Ist. Mat. I - CIA
6/10/23

Poli: tutorato.

$\mathbb{C} = \{a+ib : a,b \in \mathbb{R}\}$

Fatto: ci sono formule risolutive esplicite per equazioni polinomiali di grado 3 e 4 basate sui numeri complessi.

Numero di Nepero: $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n!} = 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} + \dots$
$= e = 2.718\dots \notin \mathbb{Q}$

e^x esponenziale; $\ln = \log_e$ $(\log = \ln)$
$\mathrm{Log} = \log_{10}$.

$z = a+ib$; $\bar{z} = a-ib$; $|z| = \sqrt{a^2+b^2} = \sqrt{z\cdot\bar{z}} \geq 0$

Oss: se $a \in \mathbb{R}$, $|a+i0| = \sqrt{a^2} =$ val. ass. di a.

Proprietà:

- $\overline{z+w} = \bar{z} + \bar{w}$

$\overline{(a+ib)+(c+id)} \overset{?}{=} \overline{(a+ib)} + \overline{(c+id)}$

$$\overline{(a+c)+i(b+d)} \qquad\qquad (a-ib)+(c-id)$$
$$= (a+c)-i(b+d) \qquad\qquad = (a+c)-i(b+d)$$

<u>ok</u>

- $\overline{z\cdot w} = \bar{z}\cdot\bar{w}$

$$\overline{(a+ib)\cdot(c+id)} \overset{?}{=} \overline{(a+ib)}\cdot\overline{(c+id)}$$
$$\overline{(ac-bd)+i(ad+bc)} \qquad\qquad (a-ib)\cdot(c-id)$$
$$= (ac-bd)-i(ad+bc) \qquad\qquad = (ac-bd)-i(ad+bc)$$

<u>Sì</u>

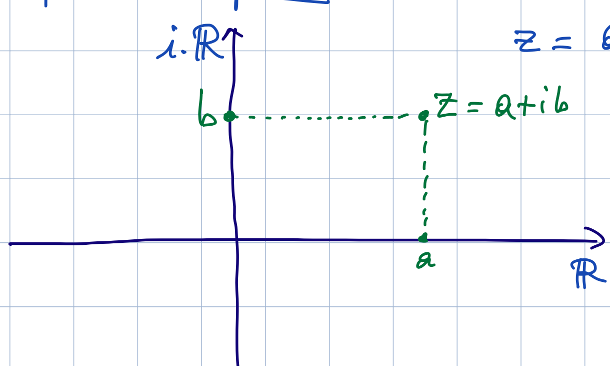
- $|z\cdot w| = |z|\cdot|w|$

$$|z\cdot w| = \sqrt{(z\cdot w)\cdot\overline{(z\cdot w)}} = \sqrt{z\cdot w\cdot\bar{z}\cdot\bar{w}}$$
$$= \sqrt{(z\cdot\bar{z})\cdot(w\cdot\bar{w})} = \sqrt{|z|^2\cdot|w|^2} = |z|\cdot|w|$$

