Geometria 25/5/22

14.1.11 $\quad C = \{ e^{3x-y} + \sin(x+2y) = 1 \}$

curva vicino a $(0,0)$ e t_p

$(0,0) \in C$

$$\frac{\partial}{\partial x}(\dots) = 3e^{\dots} + \cos(\dots) \qquad 4$$

$$\frac{\partial}{\partial y}(\dots) = -e^{\dots} + 2\cos(\dots) \qquad 1$$

$t_p = \mathrm{Span}\begin{pmatrix} -1 \\ 4 \end{pmatrix}$

14.1.12 $\quad C = \{ 2xy^2 + 3x^2y = 1 \}$

punti di C con t_p $\begin{pmatrix} -1 \\ 4 \end{pmatrix}$

Impongo t_p: $\quad (-1)\cdot(2y^2+6xy) + (4)\cdot(4xy+3x^2) = 0$

$$y^2 - 5xy - 6x^2 = 0$$

$$(y-6x)(y+x) = 0$$

$y = -x$: $\quad 2x^3 - 3x^3 = 1 \qquad x^3 = -1 \qquad x = -1$

$(-1, 1)$

$y = 6x$ ...

14.2.4(b) $\quad \kappa$ en $t=2$ pour $\alpha(t) = \begin{pmatrix} 4t - t^5 \\ 4t^2 + t^4 \end{pmatrix}$

$$\alpha' = \begin{pmatrix} 4 - 5t^4 \\ 8t + 4t^3 \end{pmatrix} \qquad \alpha'' = \begin{pmatrix} -20t^3 \\ 8 + 12t^2 \end{pmatrix}$$

$$\begin{pmatrix} -76 \\ 48 \end{pmatrix} \qquad \begin{pmatrix} -160 \\ 56 \end{pmatrix}$$

$$\kappa = \frac{\det \begin{pmatrix} -76 & -160 \\ 48 & 56 \end{pmatrix}}{\left(76^2 + 48^2\right)^{3/2}}$$

14.3.2 $\quad \kappa$ et τ pour $\alpha(s) = \begin{pmatrix} 1 + \cos(s) \\ 1 - \sin(s) \\ \cos(2s) \end{pmatrix}$

$$\alpha' = \begin{pmatrix} -\sin(s) \\ -\cos(s) \\ -2\sin(2s) \end{pmatrix} \qquad \alpha'' = \begin{pmatrix} -\cos(s) \\ \sin(s) \\ -4\cos(2s) \end{pmatrix} \qquad \alpha''' = \begin{pmatrix} +\sin(s) \\ -\cos(s) \\ 8\sin(2s) \end{pmatrix}$$

$$\alpha' \wedge \alpha'' = \quad \dots \quad = \begin{pmatrix} 4\cos^3(s) \\ 4\sin^3(s) \\ -1 \end{pmatrix}$$

$$\kappa = \frac{\|\alpha' \wedge \alpha''\|}{\|\alpha'\|^3} = \dots$$

$$\tau = \frac{\langle \alpha' \wedge \alpha'' \mid \alpha''' \rangle}{\|\alpha' \wedge \alpha''\|^2} = \dots$$

14.3.3 (b) Frénet, $\mathcal{K}$, τ in $s=0$ per

$$\alpha(s) = \begin{pmatrix} 1+2s+s^2-s^3 \\ e^{2s} \\ \sin(s) \end{pmatrix}$$

$$\alpha' = \begin{pmatrix} 2+2s-3s^2 \\ 2e^{2s} \\ \cos(s) \end{pmatrix} \qquad \alpha'' = \begin{pmatrix} 2-6s \\ 4e^{2s} \\ -\sin(s) \end{pmatrix} \qquad \alpha''' = \begin{pmatrix} -6 \\ 8e^{2s} \\ -\cos(s) \end{pmatrix}$$

$$\underbrace{\begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} \qquad \begin{pmatrix} 2 \\ 4 \\ 0 \end{pmatrix}}_{\text{ortogon.}} \qquad \begin{pmatrix} -6 \\ 8 \\ -1 \end{pmatrix}$$

ortogon. t, n

$$t = \frac{1}{3}\begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$$

$$\alpha' \wedge \alpha'' = 2\begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} \wedge \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} = 2\begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix}$$

