Ist. Mat. I - CIA
9/11/23

Confronto tra infiniti e infinitesimi (zeri)

$f,g : I \to \mathbb{R}$, $x_0 \in \bar{I}$, $\lim_{x \to x_0} f(x)$, $\lim_{x \to x_0} g(x)$ entrambi $\pm\infty$, 0.

- se $\lim_{x \to x_0} \frac{f(x)}{g(x)} = L \neq 0$ dico $f = O(g)$ "o grande"

e dico che hanno lo stesso ordine di ∞ o 0;

in un limite se sostituendo $f(x)$ con $L \cdot g(x)$ trovo il valore, allora coincide con quello originale (se invece resta F.I. non posso concludere)

Es: $\lim_{x \to 0} \frac{1-\cos(x)}{\sin^2(x)} \overset{?}{=} \lim_{x \to 0} \frac{\frac{1}{2}x^2}{(1 \cdot x)^2} = \frac{1}{2}$ sì

- se $\lim_{x \to x_0} \frac{f(x)}{g(x)} = 0$ dico che $f = o(g)$ "o piccolo"

Cioè f è un ordine di zero più alto oppure di ∞ più basso.

Es: $\log(x) = o(x)$ in $+\infty$

$1 - \cos(x) = o(\sin(x))$ in 0

Conseguenze di limiti visti:

- $\lim_{x\to 0} \frac{\sin(x)}{x} = 1$

$\sin(x) = O(x)$ in 0

$\sin(x) - x = o(x)$ in 0

$\boxed{\sin(x) = x + o(x)}$

- $\lim_{x\to 0} \frac{\sin(x) - x}{x^3} = -\frac{1}{6}$

$\sin(x) - x = O(x^3)$ in 0

$\sin(x) - x + \frac{1}{6}x^3 = o(x^3)$ in 0

$\boxed{\sin(x) = x - \frac{1}{6}x^3 + o(x^3)}$

- $\lim_{x\to 0} \frac{1-\cos(x)}{x^2} = \frac{1}{2}$

$1-\cos(x) = O(x^2)$ in 0

$\cos(x) - 1 + \frac{1}{2}x^2 = o(x^2)$ in 0

$\boxed{\cos(x) = 1 - \frac{1}{2}x^2 + o(x^2) \text{ in } 0}$

- $\lim_{x\to 0} \frac{1-\cos(x) - \frac{1}{2}x^2}{x^4} = -\frac{1}{24}$

$1-\cos(x) - \frac{1}{2}x^2 = O(x^4)$

$1-\cos(x) - \frac{1}{2}x^2 + \frac{1}{24}x^4 = o(x^4)$

$\boxed{\cos(x) = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 + o(x^4)}$

Sappiamo $\lim_{x\to 0} \frac{e^x - 1}{x} = 1$

$e^x - 1 = O(x)$

$e^x - 1 - x = o(x)$

$e^x = 1 + x + o(x)$

$\frac{e^x - 1 - x}{x^2} \overset{\text{del'H.}}{\leadsto} \frac{e^x - 1}{2x} \to \frac{1}{2}$

$e^x = 1 + x + \frac{1}{2}x^2 + o(x^2)$

$\frac{e^x - (1 + x + \frac{1}{2}x^2)}{x^3} \leadsto \frac{e^x - (1+x)}{3x^2} \to \frac{1}{6}$

$$e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + o(x^3)$$

$$\frac{e^x - \left(1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3\right)}{x^4} \rightsquigarrow \frac{e^x - \left(1 + x + \frac{1}{2}x^2\right)}{4x^3} \to \frac{1}{24}$$

$$e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + o(x^4)$$

$$= \frac{x^0}{0!} + \frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + o(x^4)$$

Esp (induz):

$$e^x = \sum_{k=0}^{n} \frac{x^k}{k!} + o(x^n) \qquad \forall n$$

$$\cosh(x) = \frac{e^x + e^{-x}}{2} = \frac{1}{2}\left(\begin{array}{l} 1 + x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \frac{1}{24}x^4 \dots \\ + 1 - x + \frac{1}{2}x^2 - \frac{1}{3}x^3 + \frac{1}{24}x^4 \dots \end{array} \right) + o(\dots)$$

$$= 1 + \frac{1}{2}x^2 + \frac{1}{24}x^4 + \dots + o(\dots)$$

$$= \sum_{k=0}^{n} \frac{x^{2k}}{(2k)!} + o(x^{2n})$$

$$\sinh(x) = \frac{e^x - e^{-x}}{2} = \sum_{k=0}^{n} \frac{x^{2k+1}}{(2k+1)!} + o(x^{2n+1})$$

