to the conclusion. We start with an easy arithmetic one.

Claim 1. *Either $(d_{ij})_{i=2,3}^{j=1,\ldots,m_i}$ includes two 1's or $(d_{2j})_{j=1,\ldots,m_2} = (3,2,\ldots,2)$ and $(d_{3j})_{j=1,\ldots,m_3} = (2,\ldots,2,1)$, up to permutation.* Assume first there are no 1's. Since d is odd we deduce that $d_{21}, d_{31} \geqslant 3$. Therefore

$$\left(d \geqslant 3 + 2(m_i - 1) \Rightarrow m_i \leqslant \frac{d-1}{2},\ i = 2,3\right) \Rightarrow m_2 + m_3 \leqslant d - 1,$$

which gives a contradiction. Suppose then that $d_{3m_3} = 1$ and there are no other 1's. Then again $m_2 \leqslant \frac{d-1}{2}$, and $m_3 \leqslant \frac{d+1}{2}$ by a similar argument. So both inequalities are equalities and the conclusion readily follows.

We are now ready to proceed with the main part of the proof, where we show that every relevant branch datum, *i.e.* one of the form

$$\big(\mathbb{S},\mathbb{S},3,d,(d-2,2),(d_{2j})_{j=1,\ldots,m_2},(d_{3j})_{j=1,\ldots,m_3}\big)$$

with odd $d \geqslant 7$ and $m_2, m_3 \geqslant 2$, is realized by some constructible diagram. The proof is separate for branch data that involve a partition of the form $(3,\ldots,3,2\ldots,2)$.

Claim 2. *If* $(d_{2j}) = (3,\ldots,3,2\ldots,2)$ *and* $(d_{3j}) = (2,\ldots,2,1,\ldots,1)$ *then the datum can be realized by an accessible constructible diagram.* Since the total number of entries in the two partitions is d, the number k of 3's in the first partition is equal to the number of 1's in the second one, and k is odd. We prove the claim by induction on k. The diagram obtained from D_4 by $(d-5)/2$ successive applications of the move μ_3 starting with the G-arc e indicated in Fig. 15 gives the base $k=1$ of the induction.

For the inductive step, notice that to get the branch datum under consideration at level k in degree d we can apply a move $\widehat{\mu}_1$ (with $j_2 = k-1$ and $j_3 = m_3 - 3$) followed by a move $\widehat{\mu}_2$ (with $i=2$ and $j_2 = m_2$) to the same datum at level $k-2$ in degree $d-4$, and the inductive assumption easily implies the conclusion.

Claim 3. *If* $(d_{2j}) = (3,2\ldots,2)$ *then the datum can be realized by a constructible diagram.* By induction on $d \geqslant 5$. For $d=5$ the conclusion is given by diagrams D_3 and D_4. For the inductive step, suppose first that $d_{31} \leqslant 2$ for all j. It follows that $(d_{3j}) = (2,\ldots,2,1)$, and this case was settled in Claim 2. Then we assume $d_{31} \geqslant 3$. So the branch datum under consideration in degree d is obtained using a move $\widehat{\mu}_3$ (with $i=3$ and $j_3=1$) from the datum with partitions

$$(3,2,\ldots,2),(d_{31}-1,d_{32},\ldots,d_{3(m_3-1)})$$

in degree $d-2$. This datum is constructible by inductive assumption, and the move is constructible because $d_{31}-1 \geqslant 2$, whence the conclusion.

Claim 4. *If* $(d_{2j}) = (3,\ldots,3,2\ldots,2)$ *then the datum can be realized by a constructible diagram.* By induction on the odd number k of 3's in (d_{2j}). The base $k=1$ is given by Claim 3. Suppose $k \geqslant 3$. Then it is easy to see that (d_{3j}) has at least k entries equal to 1. Moreover by Claim 2 we can assume $d_{31} \geqslant 3$. Then the datum under consideration at level k in degree d is obtained from the datum with partitions

$$(3, \ldots, 3, 2, \ldots, 2), (d_{31} - 1, d_{32}, \ldots, d_{3(m_3-3)})$$

at level $k-2$ in degree $d-4$ by a move $\widehat{\mu}_1$ (with $j_2 = k-1$ and $j_3 = 1$) and a move $\widehat{\mu}_2$ (with $i = 2$ and $j_2 = m_2$) . The conclusion now follows from the inductive assumption and from the remark that both moves are constructible, because $\widehat{\mu}_1$ is applied to entries which are at least 2, and $\widehat{\mu}_2$ is applied to an entry 1 created by $\widehat{\mu}_1$.