$$b = \frac{1}{3}\begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix} \qquad n = b \wedge t = \frac{1}{9}\begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix} \wedge \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} = \frac{1}{9}\begin{pmatrix} -3 \\ 6 \\ -6 \end{pmatrix}$$

$$= -\frac{1}{3}\begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$$

$$\mathcal{K} = \dots \qquad \tau = \dots$$

15.1.1 (d)

$$\int_\alpha \left(2y^2 z\,dx - 3xz^2\,dy\right)$$

$\alpha : [a,b] \to \mathbb{R}^n$

$$\alpha : \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

$$\omega = f_1\,dx_1 + \dots + f_n\,dx_n$$

$$\int_\alpha \omega = \int_a^b \left(f_1(\alpha(t))\,x_1'(t) + \dots + f_n(\alpha(t))\,x_n'(t)\right)dt$$

$\alpha : [0,1] \to \mathbb{R}^3 \qquad \alpha(t) = \begin{pmatrix} t - e^t \\ 1 - e^{2t} \\ t^2 + 3e^t \end{pmatrix}$

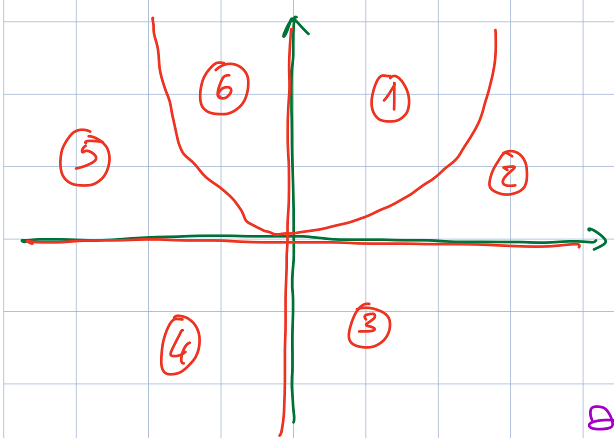
$$\int_0^1 \Big(2 \cdot (1-e^{2t})^2 \cdot (t^2+3e^t)\,(1-e^t)$$

$$-3\,(t-e^t)\cdot(t^2+3e^t)^2 \cdot (-2e^{2t}) \Big)\,dt$$

15.2.3(c) Per quali k la

$$\frac{ky-12x^2}{5xy-4x^3}\,dx + \frac{10y-4x^2}{5y^2-4x^2y}\,dy \qquad \text{è esatta?}$$

den $\quad x(5y-4x^2) \qquad y(5y-4x^2)$



ω def su $\Omega_1 \cup \ldots \cup \Omega_6$

ognuno di essi è semplicemente connesso

$$\frac{\partial}{\partial y}\left(\frac{ky-12x^2}{5xy-4x^3}\right) = \frac{\partial}{\partial x}\left(\frac{10y-4x^2}{5y^2-4x^2y}\right)$$

[...]

$$\frac{ky-12x^2}{5xy-4x^3}\,dx+\frac{10y-4x^2}{5y^2-4x^2y}\,dy$$

$$\frac{ky^2-12x^2y}{yx(5y-4x^2)}\,dx+\frac{10xy-4x^3}{xy(5y-4x^2)}\,dy$$

$$d\left(xy(5y-4x^2)\right)=d\left(5xy^2-4x^3y\right)$$

$$=\frac{5y^2-12x^2y}{xy(5y-4x^2)}\,dx+\frac{10xy-4x^3}{xy(5y-4x^2)}\,dy$$

$$\boxed{k=5}$$

$$g(x,y)=\ln\left(f(x,y)\right)$$

$$dg=\frac{\partial f/\partial x}{f}\,dx+\frac{\partial f/\partial y}{f}\,dy$$

$\lg(1+\cos^7(5x))$ $\quad g(y)$

15.3.4, $\quad Q=[0,1]\times[0,1]$ $\quad \displaystyle\int_{\partial Q}\left(x^2+y^2\right)dx+\left(2xy+e^y\right)dy$

$$=\int_Q(-2y+2y)\,dx\,dy=0$$

15.3.6 $\displaystyle\int_\alpha \cos(y)\,(dx+dz) - (x+z)\cdot\sin(y)\,dy$

$\alpha(t) = \begin{pmatrix} \vdots \end{pmatrix}$ $\qquad \alpha: [0,\pi] \to \mathbb{R}^3.$

