Geometria 28/4/22

Prop: se $A \in M_{n\times n}(\mathbb{R})$ simm. $\det(A) \neq 0$ con p autoval pos e q autoval neg. $(p+q=n)$ allora
$p = \max\{\dim(W) : \langle w|w\rangle_A > 0 \quad \forall w \in W \setminus \{0\}\}$
$q = \max\{ \quad . \; . \quad\quad < 0 \quad . \; . \; . \quad \}$.

Dim: Teo spettrale $\Rightarrow W_0 = \bigoplus$ autsp. rel. a autoval >0
$U_0 = \bigoplus \quad . \; . \; . \; . \quad\quad < 0$
$\dim(W_0) = p$, $\dim(U_0) = q$ e soddisfano ...
Se ci fosse W di $\dim > p$ su cui $\langle \cdot | \cdot \rangle_A$ è def >0 allora $W \cap U_0 \neq \{0\} \Rightarrow$ assurdo... $\blacksquare$

Cor: se $\det(A) \neq 0$ il numero di autoval pos./neg. non cambia sostituendo A con ${}^tM \cdot A \cdot M$, $\det(M) \neq 0$.

Dim: $\langle w|w\rangle_{{}^tM\cdot A\cdot M} = \langle M\cdot w | M\cdot w\rangle_A$ $\blacksquare$

Continua dim di ieri:
devo vedere che se $d_2 > 0$ allora il segno di $d_1 \cdot d_3$ è conservato tramite $A \rightsquigarrow {}^tM \cdot A \cdot M$.
Infatti i segni degli autoval sono:

$d_1 \cdot d_3 > 0$	$+\;+\;+$	$-\;-\;-$
$d_1 \cdot d_3 < 0$	$+\;+\;-$	$-\;-\;+$

Poiché $A \rightsquigarrow {}^tM \cdot A \cdot M$ mantiene segni autoval allora mantiene anche segno di $d_1 \cdot d_3$ (solo se $d_2 > 0$).

- I 4 casi coprono tutto e sono mutuamente esclusivi.
- Tramite le trasf. lecite $A \rightsquigarrow {}^tM \cdot A \cdot M$, $A \rightsquigarrow k \cdot A$ i 4 casi vengono preservati.

Resta: in ognuno dei casi mi riconduco tramite trasf. lecite al modello.

$$A = \begin{pmatrix} Q & \ell \\ {}^t\ell & c \end{pmatrix} ; \quad \exists B \text{ ortog t.c. } {}^tB \cdot Q \cdot B = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

Uso $M = \begin{pmatrix} B & 0 \\ 0 & 1 \end{pmatrix}$ e mi riconduco a

$$A = \begin{pmatrix} \lambda_1 & 0 & a \\ 0 & \lambda_2 & b \\ a & b & c \end{pmatrix}.$$

Oss: non può essere $\lambda_1 = \lambda_2 = 0$ perché $\det \begin{pmatrix} 0 & 0 & a \\ 0 & 0 & b \\ a & b & c \end{pmatrix} = 0$

Affermo che se $\lambda_1, \lambda_2 \neq 0$ tramite traslaz. ottengo $\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$${}^t\begin{pmatrix} I & v \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} Q & \ell \\ {}^t\ell & c \end{pmatrix} \begin{pmatrix} I & v \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} I & 0 \\ {}^tv & 1 \end{pmatrix} \begin{pmatrix} Q + \ell v & Qv + \ell \\ {}^t\ell & {}^t\ell v + c \end{pmatrix}$$

$$= \begin{pmatrix} * & Qv + \ell \\ * & * \end{pmatrix}$$

basta prendere $v = -Q^{-1}\ell$ $\quad (\lambda_1, \lambda_2 \neq 0 \Rightarrow \exists Q^{-1})$

Equaz. è:

$$\lambda_1 x^2 + \lambda_2 y^2 + c = 0$$

$$x = \sqrt{|c|} \cdot \frac{1}{\sqrt{|\lambda_1|}} \cdot X \qquad y = \sqrt{|c|} \cdot \frac{1}{\sqrt{|\lambda_2|}} \cdot Y$$

$$\Rightarrow \quad \pm X^2 \pm Y^2 \pm 1 = 0$$

$$\Rightarrow \quad X^2 + Y^2 + 1 = 0 \qquad \emptyset$$
$$X^2 + Y^2 - 1 = 0 \qquad \text{circonf}$$
$$X^2 - Y^2 + 1 = 0 \qquad \text{iperb.}$$

Se $\lambda_1 \neq 0$, $\lambda_2 = 0$ come sopra ottengo

$$\begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & 0 & b \\ 0 & b & c \end{pmatrix}$$

$$\lambda_1 x^2 + by + c = 0$$

$$\begin{cases} X = x \\ Y = -(by + c) \end{cases} \qquad Y = X^2. \quad \text{parab.}$$

