# Geometria 19/5/22

$\omega$ su $\Omega \subset \mathbb{R}^2$ aperto $\qquad \omega = f \cdot dx + g \cdot dy$

$\qquad f, g : \Omega \to \mathbb{R}$

$\alpha : [a,b] \to \Omega$

$$\int_\alpha \omega = \int_a^b \Big( f(\alpha(t)) \cdot X'(t) + g(\alpha(t)) \cdot Y'(t) \Big)\, dt \qquad \alpha = \begin{pmatrix} X \\ Y \end{pmatrix}$$

$U : \Omega \to \mathbb{R} \qquad dU = \frac{\partial U}{\partial x} \cdot dx + \frac{\partial U}{\partial y} \cdot dy$

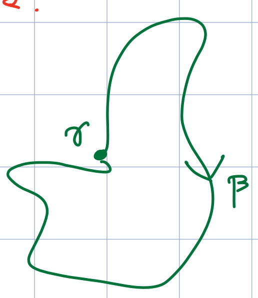
$$\int_\alpha dU = U(\alpha(b)) - U(\alpha(a))$$

$\omega = dU$ (esatta) $\Rightarrow \int_\alpha \omega$ dipende solo da estremi di $\alpha$ $\quad \forall \alpha$

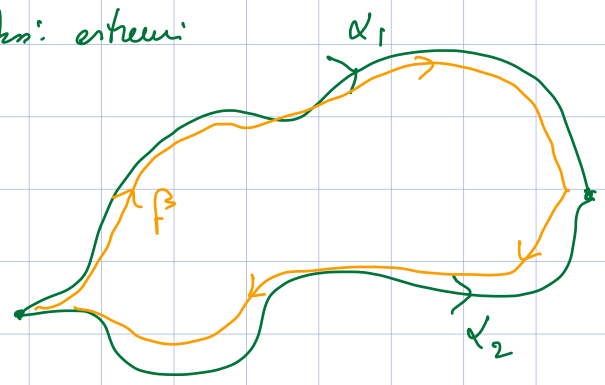
$\Updownarrow$

$\int_\beta \omega = 0 \quad \forall \beta$ curva chiusa.

$\Downarrow \quad \int_\beta \omega = \int_\gamma \omega = 0 \qquad \gamma =$ costante



$\Uparrow$ $\alpha_1, \alpha_2$ con stessi estremi




$$\int_\beta \omega = 0 \qquad \beta = \alpha_1 \cup (-\alpha_2)$$

$$\Rightarrow \int_\beta \omega = \int_{\alpha_1} \omega - \int_{\alpha_2} \omega$$

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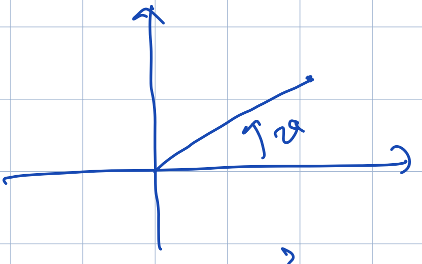
$$d(f\,dx + g\,dy) = \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dx\,dy$$

$\omega = dU \Rightarrow d\omega = 0$

esatta $\qquad$ chiusa

Q: chiusa $\overset{?}{\Rightarrow}$ esatta

No in gen: $\omega = d\vartheta$




$\mathbb{R}^2 \setminus \{0\}$

$\Lambda \subset \Omega$ , $\Lambda \cup \partial\Lambda \subset \Omega$ , $\Lambda$ limitato

$\omega$ su $\Omega$ $\qquad$ $\partial\Lambda$ = unione curve

$$\int_{\partial\Lambda} \omega = \int_{\Lambda} d\omega \qquad \text{(Gauss-Green)}$$