$U(x,y,z) = \cos(y)\cdot(x+z)$

$U(\alpha(\pi)) - U(\alpha(0)) = \ldots$

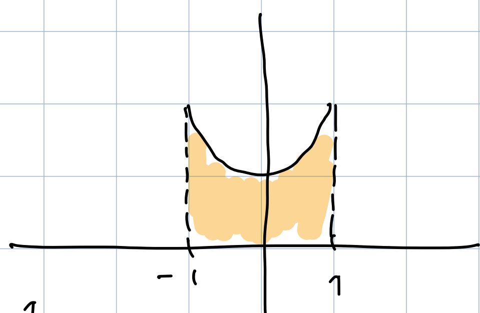
$U: \mathbb{R}^n \to \mathbb{R}$

$dU = \frac{\partial U}{\partial x_1}\cdot dx_1 + \ldots + \frac{\partial U}{\partial x_n} dx_n$

$\int_\alpha dU = U(\alpha(b)) - U(\alpha(a))$

15.3.8 $\displaystyle\int_{\partial A} x\,dy$ $\qquad A = \{ |x| \leq 1,\ 0 \leq y \leq 1+x^2 \}$

$= \displaystyle\int_A dx\,dy = \text{Area}(A)$



$= \displaystyle\int_{-1}^{1} \left(\int_0^{1+x^2} dy \right) dx = \int_{-1}^{1} (1+x^2)\,dx$

$= \left(x + \frac{1}{3}x^3\right)\Big|_{-1}^{1} = 1 + \frac{1}{3} - \left(-1-\frac{1}{3}\right) = \frac{8}{3}$

15.3.10 (i) $\omega(x,y) = \boxed{(2x + y\cdot\sin(xy))}\,dx + (x\cdot\sin(xy) - 1)\,dy$

(chiusa : ...)

$$U(x,y) = x^2 - \cos(xy) + g(y)$$

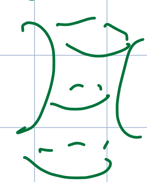
$\partial/\partial y$: 0 $\quad$ $x\sin(xy)$ $\quad$ $-y$

~~$\boxed{7}$~~ $\quad \boxed{5} \quad 5$

$x^2+(a-1)y^2+(a+2)z^2+a=0$ $\qquad$ iperb. 1 falda per $a=\dots$?

$$\begin{pmatrix} 1 & & & \\ & a-1 & & \\ & & a+2 & \\ & & & a \end{pmatrix}$$

$x^2+y^2=1+z^2$



$x^2+y^2-z^2-1=0$

autoval. di (Q) A

$(+\ +\ -)\ -$ $\iff$ $d_4>0$

$(+\ -\ -)\ +$ $\qquad$ Q non concordi

	<-2	-2 … 0	0 … 1	>1
$d_4>0$	NO	OK	NO	OK
Q nonconc.	NO	OK	OK	NO

⑥ autoval. $\begin{pmatrix} 2 & 1 \\ 10 & -1 \end{pmatrix}$ + base diag.

$\lambda_1 + \lambda_2 = 1$

$\lambda_1 \cdot \lambda_2 = -12$

$\lambda_1 = 4 \quad \begin{cases} 2x + y = 4x \\ \text{—} \end{cases} \qquad v_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$\lambda_2 = -3 \quad \begin{cases} 2x + y = -3x \\ \text{—} \end{cases} \qquad v_2 = \begin{pmatrix} -1 \\ 5 \end{pmatrix}$

⑦ $\underbrace{\begin{pmatrix} 0 & -1 & 2 \\ 1 & 0 & -2 \\ -2 & 2 & 0 \end{pmatrix}}_{\text{antisimm.}}$ coniugata a $\begin{pmatrix} 0 & a & 0 \\ -a & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ per $a = \ldots$?