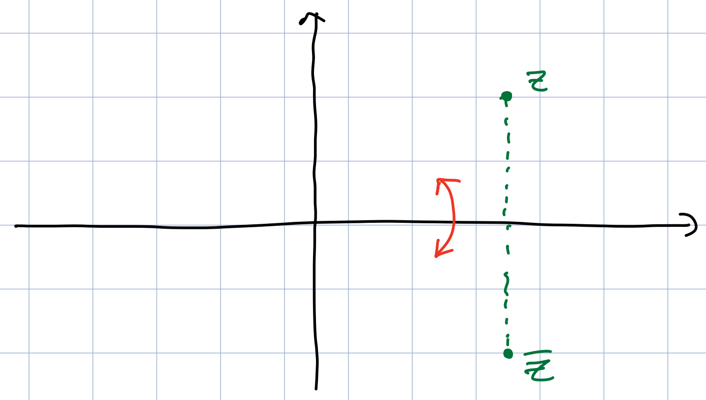
<u>Sì</u>

<u>Il piano complesso</u>: $\quad \mathbb{C} = \{a+ib : a,b\in\mathbb{R}\}$

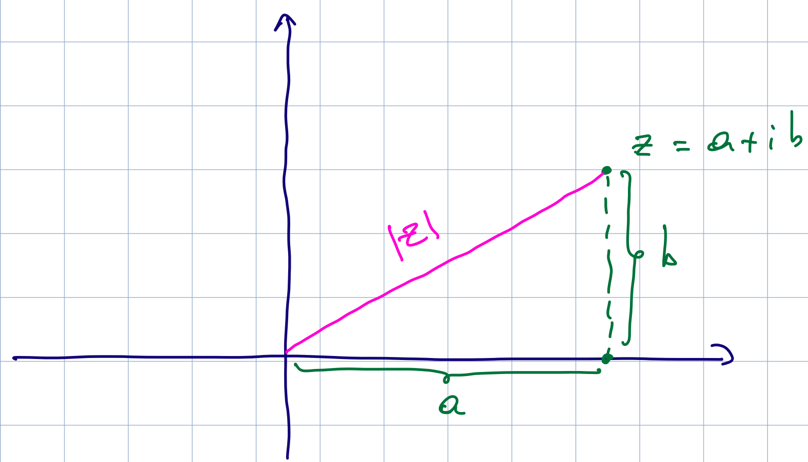
$$z = a+ib \longleftrightarrow \begin{pmatrix} a \\ b \end{pmatrix} \in \mathbb{R}^2$$