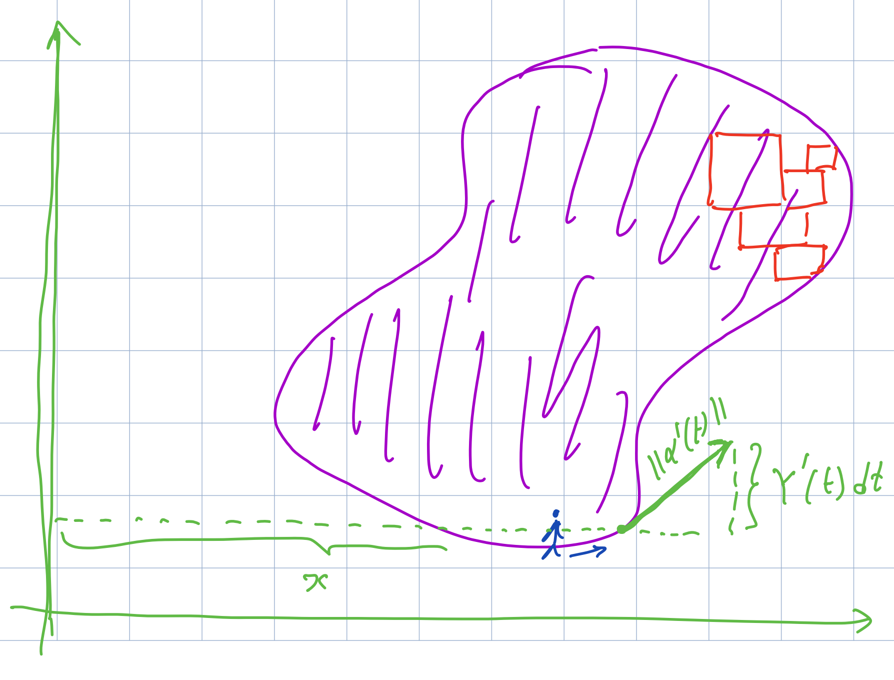
<u>Cor</u> : $\Lambda \subset \mathbb{R}^2$ limitato, $\partial\Lambda = \bigcup$ curve

$$A(\Lambda) = \int_{\partial\Lambda} -y\,dx = +\int_{\partial\Lambda} x\,dy$$

dunque l'area di $\Lambda$ si può misurare con operazioni solo su $\partial\Lambda$.

