Ist. Mat. I - C / A
23/11/23

(59) $\lim_{x \to +\infty} x^{\frac{1}{x}}$ $\quad \infty^0$

$$x^{\frac{1}{x}} = e^{\log\left(x^{\frac{1}{x}}\right)} = e^{\overbrace{\underbrace{\underbrace{\frac{1}{x}}_{0} \cdot \underbrace{\log(x)}_{+\infty}}_{0}}^{}}$$

$$\underbrace{\qquad\qquad}_{1}$$

(60) $\lim_{x \to 2} \dfrac{x^2-4}{\sqrt{\sin((x-2)^2)}}$ $\quad \frac{0}{0}$

$x^2 - 4 = (x-2)(x+2)$

dell H: $\rightsquigarrow \dfrac{2x}{\dfrac{1}{2\sqrt{\sin((x-2)^2)}} \cos((x-2)^2) \cdot 2(x-2)}$

$$= \frac{2x}{\cos((x-2)^2)} \cdot \frac{\sqrt{\sin((x-2)^2)}}{x-2}$$ non conclude

$$\frac{x^2-4}{\sqrt{\sin((x-2)^2)}} = \frac{(x-2)(x+2)}{\sqrt{(x-2)^2 + o((x-2)^2)}} \sim \frac{(x-2)(x+2)}{|x-2|} \xrightarrow[2^\pm]{} \pm 4$$

(67) $f(x) = \begin{cases} \dfrac{\sin(x)}{x} & x \neq 0 \\ 1 & x = 0 \end{cases}$ (continua in $\mathbb{R}$)

Calcolare se esistono $f'(0)$, $f''(0)$

$$f'(0) = \lim_{h \to 0} \frac{\frac{\sin(h)}{h} - 1}{h} = \lim_{h \to 0} \frac{\sin(h) - h}{h^2}$$

dH $\longrightarrow$ $\dfrac{\cos(h) - 1}{2h} \rightsquigarrow \dfrac{-\sin(h)}{2} \to 0$

Taylor $\rightsquigarrow$ $\dfrac{\left(h - \frac{1}{6}h^3 + o(h^3)\right) - h}{h^2} = -\dfrac{1}{6}h + o(h) \to 0$

$x \neq 0$ $\quad f'(x) = \left(\dfrac{\sin(x)}{x}\right)' = \dfrac{\cos(x)}{x} - \dfrac{\sin(x)}{x^2}$

$$f''(0) = \lim_{h \to 0} \frac{\frac{\cos(h)}{h} - \frac{\sin(h)}{h^2}}{h} = \lim_{h \to 0} \frac{h \cdot \cos(h) - \sin(h)}{h^3}$$

Taylor: $\dfrac{h \cdot \left(1 - \frac{1}{2}h^2 + o(h^3)\right) - \left(h - \frac{1}{6}h^3 + o(h^3)\right)}{h^3}$

$$= \frac{\cancel{h} - \frac{1}{2}h^3 - \cancel{h} + \frac{1}{6}h^3 + o(h^3)}{h^3} = -\frac{1}{3} + o(1)$$

$$\longrightarrow -\frac{1}{3}$$

In realtà:

$$\frac{\sin(x)}{x} = \frac{1}{x}\cdot\left(\sum_{k=0}^{n} \frac{(-1)^k}{(2k+1)!} x^{2k+1} + o\left(x^{2n+1}\right)\right)$$

$$= \sum_{k=0}^{n} \frac{(-1)^k}{(2k+1)!} x^{2k} + o\left(x^{2n}\right)$$

$$= \underbrace{1}_{f(0)} \underbrace{+0x}_{\frac{f'(0)}{1}} \underbrace{-\frac{1}{6}x^2}_{\frac{f''(0)}{2!}} \underbrace{+0\cdot x^3}_{\frac{f'''(0)}{3!}} + \underbrace{\frac{1}{120}x^4}_{\frac{f^{(iv)}(0)}{4!}} + o(x^4) \quad \dots$$

$$\Downarrow$$

$$f(0)=1 \quad f'(0)=0 \quad f''(0)=-\frac{1}{3} \quad f'''(0)=0 \quad f^{(iv)}(0)=\frac{4!}{120}=\frac{1}{5}$$

Stabilire convergenza (assoluta) di $\sum a_n$.