$\bar{z} = a - ib$

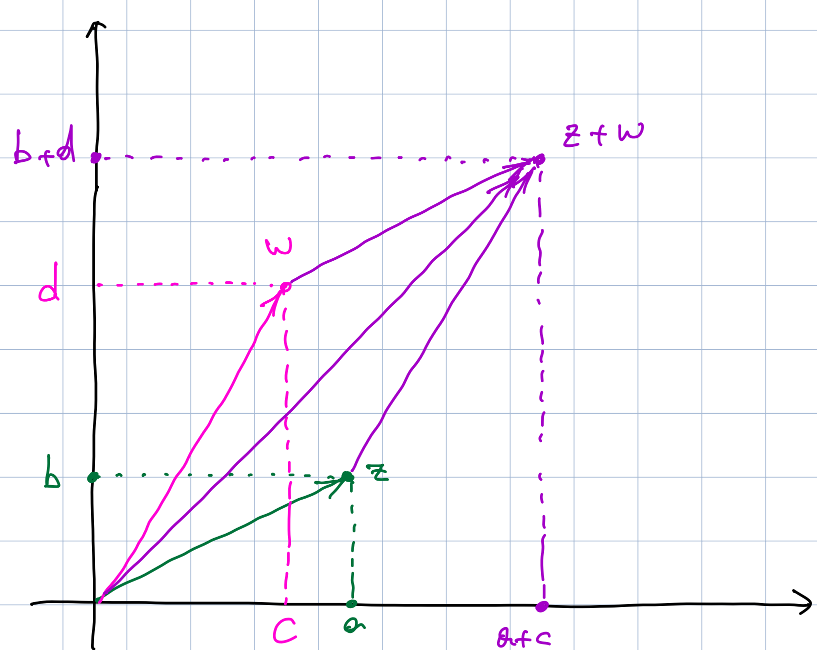


$|z| = \sqrt{a^2+b^2}$



$|z|$ = distanza di z da 0 in $\mathbb{R}^2$

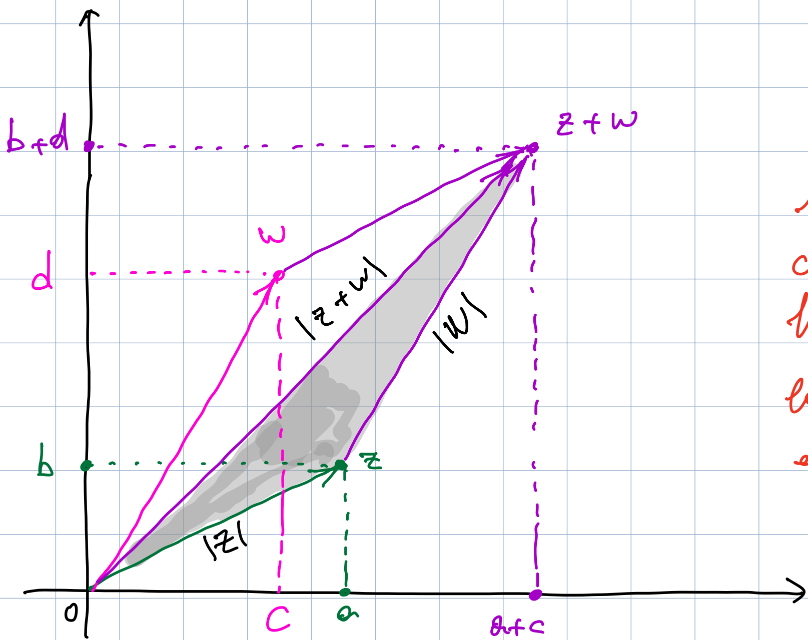
$z = a+ib$
$w = c+id$



Prop: $|z+w| \leq |z| + |w|$

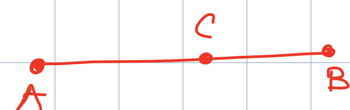
e vale $=$ se e solo se uno dei due è $k \cdot$ l'altro con $k \in \mathbb{R},\ k \geq 0$.

disuguaglianza triangolare

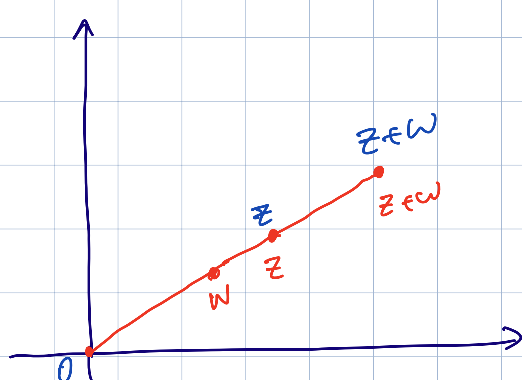


in un triangolo ciascun lato è lungo al più come la somma degli altri due.

uguale solo se



$\overline{AB} = \overline{AC} + \overline{BC}$



Oss: • su $\mathbb{C}$ non c'è una relazione sensata di ordinamento $\leq$.

• se $a \in \mathbb{R}$ $\quad a \leq |a|$

• se $z \in \mathbb{C}$ allora $|\mathrm{Re}(z)| \leq |z|$
$|\mathrm{Im}(z)| \leq |z|$

$z = a + ib \quad ; \quad |z| = \sqrt{a^2+b^2} \geq \sqrt{a^2} = |a| = |\mathrm{Re}(z)|$

Oss: $\overline{\overline{u}} = u$

Dimo disug. triang.:

$$|z+w|^2 = (z+w)\cdot\overline{(z+w)} = (z+w)\cdot(\overline{z}+\overline{w})$$

$$= z\cdot(\overline{z}+\overline{w}) + w\cdot(\overline{z}+\overline{w})$$

$$= z\cdot\overline{z} + z\cdot\overline{w} + w\cdot\overline{z} + w\cdot\overline{w}$$

$$= |z|^2 + z\cdot\overline{w} + \overline{z\cdot\overline{w}} + |w|^2$$

$$= |z|^2 + 2\,\mathrm{Re}(z\cdot\overline{w}) + |w|^2$$

$$\leq |z|^2 + 2\cdot|\mathrm{Re}(z\cdot\overline{w})| + |w|^2$$

$$\leq |z|^2 + 2\cdot|z\cdot\overline{w}| + |w|^2$$

$$= |z|^2 + 2\cdot|z|\cdot|\overline{w}| + |w|^2$$

Oss: $|\overline{u}| = |u|$

$$= |z|^2 + 2\cdot|z|\cdot|w| + |w|^2$$

$$= (|z|+|w|)^2$$

$$\Longrightarrow \quad |z+w|^2 \le (|z|+|w|)^2$$

$$\Longrightarrow \quad |z+w| \le |z| + |w| .$$

Quando vale $=$?

$$\mathrm{Re}(z\cdot\bar{w}) \geq 0$$

$$z\cdot\bar{w} \in \mathbb{R}$$

cioè se $z\cdot\bar{w} = h \in \mathbb{R}$, $h \geq 0$.