$$d(-y\,dx) = -dy\,dx = dx\,dy$$
$$d(x\,dy) = dx\,dy$$

$$\int_{\partial\Lambda} (\dots) = \int_{\Lambda} dx\,dy = A(\Lambda)$$



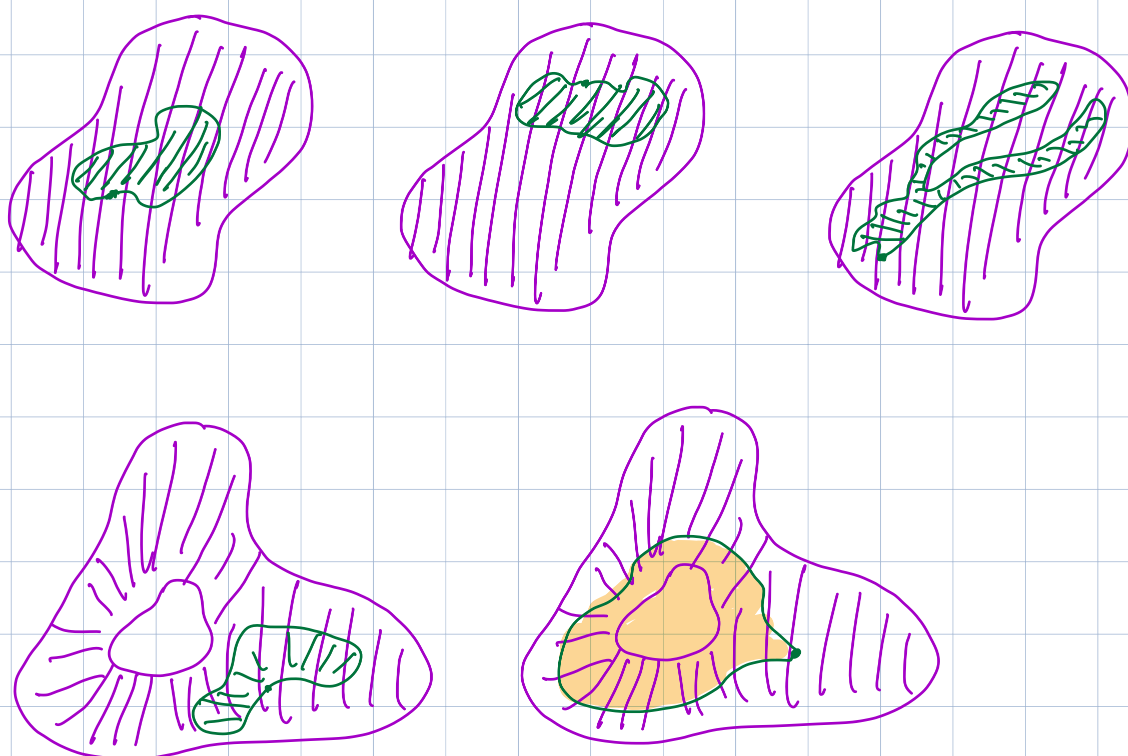

$$\int_{\partial\Lambda} x\,dy$$

$$d\omega = 0 \;\cancel{\Longrightarrow}\; \omega = dU$$

non sempre

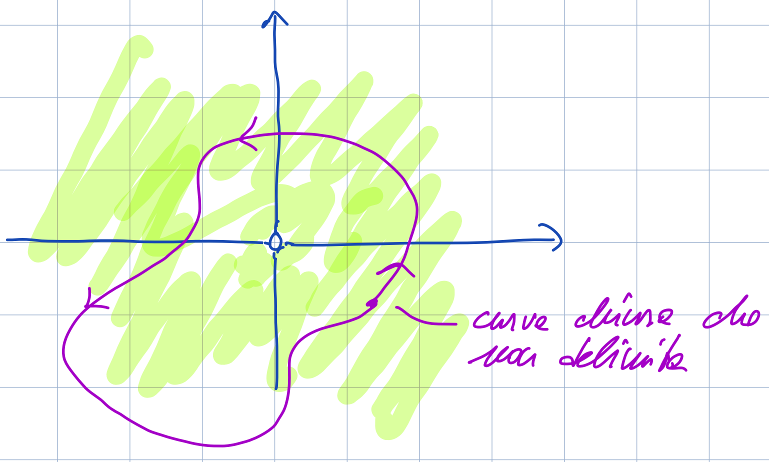
**Def:** chiamo $\Omega$ semplicemente connesso se

- ogni curva semplice chiusa $\alpha$ contenuta in $\Omega$ è bordo di un sottoinsieme $\Lambda \subset \Omega$ ($\alpha = \partial\Lambda$)
- "senza buchi"



$\alpha$ non delimita in $\Omega$ un sottoinsieme (quello che delimita in $\mathbb{R}^2$ non è contenuto in $\Omega$)

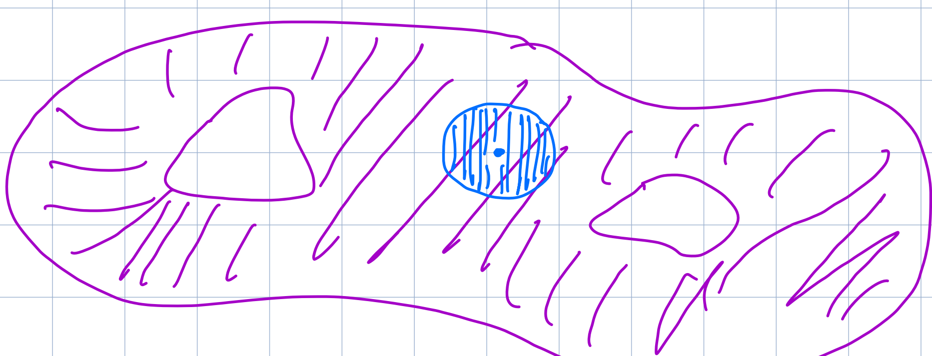
$d\vartheta$ chiusa non esatta: $\mathbb{R}^2 \setminus \{0\}$