∎

Quadriche non degeneri

$$\mathcal{L} = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 : {}^t\begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix} \cdot A \cdot \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix} = 0 \right\}$$

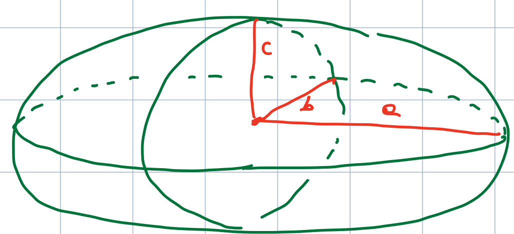
$A \in M_{4\times4}(\mathbb{R})$ simm. $\det(A) \neq 0$

Modelli:

1) $x^2+y^2+z^2+1=0$ $\qquad\qquad \emptyset$

2) $x^2+y^2+z^2=1$ $\qquad\qquad$ ellissoide

(metrico $\quad \dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=1$

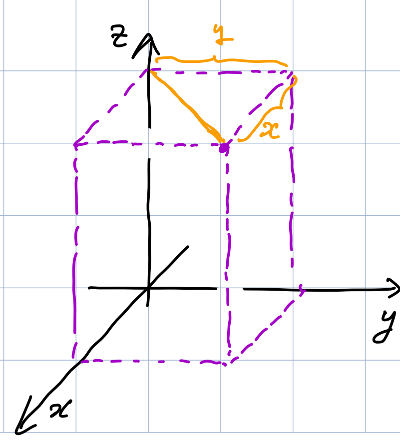


)

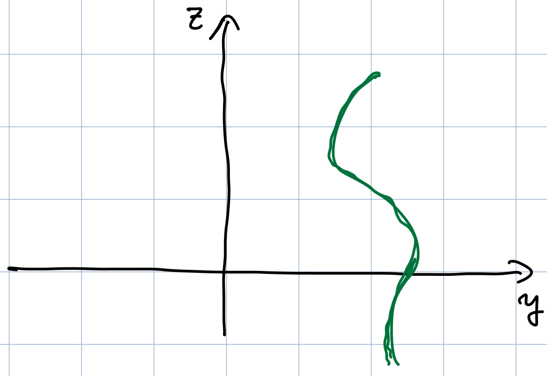
Oss: se $\mathcal{L} \subset \mathbb{R}^3$ è definito da una equazione in cui x,y compaiono solo nelle espressioni x^2+y^2 allora $\mathcal{L}$ è una superficie di rotazione intorno all'asse z.

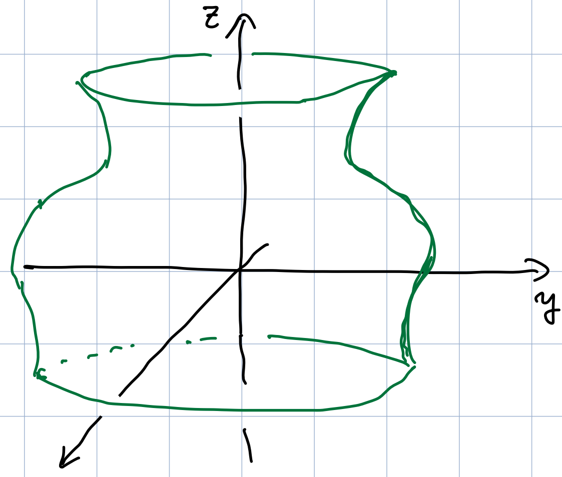
Equaz: $\quad h\left(\sqrt{x^2+y^2}, z\right)=0$

$$\sqrt{x^2+y^2} = \text{dist}\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}, \text{asse } z\right)$$

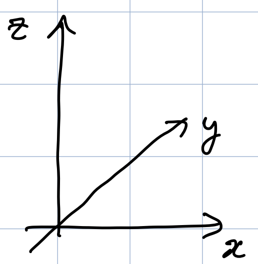


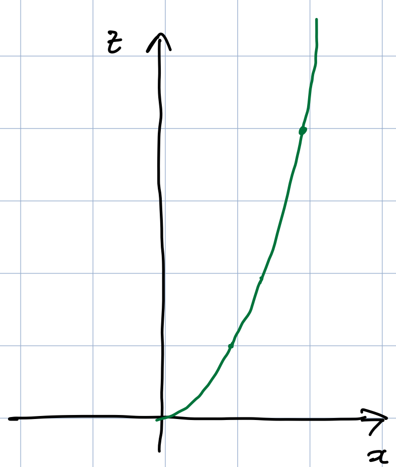
Nel piano y,z l'equaz $h(y,z)=0$ dona una certa curva