tramite una ortogonale

a t.c. gli autoval di M sono $0, \pm ia$

In generale:

$$\det\left(t \cdot I_3 - \begin{pmatrix} 0 & \alpha & \beta \\ -\alpha & 0 & \gamma \\ -\beta & -\gamma & 0 \end{pmatrix}\right)$$

$$= \det\begin{pmatrix} t & -\alpha & -\beta \\ \alpha & t & -\gamma \\ \beta & \gamma & t \end{pmatrix}$$

$$= t^3 + \alpha\beta\gamma - \alpha\beta\gamma + t\beta^2 + t\alpha^2 + t\gamma^2$$

$$= t\left(t^2 + (\alpha^2 + \beta^2 + \gamma^2)\right)$$

$\lambda_1 = 0 \qquad \lambda_{2,3} = \pm i \sqrt{\alpha^2+\beta^2+\gamma^2}$

Per noi $\qquad a = \pm \sqrt{1+4+4} = \pm 3$

$\boxed{7}$ (1) autoval $\begin{pmatrix} 2+i & 1 \\ -i & 1 \end{pmatrix}$ e base diago...

$tr = \boxed{3+i} \qquad det = \boxed{2(1+i)}$

$t^2 - (3+i)t + 2(1+i)$

$\Delta = 9+6i-1-8-8i = -2i = (1-i)^2$

$\lambda_{1,2} = \dfrac{3+i \pm (1-i)}{2} \quad \begin{cases} 2 \\ 1+i \end{cases}$

$v_1 : \quad -ix+y = 2y \qquad \begin{pmatrix} 1 \\ -i \end{pmatrix}$

$v_2 : \quad (2+i)x+y = (1+i)x \qquad \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

(2) trovare $p_A(t)$ $\quad A \in M_{3\times 3}$ $\quad$ sapendo

$tr(A) = 4 \qquad det(A) = -3 \qquad p_A(-1) = -7$

Convenzione : $\quad p_A(t) = det(t.I_n - A)$

$= t^n + \ldots\ldots$

$$p_A(t) = t^3 - 4\cdot t^2 + c\cdot t + 3$$

$$-1 - 4 - c + 3 = -7$$

$$c = 5$$

(3) Trovare tutti i $v \in \mathbb{C}^2$

$$\|v\| = 1, \quad \underbrace{v \perp \begin{pmatrix} 1+i \\ -i \end{pmatrix}, \quad v_2 \in i\cdot\mathbb{R}}_{\text{preservate sostituendo } v \text{ con } k\cdot v \quad k \in \mathbb{R}}$$

Cerco $v = \begin{pmatrix} x+iy \\ i \end{pmatrix}$ (III ✓ ma posso cambiare segno)

$$\left\langle \begin{pmatrix} x+iy \\ i \end{pmatrix} \middle| \begin{pmatrix} 1+i \\ -i \end{pmatrix} \right\rangle = 0$$

$$(x+iy)(1-i) + i\cdot(i) = 0$$

$$x + iy - ix + y - 1 = 0$$

$$\begin{cases} x + y = 1 \\ x = y \end{cases} \qquad x = y = \frac{1}{2}$$

$$v = \pm \frac{1}{\sqrt{6}} \begin{pmatrix} 1+i \\ 2i \end{pmatrix}$$

(4) Per quali z le $\begin{pmatrix} 1+i & 2i \\ 2 & z \end{pmatrix}$ ammette base ortonormale autovettori?

$\Updownarrow$

è normale

$(M^* \cdot M = M \cdot M^*)$

$$\begin{pmatrix} 1+i & 2i \\ 2 & z \end{pmatrix}\begin{pmatrix} 1-i & 2 \\ -2i & \bar{z} \end{pmatrix} = \begin{pmatrix} 1-i & 2 \\ -2i & \bar{z} \end{pmatrix}\begin{pmatrix} 1+i & 2i \\ 2 & z \end{pmatrix}$$

$$\begin{cases} 6 = 6 \quad \checkmark \\ 2(1+i) + 2i\bar{z} = (1-i)2i + 2z \\ 2(1-i) - 2iz = -2i(1+i) + 2\bar{z} \quad \checkmark \\ 4 + |z|^2 = 4 + |z|^2 \quad \checkmark \end{cases}$$