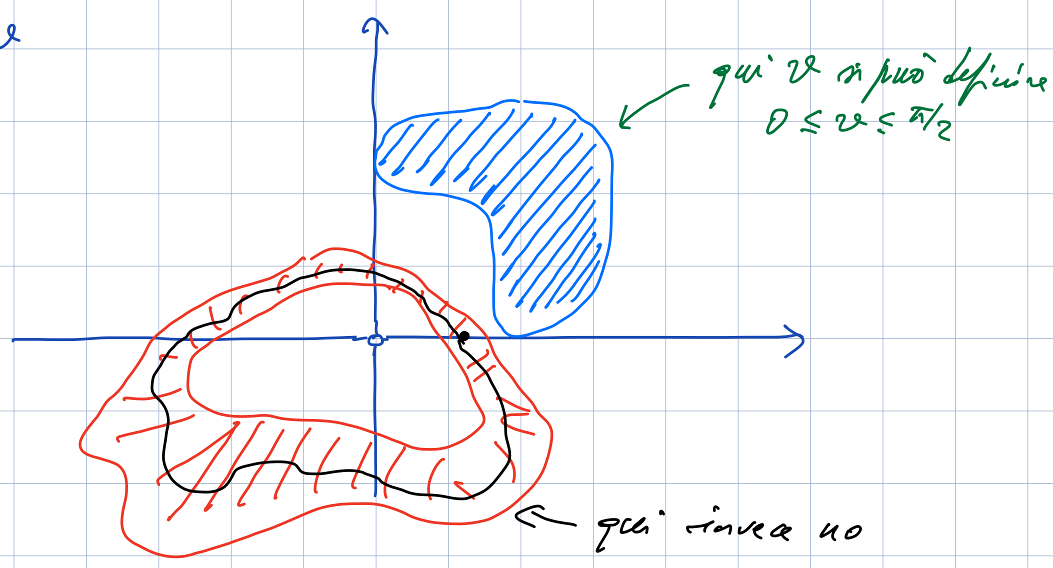

Teo: se $\Omega$ è semplicemente connesso e $\omega$ su $\Omega$ è $1$-forma chiusa allora $\omega$ è esatta su $\Omega$.

Su un insieme senza buchi le forme chiuse hanno potenziale.

Cor: ogni forma chiusa è localmente esatta (vicino a ogni punto ammette un potenziale).

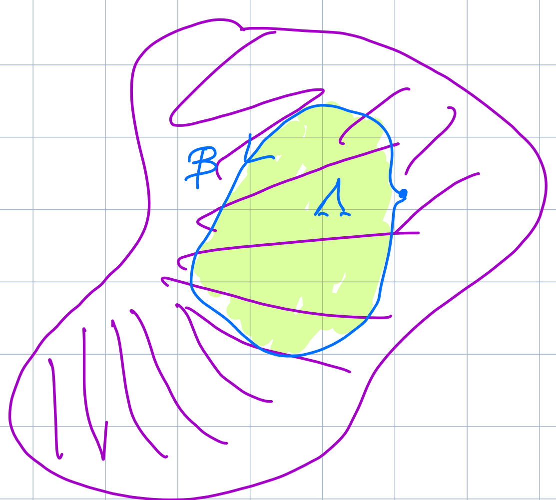


$\omega = d\vartheta$




$\omega$ chiusa su $\Omega$ sempl. connesso $\Longrightarrow \omega = dU$

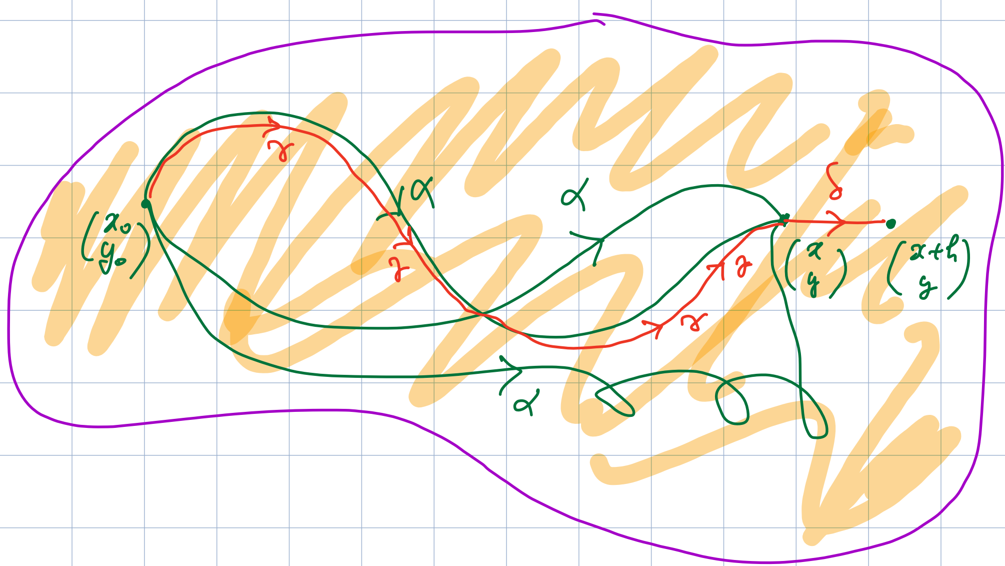
Dim: Notiamo che $\int_\beta \omega = 0 \quad \forall \beta$ chiusa:




$$\int_\beta \omega = \int_{\partial\Lambda} \omega = \int_\Lambda \underbrace{d\omega}_{=0}$$

$\Longrightarrow \int_\alpha \omega$ dipende solo da estremi.

Scelgo $\binom{x_0}{y_0} \in \Omega$ e pongo $U\binom{x}{y} = \int_\alpha \omega$ dove $\alpha$ è qualsiasi curva tra $\binom{x_0}{y_0}$ e $\binom{x}{y}$




Devo vedere che se $\omega = f \cdot dx + g \cdot dy$ ho

$$\frac{\partial U}{\partial x} = f, \qquad \frac{\partial U}{\partial y} = g$$

Calcolo $U\binom{x+h}{y}$ con $\gamma = \alpha \cup \delta$

← segmento orizzontale.

$$\frac{\partial U}{\partial x}\binom{x}{y} = \lim_{h \to 0} \frac{U\binom{x+h}{y} - U\binom{x}{y}}{h}$$

$$= \lim_{h \to 0} \frac{1}{h} \left( \cancel{\int_\alpha \omega} + \int_\delta \omega - \cancel{\int_\alpha \omega} \right)$$

$$\delta(t) = \binom{x+t}{y} \qquad 0 \le t \le h$$

$$= \lim_{h \to 0} \frac{1}{h} \int_0^h \left( f(x+t, y) \cdot 1 \cdot dt + g(x+t, y) \cdot 0 \cdot dt \right)$$

$$= \lim_{h \to 0} \frac{1}{h} \int_0^h f(x+t, y)\, dt = f(x, y).$$

$$\frac{\partial U}{\partial y} = \dots = g.$$

■

Libro arancione.

1 ① $\begin{pmatrix} k^2 & k+1 \\ 0 & k+2 \end{pmatrix}$ diago per $k = \dots$

$\lambda_1 = k^2 \qquad \lambda_2 = k+2$

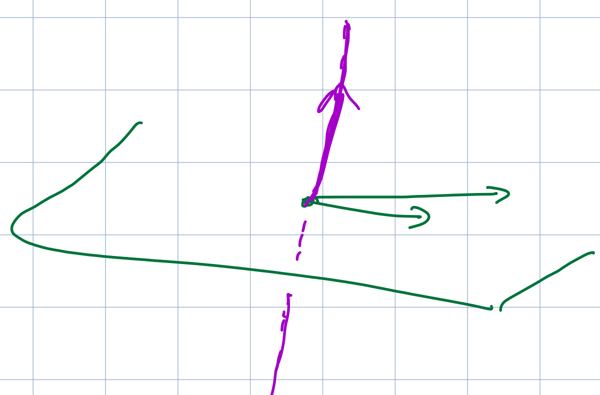
diago per $\lambda_1 \neq \lambda_2$.

$k^2 = k+2$ per $k = -1$, $k = 2$.