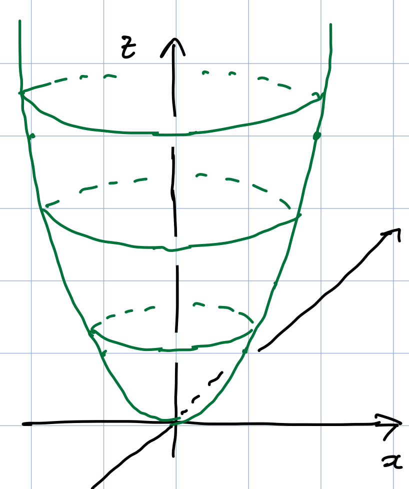




3) $z = x^2+y^2$ rotaz. intorno all'asse z della parabola $z = x^2$





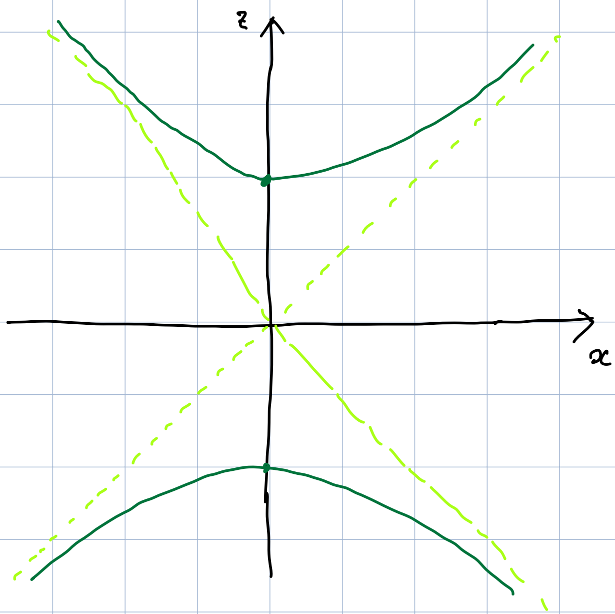


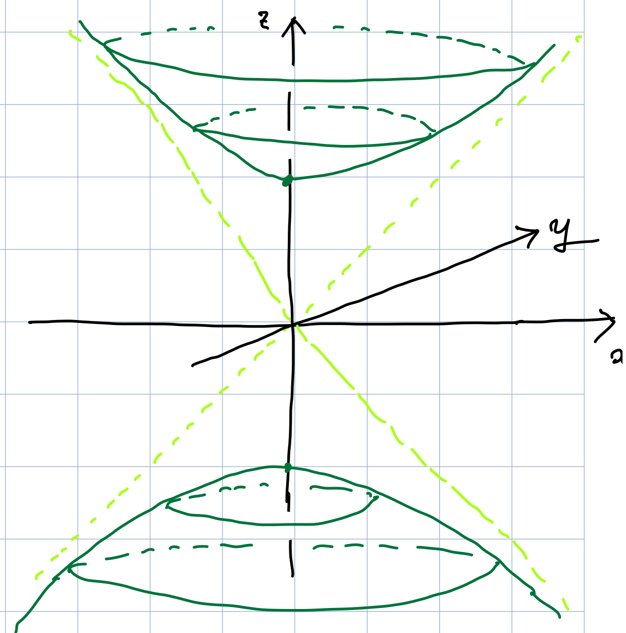
paraboloide ellittico

4) $z^2 = x^2 + y^2 + 1$

rotazione intorno all'asse z di

$z^2 = x^2 + 1 \qquad z^2 - x^2 = 1$



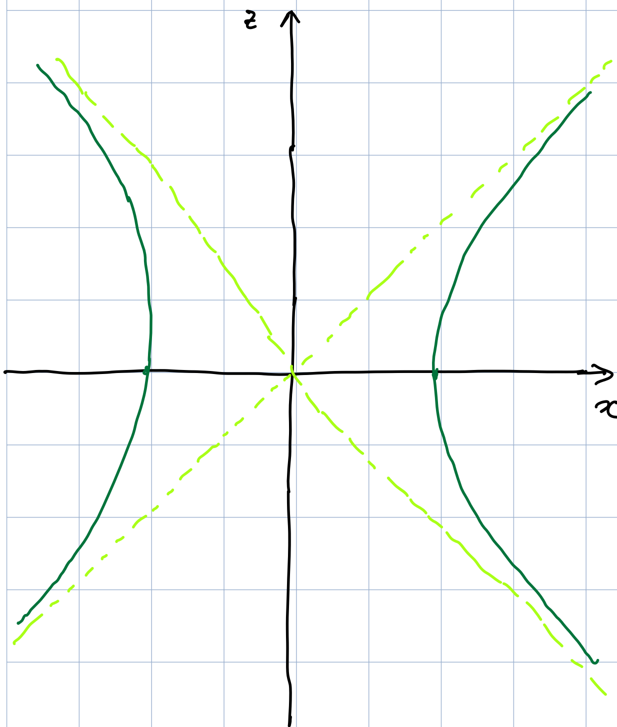


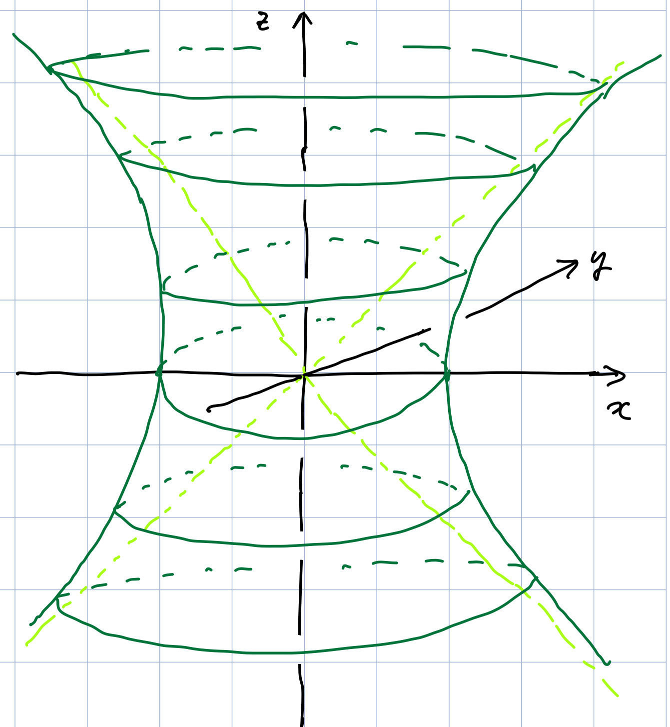
iperboloide ellittico
