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Complexity and T-invariant of Abelian and Milnor groups, and complexity of 3-manifolds

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Abstract

We investigate the notion of complexity for finitely presented groups and the related notion of complexity for three-dimensional manifolds. We give two-sided estimates on the complexity of all the Milnor groups (the finite groups with free action on S^3), as well as for all finite Abelian groups. The ideas developed in the process also allow to construct two-sided bounds for the values of the so-called T-invariant (introduced by Delzant) for the above groups, and to estimate from below the value of T-invariant for an arbitrary finitely presented group. Using the results of this paper and of previous ones, we then describe an infinite collection of Seifert three-manifolds for which we can asymptotically determine the complexity in an exact fashion up to linear functions. We also provide similar estimates for the complexity of several infinite families of Milnor groups.

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Introduction

The motivation for considering some notion of complexity for groups is its connection [19] with the problem of estimating the complexity of 3-manifolds. The main idea of the theory of complexity for 3-manifolds (introduced in [16, 17], investigated in [8], and comprehensively covered in [18]) is

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to introduce a filtration in the set of all 3-manifolds, in such a way that each level of the filtration contains only finitely many closed irreducible items. This allows to break down the task of classifying all closed 3-manifolds into an infinite collection of finite classification tasks, because complexity is also additive under connected sum, so the complexity of any closed manifold can be computed once the complexity of its irreducible summands is known. This classification program has been carried out to a remarkable extent in recent years (see [14] and the references quoted therein).

For any given manifold it is very easy to give upper bounds for its complexity, whereas lower bounds are much harder to establish. As a matter of fact, the computer programs of Martelli and Matveev [14], which manipulate special spines, provide upper bounds which experimentally are always sharp. On the other hand, the only methods currently known to obtain general lower bounds are those of [19], based on group theory (for some hyperbolic manifolds, there is also a lower estimate in terms of the volume by Anisov, see [1]). From this point of view the results of the present paper can be viewed as potential tools for constructing more lower estimates on the complexity of manifolds. This idea is specified in Lemma 1.7 below, and a concrete application is given in Theorem 4.5, where we provide two-sided bounds for the complexity of certain infinite classes of Seifert manifolds. These estimates are "asymptotically exact up to linear maps," meaning that the upper and the lower bound differ by a fixed linear function.

The notion of T-invariant, also closely related to the complexity of 3-manifolds (see [5, 6, 7] or the proof of Theorem 4.7 below), was introduced by Delzant in [5] for what appear to be completely different reasons, namely, to study hierarchical decompositions of finitely presented groups. For instance, it played a central rôle in the proof by Delzant himself and Potyagailo of the strong accessibility theorem for such groups [7]. For one-relator groups, the T-invariant was studied in [12].

The complexity and the T-invariant of a group never coincide, except for the trivial group, but they are closely related and they can be studied by similar methods. Exploiting this fact, we provide in this paper lower bounds for the complexity and the T-invariant of an arbitrary finitely presented group in terms of the order of the torsion part of its Abelianization (Theorems 2.2 and 2.7). Then, for all the members of Milnor's list [20] of finite groups with free linear action on S^3, as well as for all finite Abelian groups, we present two-sided estimates on their complexity (Theorems 3.6 and 3.10) and on the value of the T-invariant (Theorems 3.9 and 3.10, and Remark 3.12).

For some Milnor groups, the estimates we obtain show that the complexity is asymptotically given by the logarithm of the order of the torsion part of the Abelianization, up to linear maps. Using results from this paper and from previous ones we also provide similar "asymptotically exact" estimates for the complexity of certain Seifert manifolds (Theorem 4.5). We then in turn apply this theorem to derive more precise bounds on the complexity and the T-invariant of some of the other Milnor groups (Theorems 3.14 and 4.7). The interplay between the estimates on complexity and those on the T-invariant is that by combining upper and lower bounds we can deduce information on the average length of the relations in a presentation realizing the complexity (Propositions 3.7, 3.13, and 4.8). It is interesting to note that for some Milnor groups the complexity is asymptotically very close to that of the 3-manifolds whose fundamental groups they are (Theorems 4.5 and 4.7).

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1 Main definitions

In this section we define the invariants for which we will provide estimates in this paper.

Groups The notion of group complexity was introduced in [19].

Definition 1.1. Let $\langle a_1,\ldots,a_n|\ r_1,\ldots,r_m\rangle$ be a presentation of a group. The *length* of this presentation is the number $|r_1|+\ldots+|r_m|$, where $|r_i|$ is the length of the word r_i in the alphabet $a_1^{\pm1},\ldots,a_n^{\pm1}$. The *complexity* $c(G)$ of a group G admitting finite presentations is the minimum of the lengths of all such presentations.

It can be seen (by explicit enumeration of presentations of small length) that for $n \leqslant 7$ the complexity of the cyclic group of order n is equal to n. However, the groups $\mathbb{Z}/_8$ and $\mathbb{Z}/_9$ both have complexity 7, which is smaller than the order, and $c(\mathbb{Z}/_{10}) = 8$. The following presentation of $\mathbb{Z}/_{147}$, which has length 23, shows that the complexity can be significantly smaller than the order:

$$\langle a, b, c, d |\ a^4bc^4, b^3c^{-1}, a^2d^3b^{-1}, a^3d^{-1}\rangle.$$

An alternative measure of how complicated a group is, now called the T-invariant of the group, was suggested by Delzant in [5] and investigated in [6, 7]:

Definition 1.2. The *T-invariant* $T(G)$ of a finitely presented group G is the minimal number t such that G admits a presentation with t relations of length 3 and an arbitrary number of relations of length at most 2. A presentation of this type is called *triangular*.