① $\displaystyle\sum_{n=1}^{\infty} \frac{n+3}{2n^3+2n+2}$

- $\sum a_n$, $0<a_n<b_n$, $\sum b_n<+\infty$
  $\Rightarrow \sum a_n<+\infty$

- $\sum a_n$, $a_n>0$ $\quad \lim \frac{a_n}{b_n} = L \neq 0$

  $\Rightarrow \sum a_n, \sum b_n$ stesso andamento

$$\frac{n+3}{2n^3+2n+2} \sim \frac{n}{2n^3} = \frac{1}{2n^2}$$

$$\frac{\frac{n+3}{2n^3+2n+2}}{\frac{1}{n^2}} \longrightarrow \frac{1}{2}$$

- $\sum \frac{1}{n^\alpha}$ convergente $\alpha > 1$; divergente $\alpha \leq 1$

$\Longrightarrow$ poiché $\sum \frac{1}{n^2} < +\infty$ anche quella data

② $$\sum \frac{\sqrt[3]{n}}{\underbrace{\sqrt{n^2+n+1}}}$$

$$\frac{n^{1/3}}{n} = \frac{1}{n^{2/3}} \Longrightarrow \text{diverge}$$

Criteri per $\sum a_n$, $a_n > 0$

$\frac{a_{n+1}}{a_n} \longrightarrow L$ oppure $\sqrt[n]{a_n} \longrightarrow L$

$L > 1 \Longrightarrow$ diverge

$L < 1 \Longrightarrow$ converge

③ $\sum \frac{n!}{n^n}$

$$\frac{a_{n+1}}{a_n} = \frac{\frac{(n+1)!}{(n+1)^{n+1}}}{\frac{n!}{n^n}} = \frac{\frac{\cancel{n!}\cdot\cancel{(n+1)}}{(n+1)^{\cancel{n+1}}}}{\frac{\cancel{n!}}{n^n}}$$

$$= \left(\frac{n}{n+1}\right)^n = \left(\frac{n+1}{n}\right)^{-n}$$

$$= \Big(\underbrace{\left(1+\frac{1}{n}\right)^n}_{\downarrow\; e}\Big)^{-1} \longrightarrow \frac{1}{e} < 1$$

$\Longrightarrow$ converge

④ $\sum \frac{n^{1/n}}{n!}$

$n^{1/n} \longrightarrow 1$

$\frac{n^{1/n}}{n!} \sim \frac{1}{n!} \quad \Longrightarrow$ converge

⑤ $\sum \frac{2^n}{e^{2n}} = \sum \left(\frac{2}{e^2}\right)^n = \frac{1}{1-\frac{2}{e^2}} < +\infty$

⑥ $\sum \underbrace{\log\left(\frac{n+2}{n+4}\right)}_{\downarrow\; 0}$

$$\log\left(\frac{n+2}{n+4}\right) = \log\left(\frac{n+4-2}{n+4}\right) = \log\left(1-\frac{2}{n+4}\right) \sim -\frac{2}{n+4}$$

$\Longrightarrow$ diverge (a $-\infty$)

(7) $$\sum_{n=1}^{\infty} \underbrace{\cos\left(\underbrace{\frac{n+2}{n^2+n}}_{\to 0} \right)}_{\to 1}$$

$\Longrightarrow$ diverge a $+\infty$

(8) $$\sum_{n=2}^{+\infty} \log\left(\frac{n^2+2}{n^2-2}\right)$$

$$\log\left(\frac{n^2+2}{n^2-2}\right) = \log\left(1+\frac{4}{n^2-2}\right) \sim \frac{4}{n^2-2} \sim \frac{4}{n^2}$$