o a 2 falde

5) $z^2 + 1 = x^2 + y^2$

Superficie di rotazione intorno all'asse z di

$z^2 + 1 = x^2 \qquad x^2 - z^2 = 1$





iperboloide iperbolico

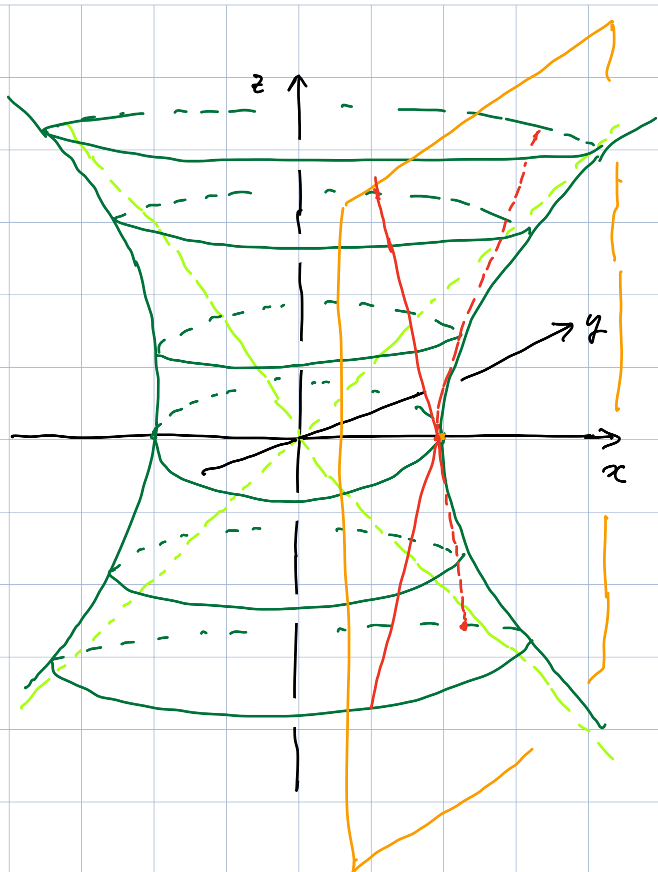
o a 1 falde

$$z^2 + 1 = x^2 + y^2$$

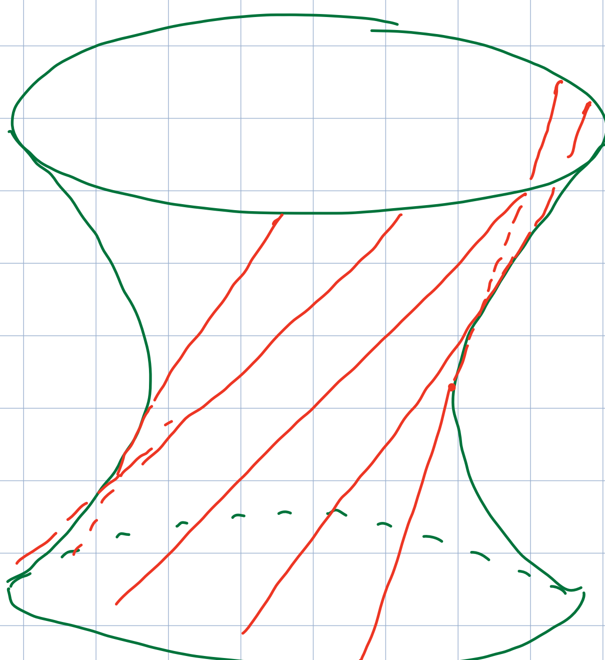
Intersecando con il piano $x = 1$

trovo $z^2 + 1 = 1 + y^2$

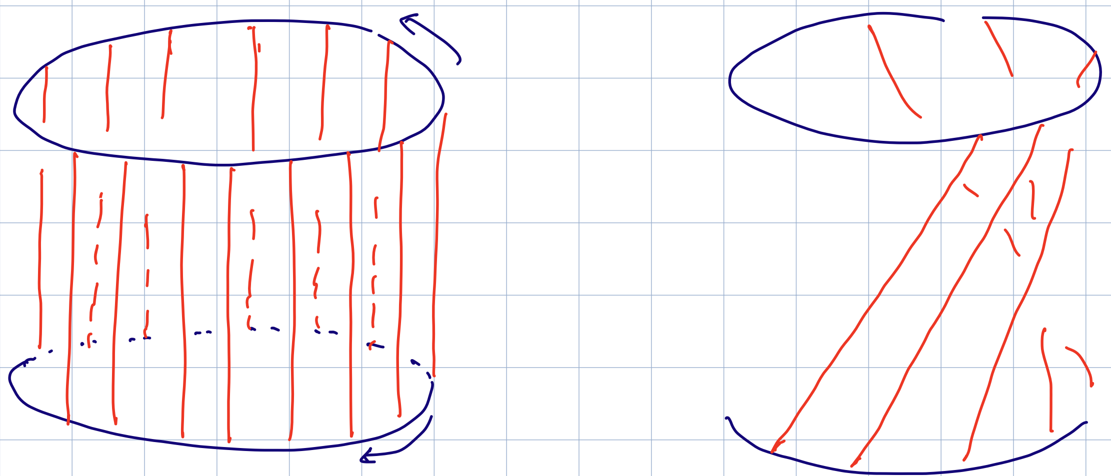
$z^2 = y^2$ $\quad z = \pm y$ due rette