The following easy fact was already noted in [5]:

Proposition 1.3.

$$T(G) = \min\left\{\sum_{i=1}^{m} \max\{|r_i| - 2, 0\} :\ G = \langle a_1, \ldots, a_n |\ r_1, \ldots, r_m\rangle\right\}.$$

3-manifolds We now review some notions related to 3-manifolds. We will use the PL category throughout.

Definition 1.4. A 2-dimensional subpolyhedron P of a closed connected 3-manifold M is called a *spine* of M if $M \setminus P$ is homeomorphic to an open 3-ball.

In particular, for every spine P of M we have $\pi_1(P) \cong \pi_1(M)$. We will consider only a particular class of spines, that we now define.

Definition 1.5. A compact polyhedron is called *special* if the following two conditions hold. First, the link of each point is homeomorphic to one of the following 1-dimensional polyhedra:

(a) a circle;

(b) a circle with a diameter;

(c) a circle with three radii.

Second, the components of set of points of type (a) are open discs, while the components of set of points of type (b) are open segments. The components just described are called *faces* and *edges*, respectively, and the points of type (c) are called *vertices*. A *special spine* of a closed manifold M is a spine of M which is a special polyhedron at the same time.

The notion of complexity for (arbitrary) 3-manifolds was introduced in [16], see also [17]. We will only need here the following partial characterization, which could also be used as a definition:

Proposition 1.6. *The* complexity $c(M)$ *of a closed irreducible manifold* $M \notin \{S^3, \mathbb{RP}^3, L_{3,1}\}$ *is the minimal number of vertices of a special spine of* M*. The complexity of the three exceptional manifolds is equal to zero.*

It turns out that there is a clear relation between the complexity of a 3-manifold and the complexity of its fundamental group. This relation is described in the following lemma, which was essentially proved in [19].

Lemma 1.7. *If a manifold* M *has a special spine* P *with* n *vertices then* $\pi_1(M)$ *has a presentation of length* $3n+3$.

Proof. We know that $\pi_1(M)$ coincides with $\pi_1(P)$. Moreover, the stratification of P into vertices, edges, and faces gives P the structure of a cell complex. So we can employ the general algorithm yielding a presentation of the fundamental group of a cell complex. The generators are the edges in the complement of a maximal tree in the 1-skeleton, so there are $n+1$ of them. The relations correspond to the faces. Since precisely 3 faces are incident to any given edge (with multiplicity), the total length of the relations is $3(n+1)$. □

According to this result, an upper bound on $c(M)$ implies an upper bound on $c(\pi_1(M))$, and a lower bound on $c(\pi_1(M))$ implies a lower bound on $c(M)$.

2 Lower estimates

In this section we establish lower estimates for the complexity and the T-invariant for an arbitrary group, whereas starting from the next section we will concentrate on Abelian and Milnor groups.

Group complexity We begin with an easy lower bound on the complexity of a group G in terms of the so-called *relation matrices* of the presentations of G. Recall that, given a presentation of G with n generators and m relations, the relation matrix associated to the presentation has size $m \times n$, and its entry in position (i, j) is the (algebraic) sum of all the exponents of the j-th generator in the i-th relation. If $X = (x_{i,j})$ is the matrix thus obtained we define its *norm* as

$$||X|| = \sum_{i=1}^{m} \sum_{j=1}^{n} |x_{i,j}|.$$

We have the following:

Lemma 2.1. *For every finitely presented group G we have*

$$c(G) \geqslant \min ||X||,$$

where the minimum is taken over the relation matrices X associated to all possible finite presentations of G.

Proof. We only need to note that $||X||$ is less than or equal to the length of the presentation to which X is associated. □

To provide more effective lower bounds on $c(G)$ we then need to give an estimate on the possible norms of the relation matrices of G. This is done in the next result, where $\mathrm{Tor}(H)$ denotes the torsion (*i.e.* finite-order) part of an Abelian group H.

Theorem 2.2. *For any finitely presented group G we have*

$$c(G) \geqslant \log_2 |\,\mathrm{Tor}(G/[G, G])|.$$

Proof. By Lemma 2.1 it is enough to show that $||X|| \geqslant \log_2 |\,\mathrm{Tor}(G/[G, G])|$ for all the relation matrices X of the presentations of G. Fix such an X and suppose its size is $m \times n$. It is well-known that X can be transformed into a matrix of the form

$$Y = \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix},$$

with D a $k \times k$ diagonal matrix, and $\det(D) \neq 0$ if $k \geqslant 1$, using a finite sequence of operations as follows:

1. Interchange two rows or columns;

2. Multiply one row or column by -1;

3. Add one row or column to a different one.

One can now see that each such operation transforms the relation matrix of a finite presentation of G into the relation matrix of some other presentation of G, which easily implies that $|\operatorname{Tor}(G/[G,G])| = |\det(D)|$.

If we set $d = |\det(D)|$ we have the obvious property that *the determinants of all the $k \times k$ submatrices of Y are multiples of d, and some of them is non-zero.* Moreover one can easily see that this property is preserved under all the inverse operations which lead from Y back to X. Therefore there is a $k \times k$ submatrix X' of X such that $\det(X')$ is a non-zero multiple of d, so in particular $|\det(X')| \geqslant d$.