$$+\frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \dots$$

Visto: $\sin(x) = x - \frac{1}{6}x^3 + o(x^3)$

$\cos(x) = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 + o(x^4)$

$$-\frac{1}{6!}x^6 + \frac{1}{8!}x^8 + \dots$$

__Ese__ :

$$\sin(x) = \sum_{k=0}^{n} \frac{(-1)^k}{(2k+1)!} \cdot x^{2k+1} + o(x^{2n+1})$$

$$\cos(x) = \sum_{k=0}^{n} \frac{(-1)^k}{(2k)!} \cdot x^{2k} + o(x^{2n})$$

——— o ———

Ricordo la convenzione $e^{i\vartheta} = \cos(\vartheta) + i \cdot \sin(\vartheta)$.

La spiego :

$$e^{i\vartheta} = \sum_{k=0}^{n} \frac{(i\vartheta)^k}{k!} + o(\vartheta^n)$$

pari $1, -1, 1, -1, 1, -1 \ldots$

$$i^k = 1 \quad i \quad -1 \quad -i \quad 1 \quad i \quad -1 \ldots$$

dispari $i \cdot (1, -1, 1, -1, \ldots)$

$$= \underbrace{\sum_{h=0}^{\ldots} \frac{(-1)^h}{(2h)!} \cdot x^{2h}}_{\cos(\vartheta)} + i \cdot \underbrace{\sum_{l=0}^{\ldots} \frac{(-1)^l}{(2l+1)!} \cdot x^{2l+1}}_{\sin(\vartheta)} + o(\ldots)$$

$$= \cos(\vartheta) + i \cdot \sin(\vartheta) + o(\vartheta^n)$$

——— o ———

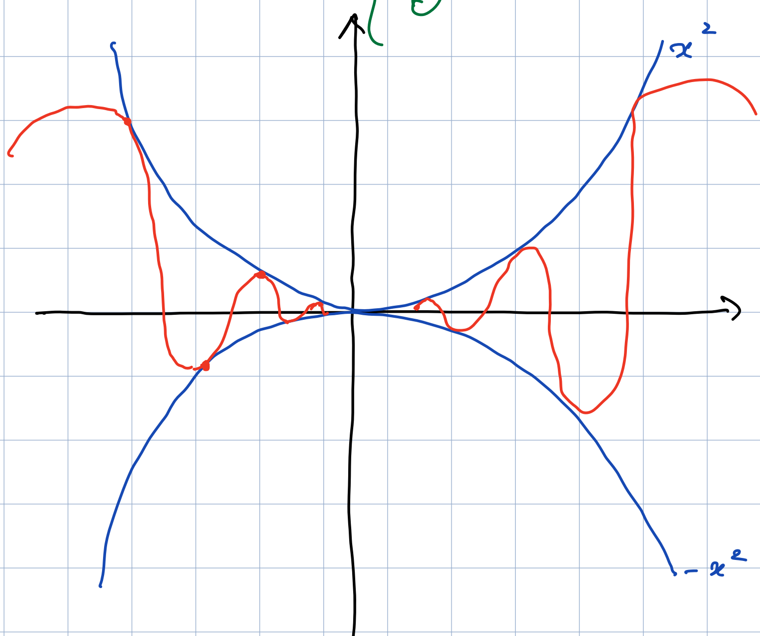
Oss: una funzione può essere ovunque derivabile con derivata non continua (anche localmente): cioè

$$f'_{\pm}(x_0) = \lim_{h \to 0^{\pm}} \frac{f(x_0+h) - f(x_0)}{h}$$

$$\lim_{x \to x_0^{\pm}} f'(x)$$

possono non coincidere.

Es:
$$f(x) = \begin{cases} x^2 \cdot \sin\left(\frac{1}{x}\right) & x \neq 0 \\ 0 & x = 0 \end{cases}$$



$\lim_{x \to 0} f(x) = 0 \Rightarrow f$ cont.

$$f'(0) = \lim_{h \to 0} \frac{f(h) - f(0)}{h} = \lim_{h \to 0} \frac{h^2 \cdot \sin\left(\frac{1}{h}\right) - 0}{h}$$

$$= \lim_{h \to 0} h \cdot \sin\left(\frac{1}{h}\right) = 0$$

$$f'(x) = D\left(x^2 \cdot \sin\left(\frac{1}{x}\right)\right)$$

$$= 2x \cdot \sin\left(\frac{1}{x}\right) + x^2 \cdot \cos\left(\frac{1}{x}\right) \cdot \left(-\frac{1}{x^2}\right)$$

$$= \underbrace{2x \cdot \sin\left(\frac{1}{x}\right)}_{\substack{\downarrow\, x \to 0 \\ 0}} + \underbrace{\cos\left(\frac{1}{x}\right)}_{\substack{\downarrow\, x \to 0 \\ \nexists}}$$

__Tuttavia__ :

__Prop__ : se $\lim\limits_{x \to x_0^{\pm}} f'(x) = L \in \mathbb{R}$

allora $\exists\, f'_{\pm}(x_0) = L$.