$\Longrightarrow$ converge

(9) $$\sum \underbrace{\sin\left(\frac{n+2}{n^3+4} \right)}_{\frac{1}{n^2}}$$

$$\frac{n+2}{n^3+4} \sim \frac{1}{n^2}$$

$\Longrightarrow$ converge

(10) $$\sum \frac{(-1)^n}{\sqrt{n}}$$

• (Leibniz) $a_n \searrow 0 \Rightarrow \sum (-1)^n a_n$ converge

$\frac{1}{\sqrt{n}} \searrow 0 \Longrightarrow$ converge

$\sum \frac{1}{\sqrt{n}} = \sum \frac{1}{n^{1/2}}$ diverge

non converge assolutamente

(11) $\sum_{n=1}^{\infty} \frac{\cos\left(\frac{\pi}{2}\cdot n\right)}{n}$

$$\begin{array}{cccccccccc} & 0 & 1 & 2 & 3 & 4 & 5 & 6 & & \\ \cos\left(\frac{\pi}{2}\cdot n\right) = & 1 & 0 & -1 & 0 & 1 & 0 & -1 & 0 & 1 \; \dots \end{array}$$

$$\sum_{n=1}^{\infty} \frac{\cos\left(\frac{\pi}{2}\cdot n\right)}{n} = \sum_{k=1}^{\infty} \frac{(-1)^k}{2k}$$

converge ma non assolutamente

(12) $\sum \frac{\sin(\log(n))}{n^2 \cdot \log(n)}$

$$\left| \frac{\sin(\log(n))}{n^2 \cdot \log(n)} \right| \leq \frac{1}{n^2 \cdot \log(n)} \leq \frac{1}{n^2}$$

$\Rightarrow$ converge assolutamente

(13) $$\sum \underbrace{\frac{\sin(n) + (-1)^n \cdot n}{n^2}}$$

$$\underbrace{\underset{\text{ass. conv.}}{\frac{\sin(n)}{n^2}} + \underset{\text{conv. ma non ass.}}{\frac{(-1)^n}{n}}}$$

conv ma non ass. conv. ($\underline{\underline{\text{Esp}}}$: formalizzare)

(14) $$\sum \frac{n \cdot 2^n}{e^{n/2}}$$

$$\sqrt[n]{\frac{n \cdot 2^n}{e^{n/2}}} = \underset{\downarrow \atop 1}{n^{1/n}} \cdot \frac{2}{\sqrt{e}} \longrightarrow \frac{2}{\sqrt{e}} > 1 \qquad \underline{\text{No}}$$

(17) $$\sum \frac{\log(n)}{n^2}$$

$$\cancel{\frac{\log(n)}{n^2} \sim \frac{1}{n^2}}$$

$$\frac{\log(n)}{n^2} = \underset{\downarrow \atop 0}{\frac{\log(n)}{\sqrt{n}}} \cdot \frac{1}{n^{3/2}} < \frac{1}{n^{3/2}} \Rightarrow \text{converge}$$

$$\boxed{\underset{\downarrow \atop 0}{\frac{\log(n)}{n}} \cdot \frac{1}{n} < \frac{1}{n} \qquad \text{non consente di concludere}}$$

(18) $\sum \frac{2^n+1}{3^n+n}$

$$\sim \left(\frac{2}{3}\right)^n \qquad \text{converge}$$

pag. 206 (73) $e^{\frac{3}{\log(x)}}$

$x>0$, $x \neq 1$ $\qquad D = (0,1) \cup (1,+\infty)$

Sempre positiva

$\lim_{x\to 0^+} = e^{0^-} = 1^-$ $\qquad \lim_{x\to 1^-} = e^{\frac{3}{0^-}} = e^{-\infty} = 0^+$