We can now observe that $|\det(X')|$ is bounded from above by the product of the Euclidean norms of the rows of X', and each such norm is bounded from above by the L^1-norm, whence by the L^1-norm of the corresponding whole row of X. Noting that each non-zero row of X has norm at least 1, and dismissing the zero rows if necessary (recall that we want to give a lower bound on $\|X\|$), we conclude that $|\det(X')|$ is bounded from above by the product of the L^1-norms of all the non-zero rows of X. Therefore

$$\begin{aligned} & |\operatorname{Tor}(G/[G,G])| = d \leqslant \prod_{i=1}^{m} \sum_{j=1}^{n} |x_{i,j}| \\ \Rightarrow \quad & \log_2 |\operatorname{Tor}(G/[G,G])| \leqslant \sum_{i=1}^{m} \log_2 \left(\sum_{j=1}^{n} |x_{i,j}| \right). \end{aligned}$$

Noting that $\log_2 n \leqslant n$ for all $n \in \mathbb{N}$ we deduce that

$$\log_2 |\operatorname{Tor}(G/[G,G])| \leqslant \sum_{i=1}^{m} \sum_{j=1}^{n} |x_{i,j}| = \|X\|,$$

and the proof is complete. $\square$

Remark 2.3. In the previous statement the base $a = 2$ of logarithms was chosen because it has the property that $\log_a(n) \leqslant n$ for all $n \in \mathbb{N}$, and the theorem remains true with any other base a having this property. One easily sees that the best lower estimate for $c(G)$ is obtained for $a = \sqrt[3]{3}$. Since we are only interested in the qualitative fact that a logarithmic lower bound exists, we will keep employing the base 2. However we will use the fact that the inequality in the previous statement is strict.

Remark 2.4. Along the proof of Theorem 2.2 we have shown that for any presentation $\langle a_1, \ldots, a_n | \ r_1, \ldots, r_m \rangle$ of a group G the following inequality is valid:

$$\log_2 |\operatorname{Tor}(G/[G,G])| \leqslant \log_2 |r_1| + \ldots + \log_2 |r_m|.$$

The T-invariant Proposition 1.3 allows to conclude immediately that $T(G) < c(G)$ for a non-trivial G. However, we will show that in many cases the invariants c and T are asymptotically equivalent. We begin with two rather easy results.

Lemma 2.5. *Let G be a finitely presented group. Then for every finite presentation of G there is a presentation of the same or smaller length which contains relations of length $\geqslant 2$ only, and all relations of length 2 are of the form x^2. Moreover, G admits a triangular presentation with exactly $T(G)$ relations of length 3 and some relations of the form x^2.*

Proof. Of course the relations of length 0 can always be omitted. Suppose the given presentation is $\langle a_1, \ldots, a_n | \ r_1, \ldots, r_t, r'_1, \ldots, r'_k \rangle$, where the lengths of all r_i are $\geqslant 3$, and the lengths of all r'_j are 1 or 2. We describe a recursive procedure to get a presentation as desired. If some r'_j has length 1, *i.e.* it is of the form x, then this x can be removed from the generators and from all the relations where it occurs. This produces a presentation of G with the same or a smaller number of relations of length $\geqslant 3$, and the total length of the presentation, as well as each relation's own length, is not increased. Hence there is a presentation of G of the same or smaller length with $\leqslant t$ relations of length $\geqslant 3$ and some number of relations of length 2.

Consider a relation of length 2. If it has the form $x^{-1}y$, then we can discard it and the generator y, replacing all occurrences of y in all the relations by x. Such a procedure increases neither the lengths of the other relations nor the total length of the presentation. So in the end we get a presentation of G of the described type and of the same or smaller length.

If the initial presentation is triangular, so is the final one, and the last assertion follows. $\square$

Proposition 2.6. *Let G be a nontrivial finitely presented group without 2-torsion. Then*

$$\frac{1}{3}c(G) \leqslant T(G) < c(G).$$

Proof. The second inequality is valid in general and was already remarked above. For the first inequality, consider a triangular presentation of G of the type described in Lemma 2.5. Suppose there is a relation of the form x^2. Then, since G does not have elements of order 2, x has to be trivial in G. Hence we can remove all occurrences of x from all the relations and remove x itself from the list of generators. As a result we get a triangular presentation of G which contains exactly $T(G)$ relations of length 3 (a priori no more than that number, but by minimality there cannot be less) and no relations of length less than 3. $\square$

Combining the previous result with Theorem 2.2 we deduce that if G has no 2-torsion then

$$T(G) \geqslant \frac{1}{3}\log_2|\operatorname{Tor}(G/[G,G])|.$$

This estimate can actually be improved to a stronger and general one:

Theorem 2.7. *Let G be a finitely presented group, and let $|\operatorname{Tor}(G/[G,G])| = 2^l(2m+1)$. Then*

$$T(G) \geqslant \log_3(2m+1).$$

Proof. Consider a triangular presentation of the type described in Lemma 2.5, with $t = T(G)$ relations of length 3 and some h relations of the form x^2. The relation matrix X of this presentation has the form

$$X = \begin{pmatrix} Y & Z \\ 2I_h & 0 \end{pmatrix},$$

where the matrix $(Y\ Z)$ has t rows, the L^1-norm of each row is 3, and I_h is the $h \times h$ unit matrix. As in the proof of Theorem 2.2, X has a square submatrix X' whose determinant is non-zero and divisible by $|\operatorname{Tor}(G/[G,G])| = 2^l(2m+1)$. On the other hand, X' consists of $t' \leqslant t$ rows of L^1-norm 3 and some $h' \leqslant h$ rows with a single non-zero entry equal to 2. It follows that $|\det(X')| = 2^{h'} \cdot \delta$ and $\delta \leqslant 3^{t'} \leqslant 3^t$. Recalling that $|\det(X')|$ is divisible by $2^l(2m+1)$, we deduce that δ is divisible by $2m+1$, whence $2m+1 \leqslant \delta \leqslant 3^t$, and the desired estimate follows. $\square$

3 Abelian and Milnor groups

In this section we will give two-sided estimates for the complexity and the T-invariant of all the Milnor groups, and in the process we will obtain similar estimates for Abelian groups.