__Dimo__ : $L = \lim\limits_{x \to x_0^+} f'(x)$

$$f'_+(x_0) = \lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{\underbrace{(x_0+h) - x_0}_{\text{Cauchy (Lagrange)}}}$$

$f'(c)$ $\qquad x_0 < c < x_0 + h$

$\downarrow$

L per $h \to 0$

$\blacksquare$

Zanichelli p. 135

Trovare dominio, limiti agli estremi, asintoti.

(31) $\dfrac{\log(x)}{\sqrt[3]{x-1}}$ $\qquad (0,1)\cup(1,+\infty)$

$$\lim_{x\to 0^+} = \frac{-\infty}{-1} = +\infty$$

$$\lim_{x\to+\infty} = \lim_{x\to+\infty} \frac{\log(x)}{x^{1/3}} = 0$$

$$\lim_{x\to 1} = \frac{0}{0} \qquad \lim_{y\to 0} \frac{\log(1+y)}{y^{1/3}} = \lim_{y\to 0} \underbrace{\underbrace{\frac{\log(1+y)}{y}}_{\to 1} \cdot \underbrace{y^{2/3}}_{\to 0}}_{0}$$

Poiché f non è definita in $x=1$ ma $\lim_{x\to 1} f(x) = 0$ posso definire

$$g:(0,\infty)\to\mathbb{R} \qquad g(x) = \begin{cases} f(x) & x\neq 1 \\ 0 & x=1 \end{cases}$$

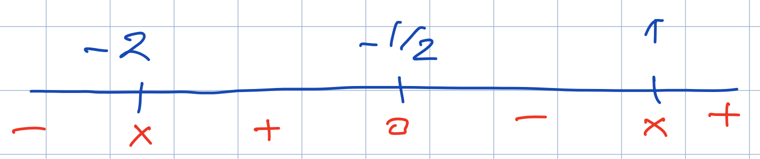
e ho che g è continua.

(32) $f(x) = \frac{\cos(x) - 1}{x}$ $\qquad \mathbb{R} \setminus \{0\}$

$$\lim_{x \to \pm\infty} f(x) = 0$$

$$\lim_{x \to 0^{\pm}} f(x) = \lim_{x \to 0^{\pm}} \underbrace{\frac{\cos(x) - 1}{x^2}}_{-1/2} \cdot x = 0$$

(33) $\log\left(\frac{2x+1}{x^2+x-2}\right)$ $\qquad \log\left(\frac{2x+1}{(x+2)(x-1)}\right)$



$$D = (-2, -1/2) \cup (1, +\infty)$$

$$\lim_{x \to -2^+} f(x) = \log\left(\frac{-3}{0^+ \cdot -3}\right) = \log(+\infty) = +\infty$$

$$\lim_{x \to (-1/2)^-} f(x) = \log\left(\frac{0^-}{\frac{3}{2} \cdot \left(-\frac{3}{2}\right)}\right) = \log(0^+) = -\infty$$

$$\lim_{x \to 1^+} f(x) = \log\left(\frac{3}{3 \cdot 0^+}\right) = \log(+\infty) = +\infty$$

$$\lim_{x \to +\infty} f(x) = \lim_{x \to +\infty} \log\left(\frac{2}{x}\right) = \log(0^+) = -\infty$$

(34) $\arctan\left(\frac{x+2}{x-1}\right)$ $\quad D = \mathbb{R} \setminus \{1\}$

$$\lim_{x\to\pm\infty} f(x) = \arctan(1) = \frac{\pi}{4}$$

$$\lim_{x\to 1^\pm} f(x) = \arctan\left(\frac{3}{0^\pm}\right) = \arctan(\pm\infty) = \pm\frac{\pi}{2}$$

(35) $x \cdot \arctan\left(\frac{1}{x}\right)$ $\quad \mathbb{R} \setminus \{0\}$

$$\lim_{x\to 0} f(x) = 0 \cdot \left(\pm\frac{\pi}{2}\right) = 0$$

$$\lim_{x\to\pm\infty} f(x) = \lim_{y\to 0} \frac{\arctan(y)}{y} = 1$$

$$\tan(x) = \frac{\sin(x)}{\cos(x)} \sim \sin(x) \sim x$$
$$\Rightarrow \arctan(x) = x$$