$\lim_{x\to 1^+} = e^{\frac{3}{0^+}} = e^{+\infty} = +\infty$ $\qquad \lim_{x\to +\infty} = e^{\frac{3}{+\infty}} = e^{0^+} = 1^+$

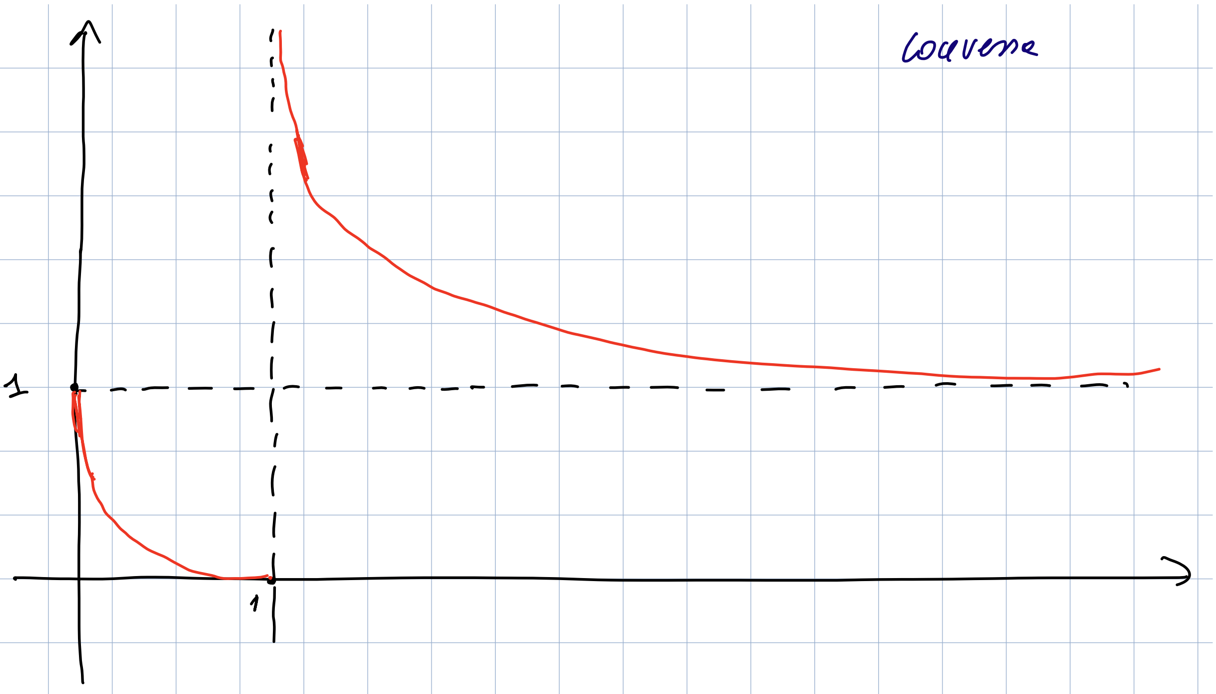
$f'(x) = e^{\frac{3}{\log(x)}} \cdot 3 \cdot \left(-\frac{1}{\log^2(x)} \cdot \frac{1}{x}\right)$ sempre <0

$\lim_{x\to 0^+} f'(x) = 1 \cdot 3 \cdot (-\infty) = -\infty$

$\lim_{x\to 1^-} f'(x) = 0 \cdot \ldots = 0^-$

$$f''(x) = e^{\frac{3}{\log(x)}} \cdot \left(\frac{9}{x^2\log^4(x)} + 6\,\frac{1}{\log^3(x)} \cdot \frac{1}{x^2} + \frac{3}{\log^2(x)} \cdot \frac{1}{x^2}\right)$$

$$= e^{\frac{3}{\log}} \cdot \frac{3}{x^2 \cdot \log^4(x)} \cdot \underbrace{\left(\log^2(x) + 2\log(x) + 3\right)}_{(\log(x)+1)^2+2} > 0$$