Milnor groups As already mentioned, Milnor classified in [20] the finite groups having a free linear action on S^3. The complete list of all such groups is as follows:

1. $Q_{4n} = \langle x, y \mid x^{-1}yxy,\ x^{-2}y^n \rangle$, $n \geqslant 2$;
2. $D_{2^k(2n+1)} = \langle x, y \mid x^{2^k},\ y^{2n+1},\ xyx^{-1}y \rangle$, $k \geqslant 3$, $n \geqslant 1$;
3.

$$\begin{aligned}
&P_{24} = \langle x, y \mid x^{-1}yxyxy,\ x^{-2}y^3,\ x^4 \rangle,\\
&P_{48} = \langle x, y \mid x^{-1}yxyxy,\ x^{-2}y^4,\ x^4 \rangle,\\
&P_{120} = \langle x, y \mid x^{-1}yxyxy,\ x^{-2}y^5,\ x^4 \rangle,\\
&P'_{8\cdot 3^k} = \langle x, y, z \mid x^{-1}yxy,\ x^{-2}y^2,\ zxz^{-1}y^{-1},\ zyz^{-1}y^{-1}x^{-1},\ z^{3^k} \rangle, \quad k \geqslant 2;
\end{aligned}$$

4. The direct product of any of the groups listed so far, or of the trivial group, with a cyclic group of coprime order.

We will start by considering the simplest case, namely, cyclic groups. It is interesting to note that many of the ideas which will be used for the other groups in the list are already present at this level.

Cyclic groups In this paragraph we give two-sided estimates for the complexity and the T-invariant of finite cyclic groups and we describe some properties of their minimal presentations. Although arbitrary finite Abelian groups are not in Milnor's list, the complexity estimates can be generalized to include them with no extra effort, so we cover them too. We begin with an upper estimate:

Proposition 3.1. *For every $p \geqslant 2$ we have $c(\mathbb{Z}/_p) < 4\log_2 p$.*

To prove this result, we first state the following evident:

Lemma 3.2. *Let p, q, r, s be non-negative integers such that $p = sq + r$. Consider a group presentation of length l involving a relation of the form $ua^p v$, where a is a generator and u, v are words in the generators. Then a new presentation of the same group is obtained by adding a new generator b and a new relation $b^{-1}a^q$, and replacing the relation $ua^p v$ by $ub^s a^r v$. The length of the new presentation is $1 + l + q + r + s - p$.*

Let us denote now by $\ell(p)$ the shortest length of a presentation of the group $\mathbb{Z}/_p$ obtained from the trivial presentation $\langle a|\ a^p\rangle$ by repeated application of the lemma just stated. Of course $c(\mathbb{Z}/_p) \leqslant \ell(p)$, so the next result implies Proposition 3.1:

Proposition 3.3. *For all $p \geqslant 2$ we have $\ell(p) < 4\log_2 p$.*

Proof. We proceed by induction on p, noting that the inequality is true for $p = 2$ and $p = 3$, because $\ell(2) = 2$ and $l(3) = 3$. For the inductive step, for $p \geqslant 4$ we apply Lemma 3.2 with $q = 2$, and we distinguish according to the parity of p. If p is even then

$$\begin{aligned} &\mathbb{Z}/_p \ = \langle a, b|\ b^{-1}a^2, b^{p/2}\rangle \\ &\Rightarrow\ \ \ell(p) \leqslant 3 + \ell(p/2) < 3 + 4\log_2(p/2) = 4\log_2 p - 1 < 4\log_2(p). \end{aligned}$$

If p is odd then

$$\begin{aligned}
\mathbb{Z}/_p &= \langle a, b |\ b^{-1}a^2, b^{(p-1)/2}a\rangle \\
\Rightarrow\quad & \ell(p) \leqslant 4 + \ell((p-1)/2) \\
& \qquad < 4 + 4\log_2((p-1)/2) = 4\log_2(p-1) < 4\log_2(p).
\end{aligned}$$

This proves the desired inequality. □

Example 3.4. Applying the procedure described in the previous proof for $p = 357$ we get the following length-27 presentation of $\mathbb{Z}/_{357}$:

$$\langle a, b, c, d, e, f, g, h |\ b^{-1}a^2, c^{-1}b^2, d^{-1}c^2, e^{-1}d^2, f^{-1}e^2, g^{-1}f^2, h^{-1}g^2, h^2gfca\rangle.$$

Remark 3.5. The estimate in Proposition 3.3, whence that in Proposition 3.1, can actually be improved using $q = 3$ rather than $q = 2$ for the repeated application of Lemma 3.2. Namely, one can show that $c(\mathbb{Z}/_p) < 6\log_3 p$, which is a slightly better bound since $\frac{6}{\log_2 3} \approx 3.786 < 4$. All other choices of q, on the other hand, produce bounds with larger constants.

In the case of an arbitrary finite Abelian group G, Proposition 3.1 implies the following result. Recall that the rank of G is the minimal number of cyclic groups which G can be expressed as the direct product of.