Vale se e solo se
- $w = 0$ e in tal caso $w = 0\cdot z$
- $w \neq 0$ e $z\cdot\bar{w}\cdot w = h\cdot w$

$$z\cdot|w|^2 = h\cdot w$$

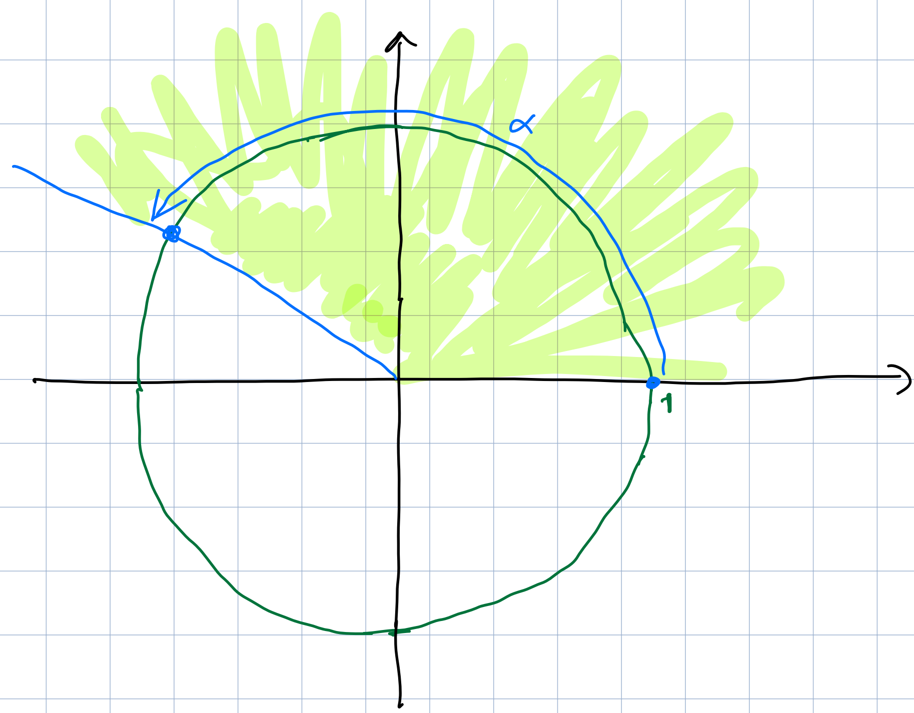
$$z = \underbrace{\left(\frac{h}{|w|^2}\right)}_{\text{reale} \geq 0} \cdot w .$$

$\blacksquare$

$\mathbb{C} \longleftrightarrow \mathbb{R}^2$ $\qquad z+w \longleftrightarrow$ somme vettori

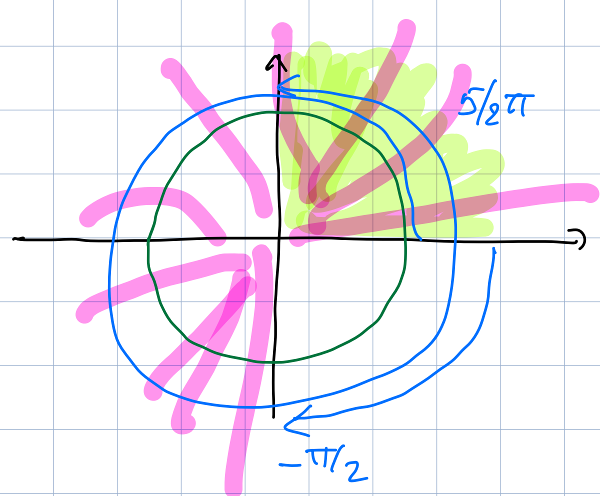
$z\cdot w \longleftrightarrow$?

Gli angoli si possono misurare in radianti:

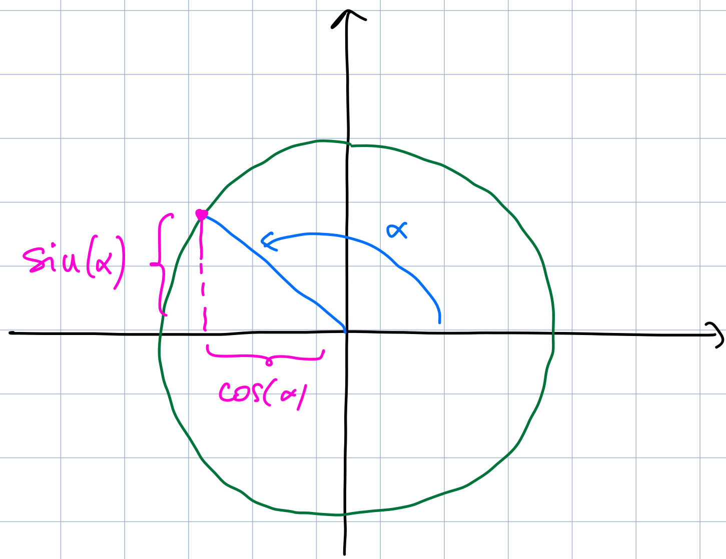


angolo piatto misura $\pi = 3.1415\ldots \notin \mathbb{R}$.