(74) $\arctan\left(\frac{x+1}{x-3}\right)+\frac{x}{4}$

$x \neq 3$

$D=(-\infty,3)\cup(3,+\infty)$

$$\lim_{x\to-\infty} \arctan\left(\frac{x+1}{x-3}\right) = \arctan(1^-) = \left(\frac{\pi}{4}\right)^-$$

$$\lim_{x\to+\infty} \arctan\left(\frac{x+1}{x-3}\right) = \arctan(1^+) = \left(\frac{\pi}{4}\right)^+$$

Asintoto obliquo $y=\frac{x}{4}+\frac{\pi}{4}$

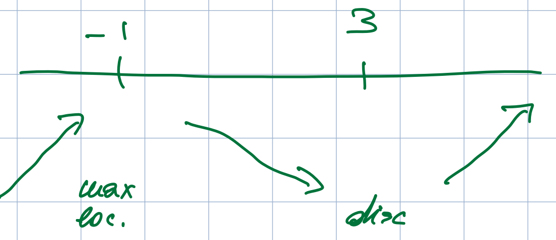
inoltre f è sotto asint. in $-\infty$ e sopra in $+\infty$

$$\lim_{x\to 3^\pm} f(x) = \arctan(\pm\infty)+\frac{3}{4} = \pm\frac{\pi}{2}+\frac{3}{4}$$

$$f'(x) = \frac{1}{1+\left(\frac{x+1}{x-3}\right)^2}\cdot\frac{1\cdot(x-3)-1\cdot(x+1)}{(x-3)^2}+\frac{1}{4}$$

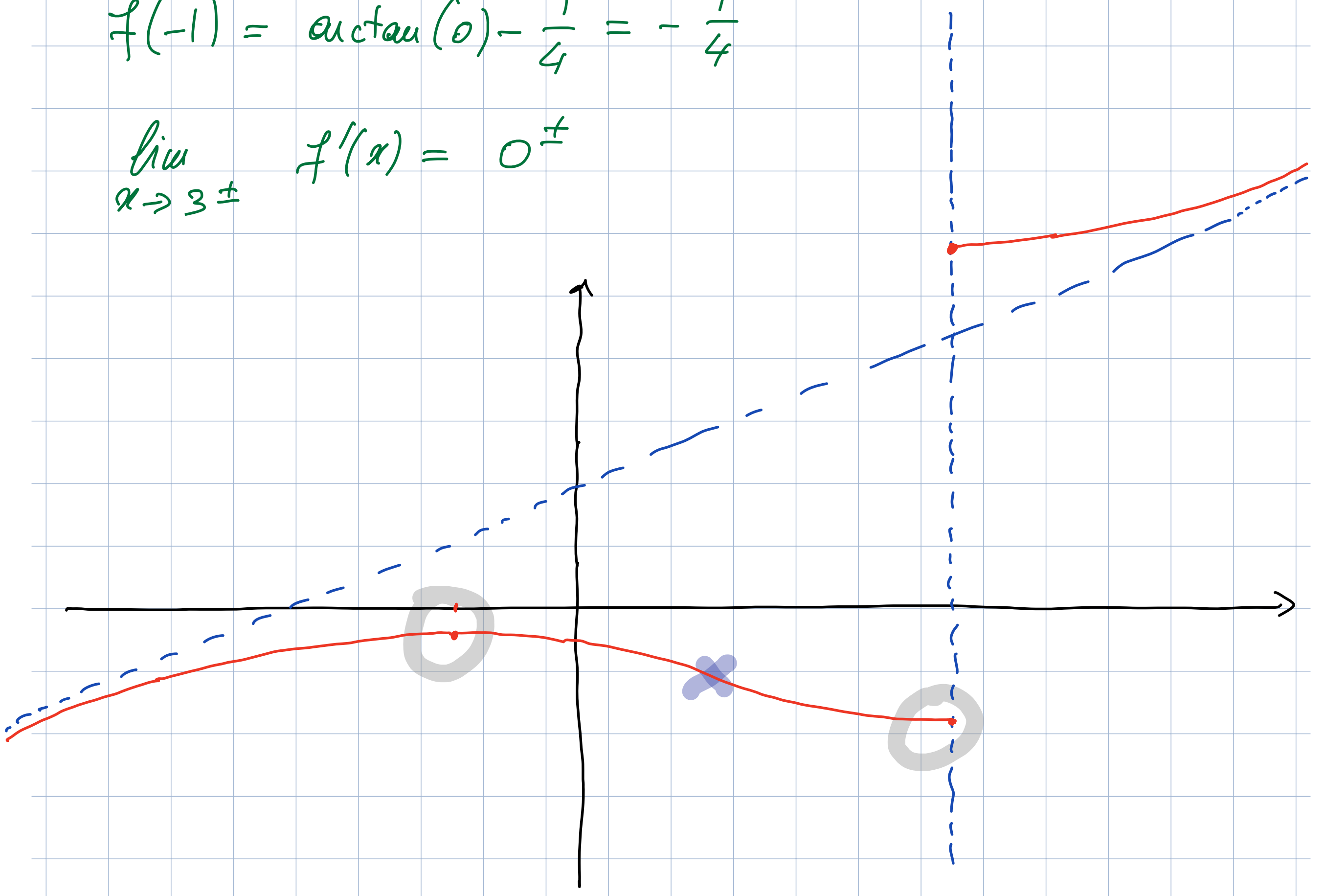
$$= -\frac{4}{(x-3)^2+(x+1)^2} + \frac{1}{4} = \frac{-16+2x^2-4x+10}{4\left((x-3)^2+(x-1)^2\right)}$$