Theorem 3.6. *Let G be a finite Abelian group of rank k. Then $c(G) < 4\log_2|G| + 2k(k-1)$.*

Proof. By assumption, G is isomorphic to $\mathbb{Z}/_{p_1} \oplus \ldots \oplus \mathbb{Z}/_{p_k}$, and $|G| = p_1 \cdot \ldots \cdot p_k$. We can obtain a presentation of G by taking the union of the presentations of $\mathbb{Z}/_{p_i}$ constructed in Proposition 3.3 and adding relations providing the commutativity. Notice that the generating set constructed in Proposition 3.3 always contains an element generating the whole group. Hence it suffices to add $k(k-1)/2$ relations of length 4 each (the commutators of all pairs of different generators). This produces a presentation of length that is strictly less than

$$\begin{aligned}
& 4\log_2 p_1 + \ldots + 4\log_2 p_k + 4\frac{k(k-1)}{2} \\
= \;& 4\log_2(p_1 \cdot \ldots \cdot p_k) + 2k(k-1) = 4\log_2|G| + 2k(k-1)
\end{aligned}$$

and the theorem is proved. □

Average length In view of Proposition 1.3, in order to estimate the T-invariant for a cyclic group, one would need a lower bound on the number of relations in a length-minimizing presentation (*i.e.* a presentation realizing the complexity). Such a bound is established in the following:

Proposition 3.7. *Suppose that* $\langle a_1, \dots, a_n | \, r_1, \dots, r_m \rangle$ *is a length-minimizing presentation of a finite cyclic group. Then*

$$\frac{|r_1| + \dots + |r_m|}{m} < 16.$$

Proof. If p is the order of G, according to Proposition 3.1 we have $\log_2 p > (|r_1| + \dots + |r_m|)/4$. Moreover $p = |\operatorname{Tor}(G/[G,G])|$, so by Remark 2.4 we have $\log_2 p \leqslant \log_2 |r_1| + \dots + \log_2 |r_m|$. Combining these two estimates we get the inequality

$$|r_1| + \dots + |r_m| < 4 \cdot \big(\log_2 |r_1| + \dots + \log_2 |r_m|\big),$$

which we divide by m to obtain

$$\frac{|r_1| + \dots + |r_m|}{m} < 4 \frac{\log_2 |r_1| + \dots + \log_2 |r_m|}{m}.$$

The right-hand side of the latter inequality is just $4 \log_2 \sqrt[m]{|r_1| \cdot \ldots \cdot |r_m|}$. Applying the Cauchy inequality between the geometric mean and the arithmetic one to the expression under the sign of logarithm, we get

$$\frac{|r_1| + \dots + |r_m|}{m} < 4 \log_2 \left(\frac{|r_1| + \dots + |r_m|}{m} \right).$$

It follows that the number $(|r_1| + \dots + |r_m|)/m$ must satisfy the inequality $x < 4 \log_2 x$. Now, the numbers satisfying this inequality lie between the two solutions to the equation $x = 4 \log_2 x$ and therefore are bounded from above by the greatest of them, which evidently is $x = 16$, because the other solution lies between 1 and 2. The desired estimate on $(|r_1| + \dots + |r_m|)/m$ follows. $\square$

Remark 3.8. Using the better estimate given by Remark 3.5 one could show that $(|r_1| + \dots + |r_m|)/m < 15$.

The values of the T-invariant for cyclic groups can now be estimated as follows.

Theorem 3.9. *For every odd p and for every integer $k \geqslant 0$ we have that*

$$\frac{1}{\log_2 3} \log_2 p \;\leqslant\; T\left(\mathbb{Z}/_{2^k p}\right) \;<\; \frac{7}{2}(\log_2 p + k).$$

Proof. The lower estimate follows from Theorem 2.7. For the upper estimate, take a length-minimizing presentation of $\mathbb{Z}/_{2^k p}$ with relations $r_1, \ldots, r_m$. Since $|r_i| \geqslant 2$, Proposition 1.3 implies that

$$T(\mathbb{Z}/_{2^k p}) \leqslant |r_1| + \ldots + |r_m| - 2m.$$

Proposition 3.7 now yields

$$|r_1| + \ldots + |r_m| - 2m \leqslant \frac{7}{8}(|r_1| + \ldots + |r_m|),$$

and $|r_1| + \ldots + |r_m| = c\left(\mathbb{Z}/_{2^k p}\right)$ by the choice of the presentation, but $c\left(\mathbb{Z}/_{2^k p}\right) < 4(\log_2 p + k)$ by Proposition 3.1, whence the conclusion. $\square$

Other Milnor groups A straight-forward application of the technique used to prove Proposition 3.3 and of Theorem 2.2 allows us to obtain some estimates on the complexity of all the other groups in Milnor's list:

Theorem 3.10. *The following estimates hold for the complexity of the Milnor groups, where in all cases the lower bound equals the base-2 logarithm of the order of the torsion of the Abelianization, and the same term appears in the upper estimate too:*

1. *For every $n \geqslant 2$ and every odd q coprime with n we have*

$$\log_2 q + 2 < c(Q_{4n} \times \mathbb{Z}/_q) < 4(\log_2 q + 2) + 4\log_2 n + 6;$$

2. *For every $k \geqslant 3$ and every coprime odds $n \geqslant 3, q \geqslant 1$, we have*

$$\log_2 q + k < c(D_{2^k n} \times \mathbb{Z}/_q) < 4(\log_2 q + k) + 4\log_2 n + 12;$$

3. For every q coprime with 2 and 3 (and 5, for the last estimate) we have

$$\begin{aligned}
&\log_2(3q) < c(P_{24} \times \mathbb{Z}/_q) < 4(\log_2(3q)) + 17;\\
&\log_2 q + 1 < c(P_{48} \times \mathbb{Z}/_q) < 4(\log_2 q + 1) + 20;\\
&\log_2 q < c(P_{120} \times \mathbb{Z}/_q) < 4\log_2 q + 25;
\end{aligned}$$

4. For every $k \geqslant 2$ and every q coprime with 2 and 3 we have

$$\log_2 q + (\log_2 3)k < c(P'_{8\cdot 3^k} \times \mathbb{Z}/_q) < 4(\log_2 q + (\log_2 3)k) + 29.$$

Proof. The lower bounds are obtained by direct application of Theorem 2.2 and Remark 2.3. To get the upper bounds, we start by writing the most straight-forward presentation for each of the groups listed in the theorem. Namely, we add to each of the standard presentations reproduced at the beginning of the present section, one generator a (corresponding to $\mathbb{Z}/_q$), a relation a^q, and the commutation relations $[x, a]$ and $[y, a]$ (and $[z, a]$ for the group $P'_{8\cdot 3^k}$).