Angoli $0 \leq \alpha \leq 2\pi$: in realtà identifico α con $\alpha + 2k\pi$ $\forall k \in \mathbb{Z}$.



Def: $\sin, \cos : \mathbb{R} \to \mathbb{R}$



Formule di addizione:

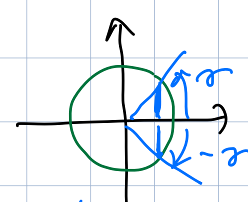
$$\cos(\alpha+\beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$$

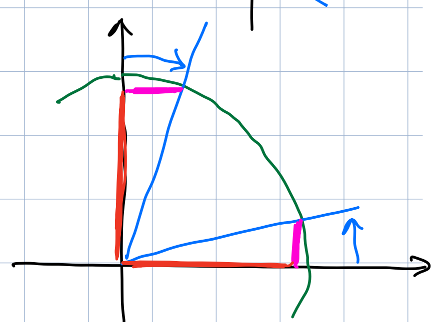
$$\sin(\alpha+\beta) = \cos(\alpha)\cdot\sin(\beta) + \sin(\alpha)\cdot\cos(\beta)$$

Seguono entrambe da:

$$\cos(\alpha-\beta) = \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)$$

usando: $\sin(-\gamma) = -\sin(\gamma)$





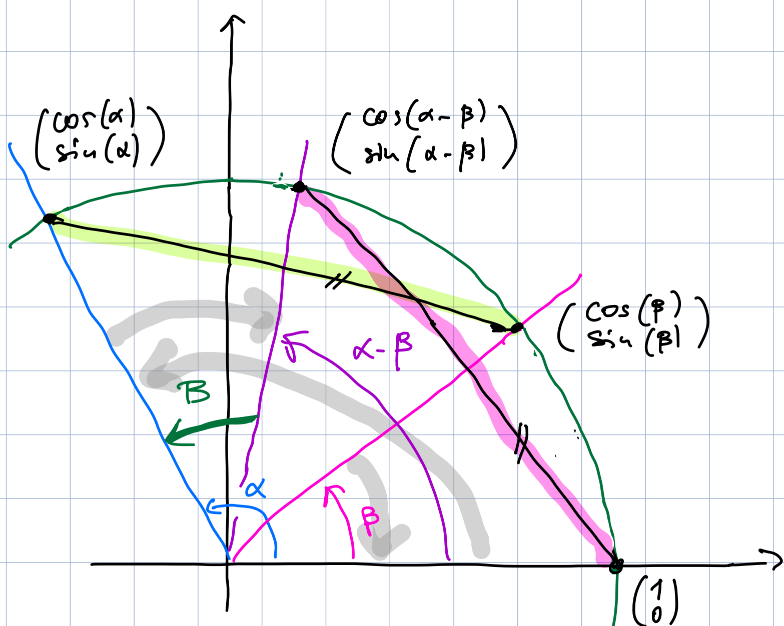
$$\sin(\gamma) = \cos\left(\frac{\pi}{2}-\gamma\right)$$

$$\cos(\gamma) = \sin\left(\frac{\pi}{2}-\gamma\right)$$

Ovvio: diff. cos $\Rightarrow$ somma cos

Esercizio: diff/somma cos $\rightarrow$ somma sin.

Dimo $\left(\cos(\alpha-\beta) = \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta) \right)$.



$$\left(1-\cos(\alpha-\beta)\right)^2 + \left(0-\sin(\alpha-\beta)\right)^2 =$$

$$= \left(\cos(\alpha)-\cos(\beta)\right)^2 + \left(\sin(\alpha)-\sin(\beta)\right)^2$$

$$\cancel{1} - 2\cos(\alpha-\beta) + \cos^2(\alpha-\beta) + \sin^2(\alpha-\beta) \quad \cancel{1}$$

$$= \cos^2(\alpha) - 2\cos(\alpha)\cos(\beta) + \cos^2(\beta) + \sin^2(\alpha) - 2\sin(\alpha)\sin(\beta) + \sin^2(\beta)$$

$$\cancel{1} \qquad \cancel{1}$$

$\Longrightarrow \cos(\alpha-\beta) = \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)$. ∎