$$1+i+i(x-iy) = (1-i)i + (x+iy)$$

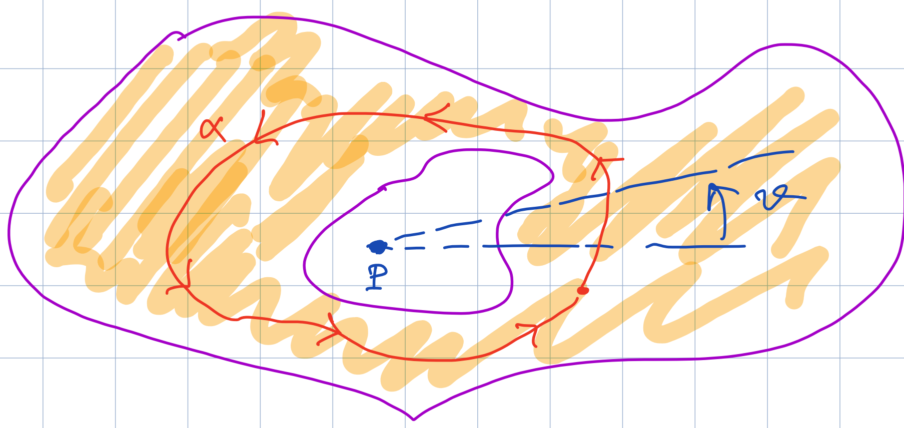
$$\cancel{1}+\cancel{i}+ix+y = \cancel{i}+\cancel{1}+x+iy$$

$$z = t\cdot(1+i)$$

(7) Stabilire se esistono forme chiuse non esatte su

$$\Omega = \{ \max\{|x|,|y|\} \leq 4 \,,\ |x|+|y| \geq 1 \}.$$

chiusa su Ω sempl. connesso $\Longrightarrow$ esatta se Ω non è sempl. connesso



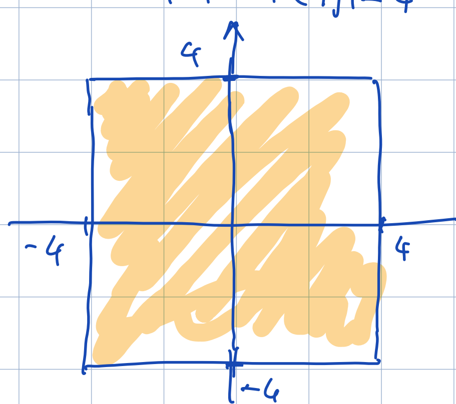
ϑ non ben def ma $d\vartheta$ sì su $\mathbb{R}^2 \setminus \{P\} \supset \Omega$

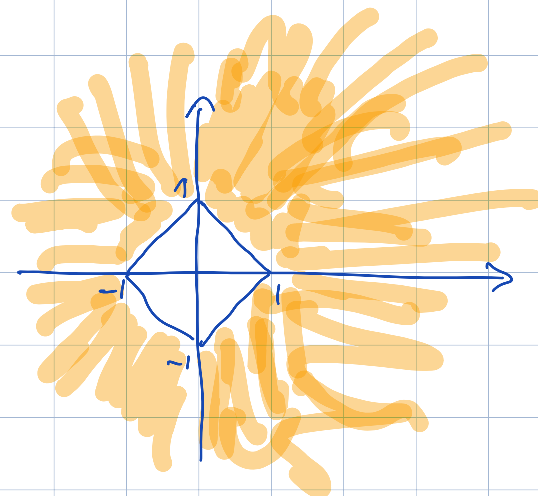
$d\vartheta$ chiusa, $\int_\alpha d\vartheta = 2\pi \neq 0$.

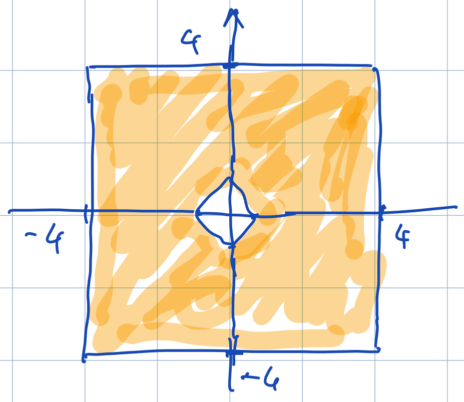
Morale: $\left(\begin{array}{l}\exists \text{ forma chiusa non} \\ \text{esatta su } \Omega\end{array}\right) \Longleftrightarrow \Omega$ ha buchi.

$$\Omega = \{ \max\{|x|,|y|\} \leq 4 \,, \quad |x|+|y| \geq 1 \}.$$

$|x| \leq 4$ e $|y| \leq 4$







Sì, esistono