Now we apply exactly the trick described in Lemma 3.2. Evidently, in case of Q_{4n} this produces a presentation of length $\ell(n) + \ell(q) + 14$. From the estimate given by Proposition 3.3, we deduce that this number is less than $4\log_2(nq) + 14$, and the conclusion easily follows. In case of $D_{2^k n}$ the trick produces a presentation of length $\ell(2^k) + \ell(n) + \ell(q) + 12$, which implies the bound stated in the theorem. For each of P_{24}, P_{48}, P_{120} we can get a presentation of that group multiplied by $\mathbb{Z}/_q$ having length $\ell(q)$ plus the length of the presentation of that group given above plus 8, and the upper bounds follow after easy calculations. Finally, for $P'_{8\cdot 3^k}$ we get a presentation of length $\ell(3^k) + \ell(q) + 29$, which gives the desired bound again. □

Remark 3.11. Slightly better numerical estimates could be shown using Remarks 2.3 and 3.5.

Remark 3.12. The above result together with Theorem 2.7 and the inequality $T(G) < c(G)$ allows to obtain upper and lower bounds on the T-invariant of all Milnor groups. Since the upper bounds thus obtained coincide with those for complexity, and the lower bounds are just an immediate consequence of Theorem 2.7, we do not spell them out here (see also below).

The complexity estimates given in Theorem 3.10(3,4) are "asymptotically exact up to linear functions," and we can now exploit this fact to give a similar estimate also for the T-invariant. We begin with the following:

Proposition 3.13. *Let P be one of the groups P_{24}, P_{48}, P_{120}, $P'_{8\cdot 3^k}$, and let q be a positive integer coprime with $|P|$. Suppose that $\langle a_1, \ldots, a_n | \, r_1, \ldots, r_m \rangle$ is a length-minimizing presentation of $P \times \mathbb{Z}/_q$. Then*

$$\frac{|r_1| + \ldots + |r_m|}{m} < 52.$$

Proof. Let G denote $P \times \mathbb{Z}/_q$. Combining the lower estimate given by Remark 2.4 with the upper estimate given by Theorem 3.10(3,4), we deduce that

$$|r_1| + \ldots + |r_m| < 4 \log_2 \big(|r_1| \cdot \ldots \cdot |r_m| \big) + 29.$$

Dividing by m, noting that $29/m \leqslant 29$, and using the Cauchy inequality as in the proof of Proposition 3.7, we deduce that

$$\frac{|r_1| + \ldots + |r_m|}{m} < 4 \log_2 \left(\frac{|r_1| + \ldots + |r_m|}{m} \right) + 29,$$

which means that the number $(|r_1| + \ldots + |r_m|)/m$ must satisfy the inequality $x < 4 \log_2 x + 29$. Any such x lies between the two solutions of the equation $x = 4 \log_2 x + 29$. Notice that the smaller of the two solutions lies between $\frac{1}{2^8}$ and $\frac{1}{2^7}$, and that 52 does not satisfy the inequality $x < 4 \log_2 x + 29$. It follows that all solutions to this inequality are less than 52. $\square$

Once again, we could slightly improve the numerical estimate given by this proposition, but we are actually only interested in the fact that a fixed upper bound exists. As announced, we use the proposition to give asymptotically exact estimates for T-invariant.

Theorem 3.14. *Let P be one of the groups P_{24}, P_{48}, P_{120}. Let q be a positive integer coprime with 2 and 3 (and 5, for $P = P_{120}$). Then*

$$\frac{1}{\log_2 3} \log_2 q \leqslant T(P \times \mathbb{Z}/_q) < \frac{50}{13} \log_2 q + 24,$$

$$\frac{1}{\log_2 3} \log_2 q + k \leqslant T(P'_{8\cdot 3^k} \times \mathbb{Z}/_q) < \frac{50}{13} (\log_2 q + \log_2 3 \cdot k) + 29.$$

Proof. Let G be one of the groups mentioned in the statement. Since q is odd, the lower bounds follow from Theorem 2.7. To get the upper bounds, we again apply Proposition 1.3 to a length-minimizing presentation of G, getting

$$T(G) \leqslant c(G) - 2m,$$

where m is the number of relations of a length-minimizing presentation of G. Now Proposition 3.13 implies that $m > \frac{c(G)}{52}$, so we finally get

$$T(G) < \frac{25}{26} c(G).$$

Combining this inequality with the upper bounds from Theorem 3.10, we get our statement. □

4 More asymptotically exact estimates

For the Milnor groups of type $P_*^{(\prime)} \times \mathbb{Z}/_q$, Theorem 3.10(3,4) provides estimates which are "asymptotically exact up to linear functions" as q (and k) tend to infinity, because the upper and lower estimates only differ by a fixed linear function. This is not the case for the other Milnor groups, *i.e.* those of Theorem 3.10(1,2), but it turns out that there are infinite families of such groups for which similar asymptotic estimates actually do hold. This section is devoted to these estimates and to some related ones, having the same property of "asymptotic exactness," for the T-invariant of the same groups and for the complexity of certain Seifert 3-manifolds. As a matter of fact, the estimates for groups depend on those for 3-manifolds, which employ results established elsewhere by more geometric methods.