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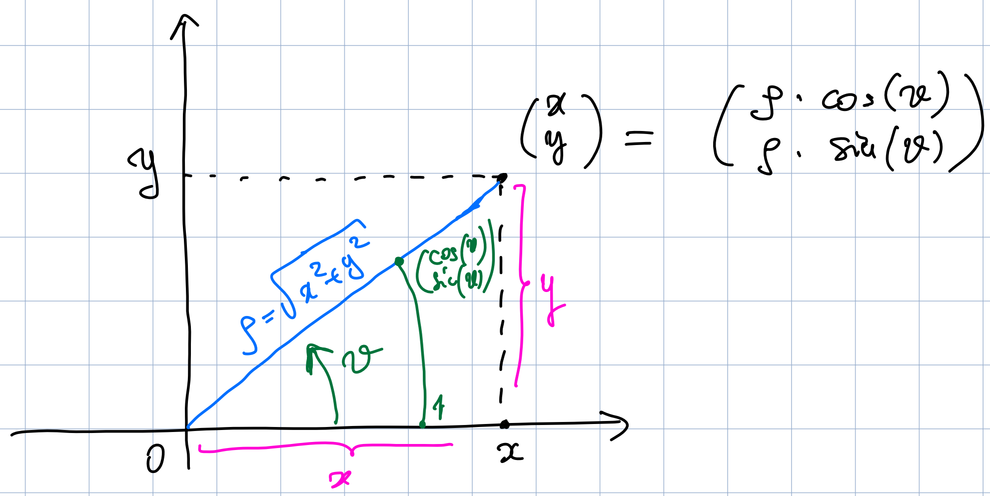
Coordinate polari in $\mathbb{R}^2$



<u>Oss</u>: • $\rho = \sqrt{x^2+y^2}$

• ϑ non è definito se $\begin{pmatrix}x\\ y\end{pmatrix} = 0$

• per $\begin{pmatrix}x\\ y\end{pmatrix} \neq 0$ ϑ è ambiguo: definito a meno della periodicità di periodo 2π.

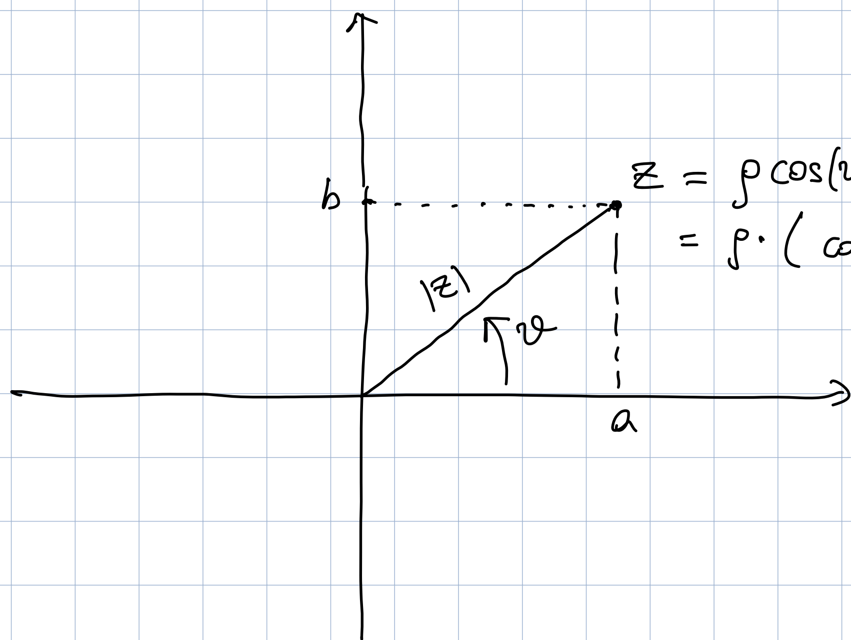
Cioè se ϑ è un angolo t.c. $\begin{pmatrix}x\\ y\end{pmatrix} = \begin{pmatrix}\rho\cdot\cos(\vartheta)\\ \rho\cdot\sin(\vartheta)\end{pmatrix}$

anche $\vartheta+2\pi$, $\vartheta+4\pi$, ...
$\vartheta-2\pi$, $\vartheta-4\pi$, ... fanno lo stesso

$$\left(\begin{array}{l}\sin(\vartheta+2k\pi)=\sin(\vartheta)\\ \cos(\vartheta+2k\pi)=\cos(\vartheta)\end{array}\qquad \forall k\in\mathbb{Z}\right)$$

$$\mathbb{C}=\{a+ib:\ a,b\in\mathbb{R}\}\longleftrightarrow\mathbb{R}^2$$

$$z=a+ib\longleftrightarrow\begin{pmatrix}a\\ b\end{pmatrix}$$



$$z=\rho\cos(\vartheta)+\rho\cdot i\cdot\sin(\vartheta)$$
$$=\rho\cdot\left(\cos(\vartheta)+i\cdot\sin(\vartheta)\right)$$