la superficie contiene anche tutte le rette da esse ottenute ruotando intorno all'asse z:

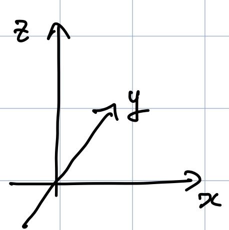


Modo per costruirlo:

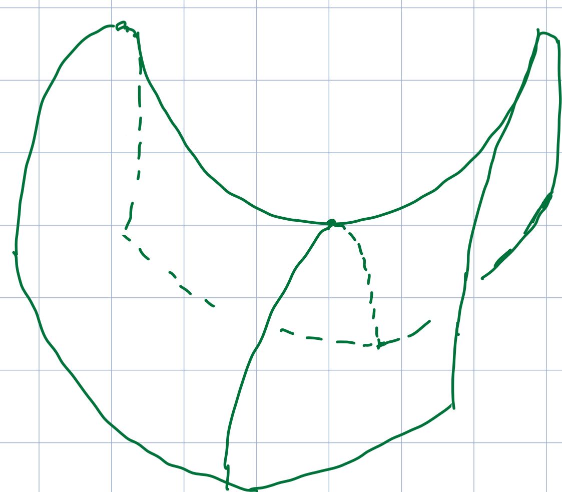


L'iperboloide iperbolico è una superficie "rigata"
(per ogni suo punto passano due rette)

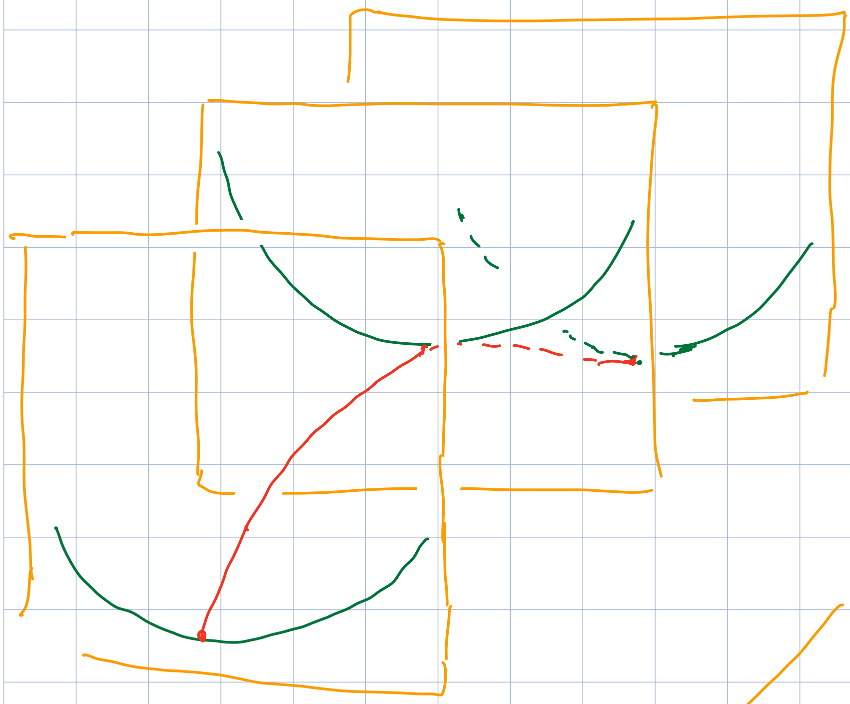
6) $z = x^2 - y^2$



$y = 0$	$z = x^2$
$x = 0$	$z = -y^2$
$y = cost$	$z = x^2 - cost^2$
$x = cost$	$z = -y^2 + cost^2$