Zaremba pairs To describe the families of Milnor groups we will deal with we must make a digression into number theory. Specifically, we need the following definition and some facts related to it.

Definition 4.1. A pair of coprime positive integer numbers (p, q) with $p > q$, is called a *Zaremba pair* if all the partial quotients a_i in the expansion of p/q into the continuous fraction

$$\frac{p}{q} = a_1 + \frac{1}{a_2 + \frac{1}{\dots + \frac{1}{a_n}}}$$

satisfy the inequality $a_i \leqslant 5$.

For example, all pairs of consecutive Fibonacci numbers are Zaremba pairs.

For all coprime $p > q > 1$ we denote now by $S(p, q)$ the sum of all the partial quotients in the expression of p/q as a continued fraction.

Proposition 4.2. *If (p, q) is a Zaremba pair then $S(p, q) \leqslant 3 \log_2 p$.*

Proof. For any coprime $p' > q' > 1$, if $a_1, \ldots, a_n$ are the partial quotients in the expression of p'/q' as a continued fraction, one can easily show by induction that

$$\begin{pmatrix} p' \\ q' \end{pmatrix} = \begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_2 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_{n-1} & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_n \\ 1 \end{pmatrix}.$$

For a Zaremba pair (p, q) only five matrices can appear in this formula

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 3 & 1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 4 & 1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 5 & 1 \\ 1 & 0 \end{pmatrix},$$

and there are only four possible "starting points" $\begin{pmatrix} a_n \\ 1 \end{pmatrix}$, namely

$$\begin{pmatrix} 2 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 3 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 4 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 5 \\ 1 \end{pmatrix}.$$

The proof now proceeds by induction on the length n of the expansion. For $n = 1$ the conclusion follows from the fact that that $m \leqslant 3 \log_2 m$ for $2 \leqslant m \leqslant 5$.

For the inductive step we note that if $2 \leqslant m \leqslant 5$ and we are given (p', q') such that $S(p', q') \leqslant 3 \log_2 p'$, setting

$$\begin{pmatrix} p'' \\ q'' \end{pmatrix} = \begin{pmatrix} m & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} p' \\ q' \end{pmatrix},$$

we have

$$\begin{aligned} S(p'', q'') &= m + S(p', q') \leqslant 3 \log_2 m + 3 \log_2 p' \\ &= 3 \log_2 (mp') \leqslant 3 \log_2 (mp' + q') = 3 \log_2 p''. \end{aligned}$$

This does not quite suffice to conclude when the expansion involves some matrix with 1 in position (1,1). However one notes that the inequality $m \leqslant 3\log_2 m$ actually holds also for $m = 6$. Therefore, if one of m', m'' is 1 and the other one is between 1 and 5, given (p', q') such that $S(p', q') \leqslant 3\log_2 p'$, if we set

$$\begin{pmatrix} p'' \\ q'' \end{pmatrix} = \begin{pmatrix} m' & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} m'' & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} p' \\ q' \end{pmatrix}$$

we have

$$\begin{aligned} S(p'', q'') &= m' + m'' + S(p', q') \leqslant 3\log_2(m' + m'') + 3\log_2 p' \\ &= 3\log_2((m' + m'')p') \leqslant 3\log_2((m' + m'')p' + m'q') = 3\log_2 p''. \end{aligned}$$

This argument suffices to prove the inequality for all Zaremba pairs except those of type $(a+1, a)$, for which the conclusion is obvious. $\square$

The conditions of Definition 4.1 may appear to be rather restrictive. However, the following fact is conjectured by numerical analysts: *there exists a constant B with the property that for every p there exists $1 < q < p$ coprime with p such that all partial quotients in the expansion of p/q as a continued fraction are not greater than B.* (This statement is known as *Zaremba's conjecture*, see [23]. Its motivation is to find optimal lattice points for numerical integration, see also [13]). Cusick conjectured in [4] that $B = 5$. So far Zaremba's conjecture has been proved only in a few particular cases. Niederreiter proved it for powers of 2 and 3 [21], and Yodphotong and Laohakosol proved it for powers of 6 [22]. On the other hand, it is known that there are actually "many" Zaremba pairs [9, 10]. We will also use the following weaker definition.

Definition 4.3. A pair of coprime positive integers (p, q) with $p > q$ is called a *weak Zaremba pair* if the partial quotients $a_1, \ldots, a_n$ in the expansion of p/q into continuous fraction satisfy the inequality $a_1 + \ldots + a_n \leqslant 5n$.

Weak Zaremba pairs were investigated in [3, 11, 2], where it was shown that they are also not infrequent.

Proposition 4.4. *If (p, q) is a weak Zaremba pair then $S(p, q) \leqslant 10\log_2 p$.*

Proof. An easy induction argument shows that $n \geqslant 2\log_2 p$, whence the conclusion at once. $\square$

Asymptotically exact estimates for manifolds In this paragraph we consider the complexity of certain Seifert manifolds and of their fundamental groups, which in some cases allows to obtain better bounds than those provided by Theorem 3.10. We employ for Seifert manifolds the same notation as in [15]. Namely, if F is a closed surface, t is an integer, and $(p_1, q_1), \ldots, (p_k, q_k)$ are coprime pairs of integers with $|p_i| \geqslant 2$, then

$$\big(F; (p_1, q_1), \ldots, (p_k, q_k), t\big)$$

denotes the (oriented) Seifert manifold obtained from $F \times S^1$ or from $F \tilde{\times} S^1$, if F is nonorientable, by removing $k + 1$ solid fibred tori and performing Dehn filling on the resulting boundary components with slopes $p_1 a_1 + q_1 b_1, \ldots, p_k a_k + q_k b_k, a_{k+1} + t b_{k+1}$. Here the a_i's are contained in a section of the bundle, the b_i's are fibres, and each a_i, b_i is a positive basis in homology.