Oss: $\rho=|z|$ è il modulo

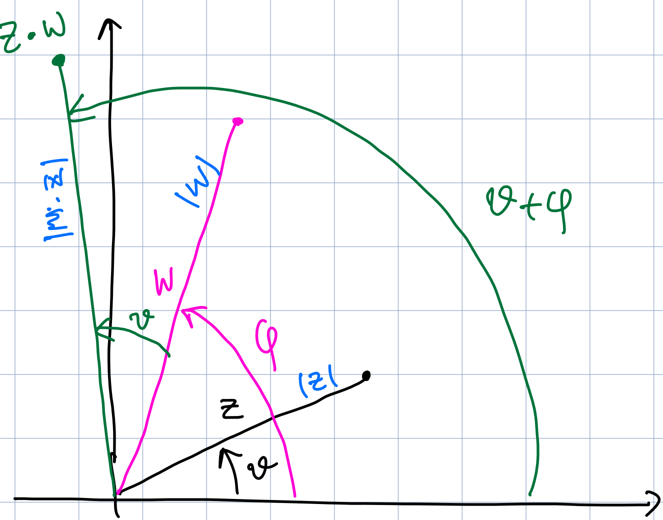
ϑ lo chiamo argomento (ambiguo).

$$z=|z|\cdot\left(\cos(\vartheta)+i\cdot\sin(\vartheta)\right)$$
$$w=|w|\cdot\left(\cos(\varphi)+i\cdot\sin(\varphi)\right)$$

$$\boxed{z\cdot w}=\underbrace{|z|\cdot|w|}_{|z\cdot w|}\cdot\Big(\underbrace{\left(\cos(\vartheta)\cdot\cos(\varphi)-\sin(\vartheta)\cdot\sin(\varphi)\right)}_{\cos(\vartheta+\varphi)}$$
$$+\,i\underbrace{\left(\cos(\vartheta)\cdot\sin(\varphi)+\sin(\vartheta)\cdot\cos(\varphi)\right)}_{\sin(\vartheta+\varphi)}\Big)$$

$$= \boxed{|z \cdot w| \cdot \left(\cos(\vartheta+\varphi) + i \cdot \sin(\vartheta+\varphi)\right)}$$

$$\Longrightarrow \arg(z \cdot w) = \arg(z) + \arg(w).$$



$$E(\vartheta) = \cos(\vartheta) + i \cdot \sin(\vartheta)$$ (numero di modulo 1 e argomento ϑ)

$$\boxed{E(\vartheta) \cdot E(\varphi) = E(\vartheta+\varphi)}$$

Ricordo: $A(x) = a^x$

$$\boxed{A(x) \cdot A(y) = A(x+y)} = a^x \cdot a^y = a^{x+y}$$

Idea: potrebbe essere che E è anch'essa una esponenziale?

$E(\vartheta) = \cos(\vartheta) + i \cdot \sin(\vartheta)$

$\Longrightarrow$ certamente non è $E(\vartheta) = a^{\vartheta}$ con $a \in \mathbb{R}$

<u>Fatto</u>: si può identificare E con l'esponenziale di base e^{i}; cioè <u>conveniamo</u>

$$e^{i\vartheta} = \cos(\vartheta) + i \cdot \sin(\vartheta)$$

$$\left(e^{i}\right)^{\vartheta} = e^{i\vartheta}$$

<u>Visto</u>: $e^{i(\vartheta+\varphi)} = e^{i\vartheta} \cdot e^{i\varphi}$

Formula di Eulero: $e^{i\pi} + 1 = 0$.

Cioè $e^{i\pi} = -1$ infatti

$$e^{i\pi} = \cos(\pi) + i \cdot \sin(\pi) = -1 + i \cdot 0 = -1.$$