paraboloide iperbolico o a sella

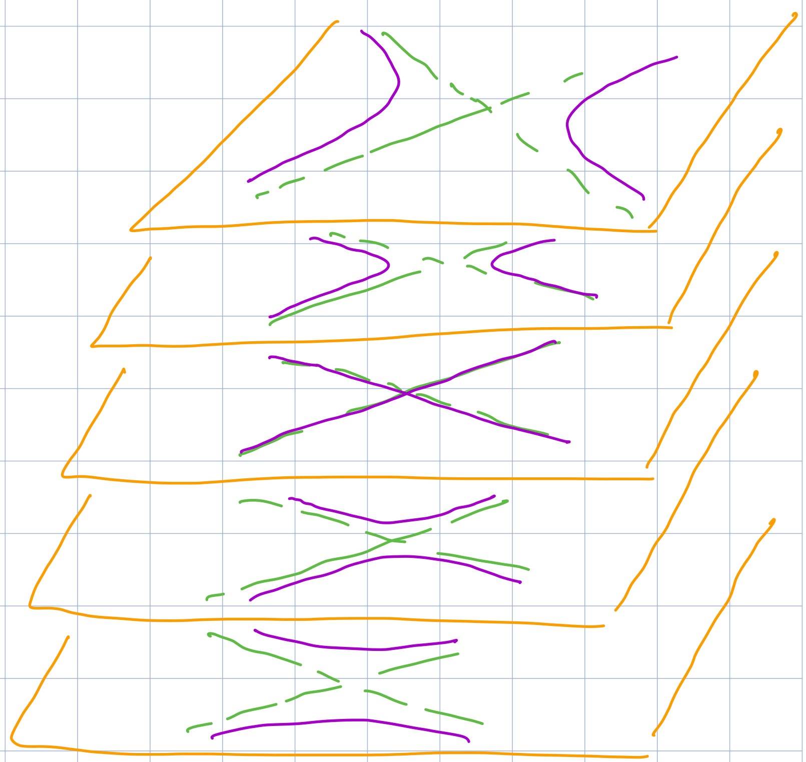


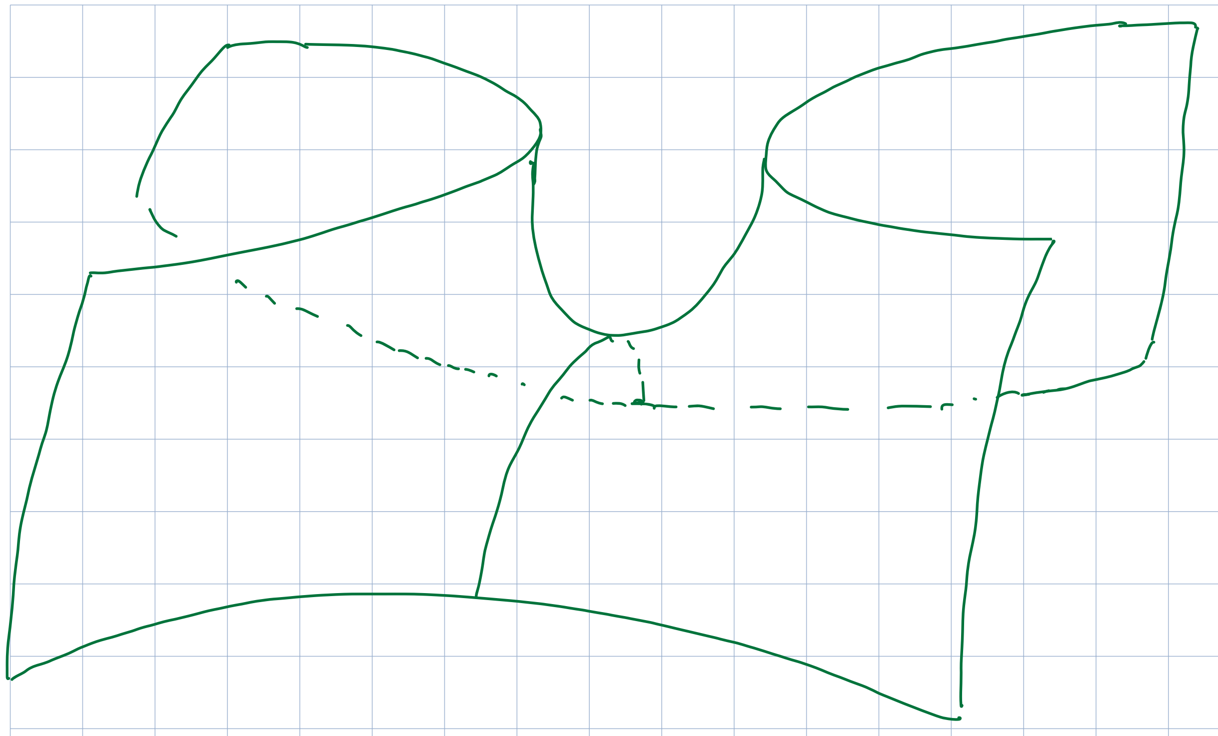
$z = x^2 - y^2$

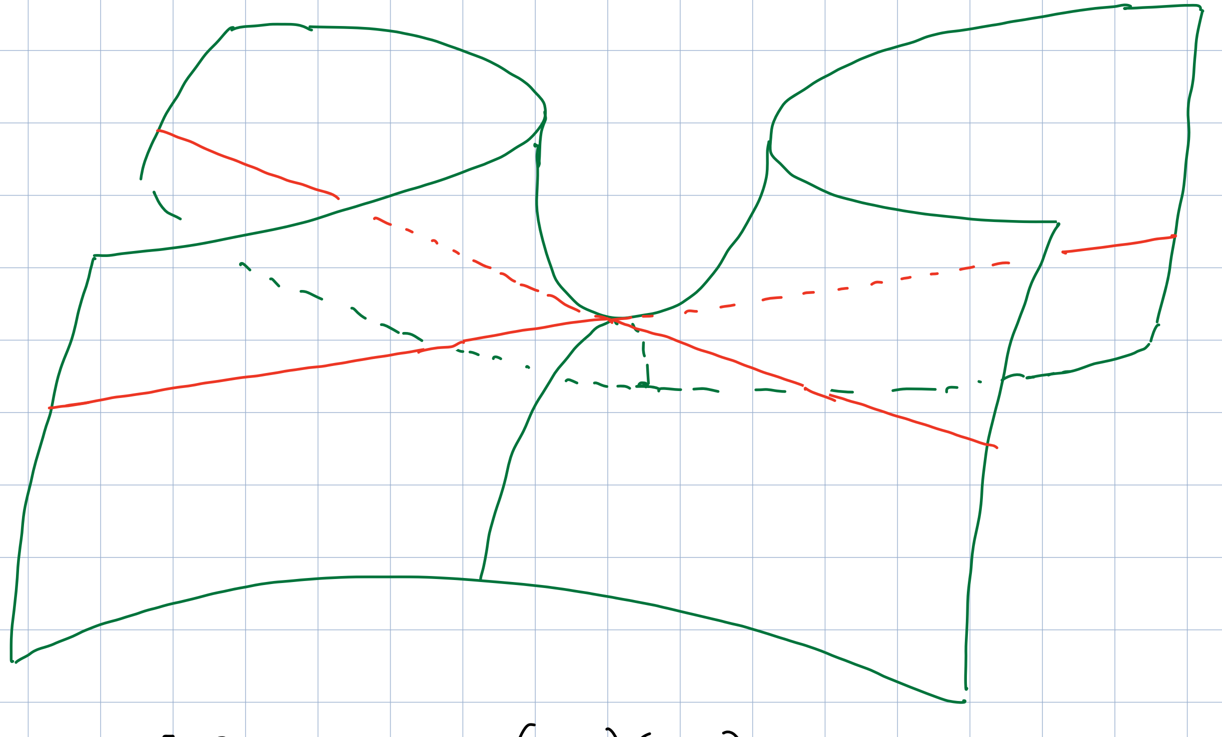
$z = cost$

$x^2 - y^2 = cost$ iperbole

o due rette

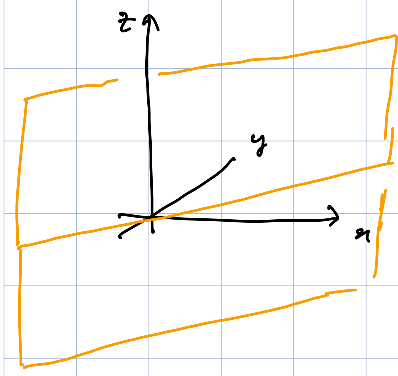


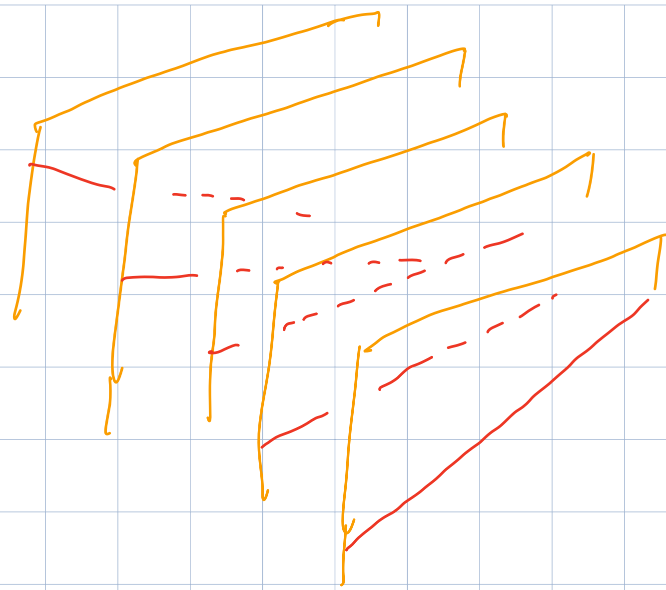


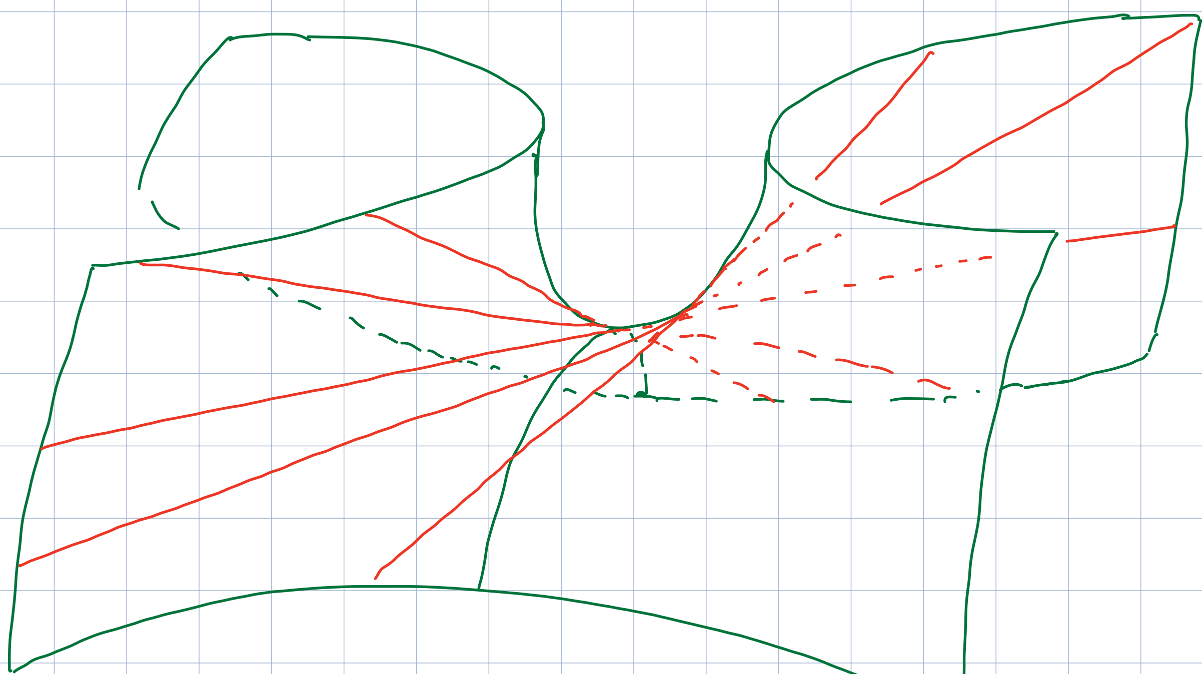


$z = x^2 - y^2$ $\qquad\qquad$ $z = (x+y)(x-y)$