To conclude the proof, consider a branch datum as specified before Claim 2, and let us prove by induction on d that it is realized by a constructible diagram. The base step follows from Fig. 15. To proceed, assume that no partition is $(3, \ldots, 3, 2 \ldots, 2)$, which is not restrictive by Claim 4. Claim 1 then implies that there are at least two 1's in $(d_{ij})_{i=2,3}^{j=1,\ldots,m_i}$. If only (d_{2j}) contains 1's then $d_{31} \geqslant 4$ by the assumption just made, so the datum is obtained from a move $\widehat{\mu}_2$ (with $i = 3$ and $j_3 = 1$) from

$$(d_{21}, \ldots, d_{2(m_2-2)}), (d_{31} - 2, \ldots, d_{3m_3})$$

and the move is constructible because $d_{31} - 2 \geqslant 2$, whence the conclusion.

Let us then assume that both (d_{2j}) and (d_{3j}) contain some 1. If one of these partitions, say (d_{2j}), is $(2, \ldots, 2, 1)$, then we see that $d_{31} \geqslant 3$, and the datum is obtained from

$$(2, \ldots, 2, 1), (d_{31} - 1, \ldots, d_{3(m_3-1)})$$

by a move $\widehat{\mu}_3$ (with $i = 3$ and $j_3 = 1$), which is constructible because $d_{31}-1 \geqslant 2$. If neither (d_{2j}) nor (d_{3j}) is $(2, \ldots, 2, 1)$, then the datum is obtained from

$$(d_{21} - 1, \ldots, d_{2(m_2-1)}), (d_{31} - 1, \ldots, d_{3(m_3-1)})$$

by a move $\widehat{\mu}_1$ (with $j_2 = j_3 = 1$), and the move is constructible because either $d_{i1} - 1 \geqslant 2$ or there are two 1's to choose from, one of which must be accessible by Lemma 5.1. $\square$

Remark 5.2. Theorem 1.20 does not extend to even d. For instance, Theorem 1.14 implies that the branch datum $\big(\mathbb{S}, \mathbb{S}, 3, 8, (6, 2), (2, 2, 2, 2), (5, 1, 1, 1)\big)$ is exceptional.

Coverings by the torus We now turn to the case of $\mathbb{T}$, where the result is even stronger.

Proof of 1.21. Note first that compatibility means $m_2 + m_3 = d - 2$. Exceptionality of $\big(\mathbb{T}, \mathbb{S}, 3, 6, (4,2), (3,3), (3,3)\big)$ was proved experimentally, so we will exclude this datum.

We now follow the same line of reasoning as in the proof of Theorem 1.20. To this end we introduce a move μ on diagrams. The move depends on the choice of a G-arc joining Γ_{11} to some Γ_{ij_i}, and its effect is described, depending on whether $i = 2$ or $i = 3$, by either the top or the bottom part of Fig. 16. Of course the effect of μ on the partitions (d_{2j}) and (d_{3j}) is the move $\widehat{\mu}$ which consists in replacing d_{ij_i} by $d_{ij_i} + 1$ and reordering, and appending 1 to the other partition. Not every move $\widehat{\mu}$ comes from some μ, but it does if applied to a datum realized by an accessible diagram. The following two facts are repeatedly use below:

- A diagram realizing a datum with $d_{ij} \geqslant 2$ for all i, j is accessible;
- μ transforms accessible diagrams into accessible diagrams.

As in the previous proof we proceed by proving a series of claims.

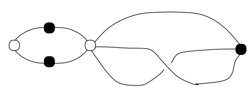
Claim 1. *If d is odd and $d_{ij} \geqslant 2$ for all i, j then (d_{2j}) is $(3, 2, \ldots, 2)$ and (d_{3j}) is either $(5, 2, \ldots, 2)$ or $(4, 3, 2, \ldots, 2)$ up to permutation.* It follows from the assumptions that $m_2, m_3 \leqslant (d-1)/2$. Since their sum is $d - 2$, the only option is that one be $(d-1)/2$ and the other $(d-3)/2$. The partitions of the statement of the claim are the only ones having these lengths and not containing 1's. A similar argument proves the next:

Claim 2. *If d is even and $d_{ij} \geqslant 2$ for all i, j then one of the following happens:*

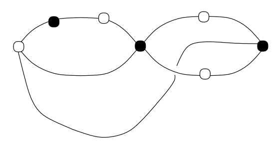
- $(d_{2j}), (d_{3j}) \in \{(4, 2, \ldots, 2), (3, 3, 2, \ldots, 2)\}$*;*
- *Up to permutation, $(d_{2j}) = (2, \ldots, 2)$ and*
  $(d_{3j}) \in \{(6, 2, 2, \ldots, 2), (5, 3, 2, \ldots, 2), (4, 4, 2, \ldots, 2)\}$.