Theorem 4.5. *Let (p, q) be a Zaremba pair. Then*

$$\frac{2}{\log_2 5} \log_2 p - 1 \leqslant c(L_{p,q}) \leqslant 3 \log_2 p - 3,$$

$$\frac{2}{\log_2 5} \log_2 q \leqslant c(S^2; (2,1), (2,1), (p,q), -1) < 3 \log_2 q + 9.$$

Proof. The proof of the upper bound in the first formula is by direct application of [15, Theorem 1.11] and Proposition 4.2 above, because, with the notation of [15], we have $S(p, q) = |p, q| + 1$. For the manifold M of the second formula, [15, Theorem 1.11] and [15, Theorem 2.5] yield the bound

$$c(M) \leqslant S(p, q) + 1.$$

Since (p, q) is a Zaremba pair, Proposition 4.2 implies that $S(p, q) \leqslant 3 \log_2 p$. However, as for any Zaremba pair, $p/q < a_1 + 1 \leqslant 6$. Hence

$$c(M) < 3 \log_2 q + 3 \log_2 6 + 1 = 3 \log_2 q + \log_2 27 + 4.$$

Since $c(M)$ is an integer, we conclude that it actually does not exceed $3 \log_2 q + ([\log_2 27] + 1) + 4 = 3 \log_2 q + 9$, as desired.

The lower bounds in both formulae follow directly from [19, Theorem 1], because the order of the first homology group is p for $L_{p,q}$, and it is $4q$ for the manifold in the second formula. This proves the theorem. $\square$

Remark 4.6. If (p,q) is a weak Zaremba pair and M is the lens space with parameters (p,q) then the complexity of M still depends on the order of the first homology group of M logarithmically, as in point 1 of the previous theorem, except that the constants are worse. Namely, we have

$$\frac{2}{\log_2 5}\log_2 p - 1 \leqslant c(M) \leqslant 10\log_2 p - 3.$$

Exact asymptotic estimates for groups Theorem 4.5 serves as a tool to obtain good estimates on the complexity and the T-invariant for some of the Milnor groups in Theorem 3.10(1,2).

Theorem 4.7. *1. Let (n,q) be a Zaremba pair with odd q. Then*

$$\begin{array}{rcccl} \log_2 q + 2 & < & c(Q_{4n}\times \mathbb{Z}/_q) & < & 8(\log_2 q + 2) + 9, \\ \frac{1}{\log_2 3}\log_2 q & \leqslant & T(Q_{4n}\times \mathbb{Z}/_q) & < & 6\log_2 q + 18. \end{array}$$

2. Let n,h,s be integers, with $h,n \geqslant 3$ and n,s coprime, and s odd. Let $q = 2^{h-2}s$ and suppose that (n,q) is a Zaremba pair. Then:

$$\begin{array}{rcccl} \log_2 s + h & < & c(D_{2^h n}\times \mathbb{Z}/_s) & < & 8(\log_2 s + h) + 15, \\ \frac{1}{\log_2 3}\log_2 s & \leqslant & T(D_{2^h n}\times \mathbb{Z}/_s) & < & 6(\log_2 s + h) + 6. \end{array}$$

In both the estimates on the complexity of the group, the lower bound equals the base-2 logarithm of the order of the torsion of the Abelianization, and the same term appears in the upper bound too.

Proof. To begin, we notice that if (n,q) is a Zaremba pair then $n < 6q$. Combining this inequality with the estimates obtained in Theorem 3.10, we get the upper bounds on complexity. The lower bounds on the complexity and the T-invariant come from Theorems 2.2 and 2.7 respectively.

To get the upper estimates on the T-invariant, we first prove the following assertion, first remarked by Delzant himself: *If M is a closed 3-manifold then $T(\pi_1(M))$ does not exceed twice the number of 3-simplices in any triangulation of M.* To see this, we note that a triangulation gives in particular a cell decomposition of M, so we can employ the general algorithm yielding a presentation of the fundamental group. The relations then correspond to faces, and each face is a triangle, therefore the length of any relation is no more than 3. Since M is a closed 3-manifold, the number of faces is twice the number of 3-simplices, and the conclusion follows.

The fact just stated also holds for "singular triangulations," *i.e.* triangulations with multiple and self-adjacencies. Therefore it follows from [18, Theorem 2.2.4] that if M is irreducible and $c(M) > 0$ then $T(\pi_1(M)) \leqslant 2c(M)$. Now we note that the groups in the statement occur as fundamental groups of Seifert manifolds of type $(S^2; (2,1), (2,1), (n,q), -1)$ (see, for instance, [18, Chapter 2]). In particular, for odd q this manifold has fundamental group $Q_{4n} \times \mathbb{Z}/_q$, and for even q it has fundamental group $D_{2^h n} \times \mathbb{Z}/_s$ with h, s as in the statement. The conclusion then easily follows from Theorem 4.5. $\square$

Using Theorem 4.7, an argument similar to that used in the proofs of Propositions 3.7 and 3.13 can now be employed to establish the following:

Proposition 4.8. *Let G be one of the groups of Theorem 4.7, and let $\langle a_1, \ldots, a_n | \, r_1, \ldots, r_m \rangle$ be a length-minimizing presentation of G. Then*

$$\frac{|r_1| + \ldots + |r_m|}{m} < 64.$$

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