Se interseco con il piano $x - y = \text{cost}$ trovo $z = \text{cost}\,(x+y)$
retta







Modelli affini delle quadriche non degeneri $\mathcal{Q}$

1) $x^2+y^2+z^2+1=0$ $\quad \emptyset$

2) $x^2+y^2+z^2=1$ $\quad$ ellissoide

3) $z=x^2+y^2$ $\quad$ parab. ell.

4) $z^2=x^2+y^2+1$ $\quad$ iperb. ell. 2 falde

5) $z^2 = x^2+y-1$ iperb iperb. 1 falda

6) $z = x^2-y^2$ parab. iperb.

$\emptyset$ $\overline{\mathcal{L}} : x^2+y^2+z^2+w^2=0$ $\emptyset$ $\mathcal{L}_\infty : x^2+y^2+z^2=0$ $\emptyset$

ellissoide $\overline{\mathcal{L}} : x^2+y^2+z^2=w^2$ ellissoide proiettivo

$\mathcal{L}_\infty : x^2+y^2+z^2=0$ $\emptyset$

parab. ell: $\overline{\mathcal{L}} : zw = x^2+y^2$ $\begin{cases} z = u+v \\ w = u-v \end{cases}$

$u^2-v^2 = x^2+y^2$

$x^2+y^2+v^2=u^2$ ellissoide proiettivo

$\mathcal{L}_\infty : x^2+y^2 = 0$ $[0:0:1]$ un pto.