Claim 3. *If $d \geqslant 7$ and $d_{ij} \geqslant 2$ for all i, j then the datum is realizable.* This is proved using the techniques of Section 3.

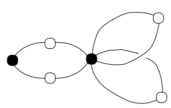
We present in Fig. 19 dessins d'enfants proving realizability of all the desired data except that with partitions $(3, 2, 2), (4, 3), (5, 2)$. For each dessin we have only drawn the minimal number of vertices it should have: an infinite series is obtained by adding pairs of vertices of opposite colors on any of the



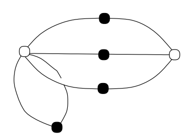
$(3,2,\ldots,2),\ (5,2,\ldots,2),\ d \geqslant 7$ odd



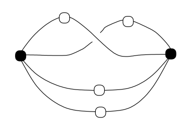
$(3,2,\ldots,2),\ (4,3,\ldots,2),\ d \geqslant 9$ odd



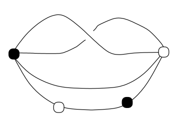
$(2,2,\ldots,2),\ (6,2,\ldots,2)$
$d \geqslant 8$ even



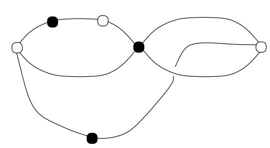
$(2,2,\ldots,2),\ (5,3,\ldots,2)$
$d \geqslant 8$ even



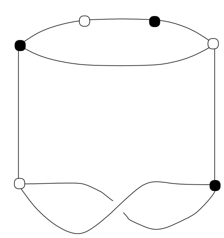
$(2,2,\ldots,2),\ (4,4,\ldots,2)$
$d \geqslant 8$ even



$(4,2,\ldots,2),\ (4,2,\ldots,2)$
$d \geqslant 6$ even



$(4,2,\ldots,2),\ (3,3,2,\ldots,2)$
$d \geqslant 8$ even



$(3,3,2,\ldots,2),\ (3,3,2,\ldots,2)$
$d \geqslant 8$ even

Figure 19: Dessins d'enfants on the torus for branch data not involving 1.

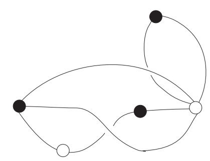


Figure 20: A dessin d'enfant for the branch datum with partitions $(3,2,2),(4,3),(5,2)$.

edges not incident to the length-4 disc that one easily sees in each picture. The branch datum with partitions $(3,2,2),(4,3),(5,2)$ is realized in Fig. 20.

CLAIM 4. *If $d \leqslant 6$ then the datum is realizable by an accessible diagram.* The first relevant d is 4, where there is only the datum with partitions $(2,2),(4),(4)$. This datum is realizable by Theorem 1.8, and we conclude because $d_{ij} \geqslant 2$ for all i,j. For $d=5$ we have $(3,2),(3,2),(5)$ and $(3,2),(4,1),(5)$. Both are realizable by Theorem 1.8, the former has $d_{ij} \geqslant 2$ for all i,j, and the latter is obtained from $(2,2),(4),(4)$ by a move $\widehat{\mu}$, whence the conclusion. For $d=6$ we know experimentally that only the excluded datum is exceptional. The data with $d_{ij} \geqslant 2$ for all i,j are realizable by accessible diagrams, and those involving a 1 are obtained via a move $\widehat{\mu}$, and the conclusion follows from the case $d=5$.
CLAIM 5. *If $d \geqslant 7$ then the datum is realizable by an accessible diagram.* By induction. The base step $d=7$ and the inductive step are essentially identical. For both, either $d_{ij} \geqslant 2$ for all i,j, whence the conclusion from Claim 3, or there is some 1, so the datum is obtained via a move $\widehat{\mu}$ from a datum in degree $d-1$. For $d=7$ we use Claim 4 and the easy fact that the exceptional datum can always be avoided, while for $d \geqslant 8$ the conclusion is immediate from the inductive assumption. $\square$

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