$$= \frac{x^2-2x-3}{\cdots} = \frac{(x-3)(x+1)}{\cdots}$$



$$f(-1) = \arctan(0) - \frac{1}{4} = -\frac{1}{4}$$

$$\lim_{x \to 3^\pm} f'(x) = 0^\pm$$



Calcolare Taylon II per $\log(1+\sin(x))$

- $f(0), f'(0), \ldots, f^{(v)}(0)$ + sostituire $\underline{\underline{OK}}$
- $\log(1+t) = t - \frac{t^2}{2} + \frac{t^3}{3} - \frac{t^4}{4} + \frac{t^5}{5} + o(t^5)$

$$\sin(x) = x - \frac{1}{6}x^3 + \frac{1}{120}x^5 + o(x^5)$$

$$\log(1+\sin(x)) = \left(x - \frac{1}{6}x^3 + \frac{1}{120}x^5 + o(x^5)\right)$$

$$-\frac{1}{2}(\ldots)^2 + \frac{1}{3}(\quad)^3 - \frac{1}{4}(\ldots)^4 + \frac{1}{5}(\ldots)^5 + o\left((\ldots)^5\right)$$

$$= \left(x - \frac{1}{6}x^3 + \frac{1}{120}x^5\right) - \frac{1}{2}\left(x^2 - \frac{1}{3}x^4\right) + \frac{1}{3}\left(x^3 - \frac{1}{2}x^5\right)$$

$$- \frac{1}{4}x^4 + \frac{1}{5}x^5 + o(x^5)$$

$$= x - \frac{1}{2}x^2 + \left(-\frac{1}{6} + \frac{1}{3}\right)x^3 + \left(\left(-\frac{1}{2}\right)\cdot\left(-\frac{1}{3}\right) + \left(-\frac{1}{4}\right)\right)x^4$$

$$+ \left(\frac{1}{120} + \frac{1}{3}\cdot\left(-\frac{1}{2}\right) + \frac{1}{5}\right)x^5 + o(x^5)$$