(36) $x \cdot \log\left(\frac{2x^2+3}{x^2+x+1}\right)$ $\quad D = \mathbb{R}$

$$\lim_{x\to\pm\infty} f(x) = \pm\infty \cdot \log(2) = \pm\infty$$

$$\lim_{x\to\pm\infty} \frac{f(x)}{x} = \log(2) \overset{?}{=} m$$

$$q \overset{?}{=} \lim_{x\to\pm\infty} (f(x) - m\cdot x) = \lim_{x\to\pm\infty} x \cdot \left(\log\left(\frac{2x^2+3}{x^2+x+1}\right) - \log(2)\right)$$

$\infty \cdot 0$

$$= \lim_{x \to \pm\infty} \frac{\log\left(\frac{2x^2+3}{2(x^2+x+1)}\right)}{\frac{1}{x}}$$

$$= \lim_{x \to \pm\infty} \underbrace{\frac{\log\left(1+\frac{-2x+1}{2x^2+2x+2}\right)}{\frac{-2x+1}{2x^2+2x+2}}}_{\frac{\log(1+y)}{y} \xrightarrow{y \to 0} 1} \cdot \underbrace{\frac{(-2x+1)\cdot x}{2x^2+2x+2}}_{-1}$$

Asintoto obliquo $\quad y = \log(2)\cdot x - 1$

(37) $\quad 2x \cdot e^{-\frac{1}{x}} \qquad D = \mathbb{R} \setminus \{0\}$

$$\lim_{x \to \pm\infty} f(x) = \pm\infty \cdot e^{0^{\mp}} = \pm\infty \qquad *$$

$$\lim_{x \to 0^-} f(x) = 0^- \cdot e^{+\infty} = 0 \cdot \infty \qquad FI$$

$$\lim_{x \to 0^-} f(x) = \lim_{y \to +\infty} \frac{e^y}{2\cdot(-y)} = -\infty$$

$$\lim_{x \to 0^+} f(x) = 0^+ \cdot e^{-\infty} = 0 \cdot 0 = 0$$

* $\lim_{x \to \pm\infty} \frac{f(x)}{x} = 2 \cdot e^0 = 2 \ \exists \ m$

$$q \ \exists \lim_{x \to \pm\infty} \left(f(x) - m \cdot x\right) = \lim_{x \to \pm\infty} 2x \cdot \left(e^{-\frac{1}{x}} - 1\right)$$

$$= \lim_{x \to \pm\infty} 2x \cdot \left(\left(1 - \frac{1}{x} + \frac{1}{2} \cdot \frac{1}{x^2} + o\left(\frac{1}{x^2}\right)\right) - 1\right)$$

$$= \lim_{x \to \pm\infty} \left(-2 + \frac{1}{x} + o\left(\frac{1}{x}\right)\right) = -2$$

Asintoto obliquo: $y = 2x - 2$

(38) $x^2 \cdot e^{-|x|}$ $\mathbb{R}$

$\lim_{x \to \pm\infty} f(x) = 0$ (funz. pari)

(39) $\sqrt[3]{x} \cdot \frac{\log |x|}{\log |x+1|}$ $x \neq 0, \ x \neq -1, \ x \neq -2$

$\lim_{x \to \pm\infty} f(x) = \pm\infty \cdot 1 = \pm\infty$

$\lim_{x \to \pm\infty} \frac{f(x)}{x} = 0 \cdot 1 = 0$ asintoto orizzontale $y = 0$.

$$f(x) = \sqrt[3]{x} \cdot \frac{\log|x|}{\log|x+1|}$$

$$\lim_{x \to 0^{\pm}} f(x) = 0^{\pm} \cdot \frac{-\infty}{0} \qquad F.I.$$

$$\lim_{x \to 0^{\pm}} \sqrt[3]{x} \cdot \frac{\log|x|}{\log(1+x)} =$$

$$= \underbrace{\lim_{x \to 0^{\pm}} \underbrace{\frac{x}{\log(1+x)}}_{\to 1} \cdot \underbrace{\frac{\sqrt[3]{x}}{x}}_{\frac{1}{\sqrt[3]{x^2}} \to +\infty} \cdot \underbrace{\log|x|}_{-\infty}}_{-\infty}$$

$$f(x) = \sqrt[3]{x} \cdot \frac{\log|x|}{\log|x+1|}$$

$$\lim_{x \to (-1)^{\pm}} f(x) = (-1) \cdot \frac{0}{\infty} = 0$$

$$\lim_{x \to (-2)^{\pm}} f(x) = \left(-\sqrt[3]{2}\right) \cdot \frac{\log(2)}{-\infty} = 0$$

$$f(x) = \sqrt[3]{x} \cdot \frac{\log|x|}{\log|x+1|}$$