$k = -1 \quad \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad k = 2, \begin{pmatrix} 4 & 3 \\ 0 & 4 \end{pmatrix}$

$$\boxed{k \neq 2}$$

② Trovare i vett. $\perp$ Span$\left(\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 4 \\ 1 \\ -5 \end{pmatrix}\right)$ con somma componenti 1.



$$\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \wedge \begin{pmatrix} 4 \\ 1 \\ -5 \end{pmatrix} = \begin{pmatrix} -9 \\ 1 \\ -7 \end{pmatrix}$$

$-9 + 2 + 7$ ✓

$-36 + 1 + 35$ ✓

$$\boxed{\frac{1}{15}\begin{pmatrix} 9 \\ -1 \\ 7 \end{pmatrix}}$$

(4) $x^2+4xy+(k-1)y^2+4x+2ky+8=0$ ellisse per $k = \dots$

$$\begin{pmatrix} 1 & 2 & 2 \\ 2 & k-1 & k \\ 2 & k & 8 \end{pmatrix}$$

ellisse : $d_2>0,\ d_1 \cdot d_3<0.$

$$\begin{cases} k-1-4>0 \\ 8(k-1)+4k+4k-4(k-1)-32-k^2<0 \end{cases}$$

$$\begin{cases} k>5 \\ k^2-12k+36>0 \end{cases} \qquad \begin{cases} k>5 \\ (k-6)^2>0 \end{cases}$$

$$\boxed{k>5,\ k\neq 6}$$

(5) Tipo affine di $z^2-8xy+2xz-4yz+3x+4z+1=0$

Parab. iperb. $\longrightarrow$ 0

$$\begin{pmatrix} 0 & -4 & 1 & 3/2 \\ -4 & 0 & -2 & 0 \\ 1 & -2 & 1 & 2 \\ 3/2 & 0 & 2 & 1 \end{pmatrix}$$

$d_2 < 0$

$d_3 = 0+8+8$
$-0-16-0=0$

Parab. iperb. $\longrightarrow$ 2

$$d_4 = \begin{vmatrix} -2 & 0 & -1 & -5/2 \\ -4 & 0 & -2 & 0 \\ 1 & -2 & 1 & 2 \\ 3/2 & 0 & 2 & 1 \end{vmatrix}$$

$$= -(-2) \cdot \begin{vmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{vmatrix} = \dots \neq 0$$

Parab. iperb. $\longrightarrow 3$

(6) sgn $Hf(0,0)$ $\qquad f(x,y) = \frac{3}{2}x^2 + \frac{1}{2}y^2 + e^{x-y} \cdot \cos(x+y)$

$$\frac{\partial f}{\partial x} = 3x + 0 + e^{x-y} \cdot \cos(x+y) + e^{x-y} \cdot (-\sin(x+y))$$

$$\frac{\partial^2 f}{\partial x^2} = \underset{3}{3} + \underset{+1}{e^{\dots} \cdot \cos(\dots)} + \underset{+0}{e^{\dots}(-\sin(\dots))} + \underset{+0}{e^{\dots} \cdot (-\sin(\dots))} + \underset{-1}{e^{\dots}(-\cos(\dots))}$$

$$\frac{\partial^2 f}{\partial x \partial y} = \underset{0}{0} + \underset{-1}{(-e^{\dots}) \cdot \cos(\dots)} + \underset{0}{e^{\dots} \cdot (-\sin(\dots))} + \underset{0}{(-e^{\dots}) \cdot (-\sin(\dots))}$$
$$+ \underset{-1}{e^{\dots} \cdot (-\cos(\dots))}$$

$$\frac{\partial f}{\partial y} = y - e^{x-y} \cdot \cos(x+y) + e^{x-y} \cdot (-\sin(x+y))$$

$$\frac{\partial^2 f}{\partial y^2} = \underset{1}{1} + \underset{1}{e^{\dots} \cdot \cos(\dots)} - \underset{0}{e^{\dots}(-\sin(\dots))} - \underset{0}{e^{\dots}(-\sin(\dots))}$$
$$+ \underset{-1}{e^{\dots} \cdot (-\cos(\dots))}$$

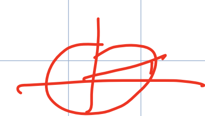
$$Hf = \begin{pmatrix} 3 & -2 \\ -2 & 1 \end{pmatrix}$$

$d_1 > 0 \qquad d_2 < 0 \qquad + \quad -$

⑦ $\displaystyle\int_\alpha \left(3x^2y^2\,dx + 2x^3y\,dy\right)$ $\qquad \alpha(t) = \begin{pmatrix}\cos\left(\frac{\pi}{4}\cdot t^2\right)\\ \cos\left(\frac{\pi}{6}\cdot t^2\right)\end{pmatrix}$ $\quad t\in[0,1]$

$$\int_0^1 \left(3\cos^2\left(\frac{\pi}{4}\cdot t^2\right)\cdot\cos^2\left(\frac{\pi}{6}\cdot t^2\right)\cdot\left(-\sin\left(\frac{\pi}{4}t^2\right)\right)\cdot\frac{\pi}{2}\cdot t \right.$$
$$\left. + \;\ldots\ldots \right) dt$$

$\alpha$ chiusa? $\qquad \alpha(0) = \begin{pmatrix}1\\1\end{pmatrix} \qquad \alpha(1) = \begin{pmatrix}1/\sqrt{2}\\ \sqrt{3}/2\end{pmatrix}$ NO



$3x^2y^2\,dx + 2x^3y\,dy$ esatta?

def. su $\mathbb{R}^2$: basta chiusa

$$\frac{\partial}{\partial x}\left(2x^3y\right) = \frac{\partial}{\partial y}\left(3x^2y^2\right)$$
$$6x^2y \qquad\qquad 6x^2y$$

Sì

$U(x,y) = x^3y^2 \qquad \displaystyle\int_\alpha \omega = U\begin{pmatrix}1/\sqrt{2}\\ \sqrt{3}/2\end{pmatrix} - U\begin{pmatrix}1\\1\end{pmatrix}$

$$= \frac{1}{2\sqrt{2}}\cdot\frac{3}{2} - 1 = \frac{3}{8}\sqrt{2} - 1$$

3 ① autoval di $\begin{pmatrix} 1 & 2 \\ 1+i & i \end{pmatrix}$ e base che diagonalizza.

$tr = 1+i \qquad det = i - 2 - 2i = -i-2$

$p_A(z) = z^2 - (1+i)z - (i+2)$

$\Delta = 1+2i-1+4i+8 = 6i+8$

$\sqrt{\Delta} = a+ib$

$\begin{cases} a^2-b^2 = 8 \\ ab = 3 \end{cases} \qquad \begin{matrix} a=3 \\ b=1 \end{matrix}$

$\lambda_{1,2} = \dfrac{1+i \pm (3+i)}{2} \begin{matrix} \nearrow \\ \searrow \end{matrix} \boxed{\begin{matrix} 2+i \\ \\ -1 \end{matrix}}$ $\qquad A = \begin{pmatrix} 1 & 2 \\ 1+i & i \end{pmatrix}$

$v_1: \begin{cases} x+2y = (2+i)x \\ \cancel{(1+i)x + iy = (2+i)y} \end{cases} \qquad \boxed{v_1 = \begin{pmatrix} 2 \\ 1+i \end{pmatrix}}$

$v_2: \begin{cases} x+2y = -x \\ \dots \end{cases} \qquad \boxed{v_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}}$

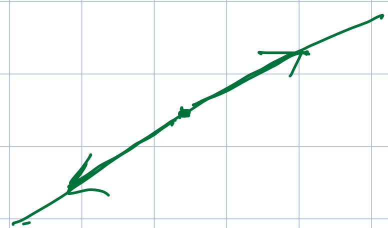
② Trovare tutti i $v \in \mathbb{R}^3$

- unitari
- somma comp. nulle
- $\perp \begin{pmatrix} 4 \\ -5 \\ 3 \end{pmatrix}$

preservate moltiplicando per $k \in \mathbb{R}$

Dunque impongo prima loro e poi normalizzo.

$$\begin{cases} x+y+z=0 \\ 4x-5y+3z=0 \end{cases} \qquad \begin{pmatrix}1\\1\\1\end{pmatrix} \wedge \begin{pmatrix}4\\-5\\3\end{pmatrix} = \begin{pmatrix}8\\1\\-9\end{pmatrix}$$



$$\boxed{\pm \frac{1}{\sqrt{64+1+81}} \begin{pmatrix}8\\1\\-9\end{pmatrix}}$$

③ Trovare intersezione in $\mathbb{P}^2(\mathbb{R})$ di

$$\{ [1-t : 3t : 7-t] : t \in \mathbb{R} \}$$
$$\{ [1-t : 1+t : -2t] : t \in \mathbb{R} \}$$

$$\cancel{(1-t, 3t, 7-t) = (1-t, 1+t, -2t)}$$