Geometria 17/3/22

Libro eserciziario — temi d'esame

Visto: bil. simm. su $\mathbb{R}^n$ sono $\langle \cdot | \cdot \rangle_A$ A simm.

$$(x|y)_A = {}^t x \cdot A \cdot y \quad : \text{ quali sono def. pos.?}$$

$n=2$: se e solo se $a_{11}>0$, $\det(A)>0$.

$n=3$ vedremo.

NO facile: • $\begin{pmatrix} \le 0 & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix} \begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \le 0 & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix} \begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \le 0 \end{pmatrix}$

$$\begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix} \qquad \begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix} \qquad \begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}$$

$$\det \square \le 0 \qquad \det \square \le 0 \qquad \det \square \le 0$$

Es: $\begin{pmatrix} 7 & 4 & 2 \\ 4 & \boxed{-3} & 1 \\ 2 & 1 & 5 \end{pmatrix} \qquad \begin{pmatrix} \boxed{4} & 2 & \boxed{7} \\ 2 & 3 & 1 \\ \boxed{7} & 1 & \boxed{6} \end{pmatrix}$

9.1.8 (b) $\begin{pmatrix} 1 & 2 & 0 \\ 2 & 5 & -1 \\ 0 & -1 & 3 \end{pmatrix}$

$$\left\langle \begin{pmatrix} x \\ y \\ z \end{pmatrix} \middle| \begin{pmatrix} x \\ y \\ z \end{pmatrix} \right\rangle_A = x^2 + 5y^2 + 3z^2 + 4xy + 0xz - 2yz$$

$$= (x+2y)^2 + (y-z)^2 + 2z^2 \geq 0$$

nullo solo se $\begin{cases} x+2y=0 \\ y-z=0 \\ z=0 \end{cases}$ cioè $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$.

$\Rightarrow$ def. pos.

9.1.8 (e) $\begin{pmatrix} 1 & -2 & 1 \\ -2 & 5 & 3 \\ 1 & 3 & 2 \end{pmatrix}$

$$x^2 + 5y^2 + 2z^2 - 4xy + 2xz + 6yz$$

Cerco x,y,z t.c. esso sia neg. x,y concord. x,z discord. y,z discord.

$\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$: $1+5+2-4-2-6 = -4 < 0$

non def. pos.

9.1.9. Data $A \in M_{n \times n}(\mathbb{R})$ simm. porre B con $(B)_{ij} = (-1)^{i+j}(A_{ij})$.
Provare che $\langle \cdot | \cdot \rangle_B$ è def. pos $\iff$ lo è $\langle \cdot | \cdot \rangle_A$.

$$\boxed{\langle x|x\rangle_B} = {}^t x \cdot B \cdot x = \sum_{i=1}^{n} \sum_{j=1}^{n} x_i \cdot (B)_{ij} \cdot x_j$$

$$= \sum_{i=1}^{n} x_i \cdot (-1)^{i+j} \cdot (A)_{ij} \cdot x_j$$

$$= \sum_{i=1}^{n} (-1)^i \cdot x_i \cdot (A)_{ij} \cdot (-1)^j \cdot x_j = \dots$$

Dunque se $T: \mathbb{R}^n \to \mathbb{R}^n$ $\boxed{T(x) = \begin{pmatrix} -x_1 \\ +x_2 \\ -x_3 \\ +x_4 \\ -x_5 \\ \vdots \end{pmatrix}}$

$$\dots = \sum_{i=1}^{n} (Tx)_i \cdot (A)_{ij} \cdot (Tx)_j$$

$$= \boxed{\langle Tx|Tx\rangle_A}$$

conclusione immediata.

9.2.1: Data $f: V \times V \to \mathbb{R}$ bilineare simmetrica posso $q: V \to \mathbb{R}$
$q(v) = f(v,v)$.
Conoscendo q trovare f.

$\langle v|w\rangle = \frac{1}{2}\left(\|v+w\|^2 - \|v\|^2 - \|w\|^2\right)$

(b) $\mathbb{R}^2$ $q(x) = 3x_1^2 + 2x_1x_2 - x_2^2$

$f = \langle \cdot | \cdot \rangle_A$ $\qquad A = \begin{pmatrix} 3 & 1 \\ 1 & -1 \end{pmatrix}$

(c) $\mathbb{R}^3$ $q(x) = -2x_1^2 + x_2^2 - 5x_3^2 - x_1x_2 + 3x_1x_3 + 2x_2x_3$

$f = \langle \cdot | \cdot \rangle_A$ $\qquad A = \begin{pmatrix} -2 & -1/2 & 3/2 \\ -1/2 & 1 & 1 \\ 3/2 & 1 & -5 \end{pmatrix}$.

9.2.3. $\boxed{\mathbb{R}_{\leq 2}[t]}$; $\langle p(t) | q(t) \rangle = p(0)\cdot q(0) + p(1)\cdot q(1) + p(2)\cdot q(2)$.

Trovare vettore di norma $\sqrt{5}$ ortog. a $1+t$ e $1+t^2$.

Verifichiamo che è prod. scal.

- bilineare ✓
- simm ✓
- def. pos. $\langle p(t) | p(t) \rangle = p(0)^2 + p(1)^2 + p(2)^2 \geq 0$

  nulla solo se $p(0) = p(1) = p(2) = 0$

  poiché p ha grado ≤ 2 concludo che $p(t) = 0$.

$p(t) = a_0 + a_1 \cdot t + a_2 \cdot t^2$ imponiamo che sia ortog. a $1+t$ e $\underline{1+t^2}$:

$$\langle p(t) | q(t) \rangle = p(0)\cdot q(0) + p(1)\cdot q(1) + p(2)\cdot q(2)$$

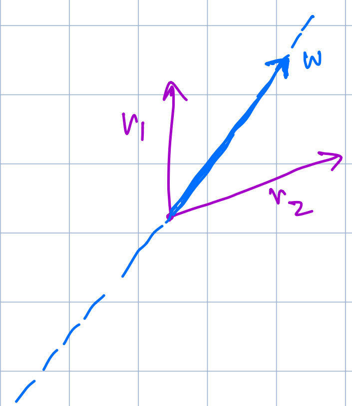
$$\begin{cases} a_0 \cdot 1 + (a_0 + a_1 + a_2)\cdot 2 + (a_0 + 2a_1 + 4a_2)\cdot 3 = 0 \\ a_0 \cdot 1 \quad (a_0 + a_1 + a_2)\cdot 2 + (a_0 + 2a_1 + 4a_2)\cdot 5 = 0 \end{cases}$$

$$\begin{cases} a_0 + 2(a_0 + a_1 + a_2) = 0 \\ a_0 + 2a_1 + 4a_2 = 0 \end{cases} \qquad \begin{cases} 3a_0 + 2a_1 + 2a_2 = 0 \\ a_0 + 2a_1 + 4a_2 = 0 \end{cases}$$

$$\begin{cases} a_2 = a_0 \\ 2a_1 + 5a_0 \end{cases} \qquad a_0 = 2 \quad a_1 = -5 \quad a_2 = 2$$

$$\boxed{2 - 5t + 2t^2}$$

$$\| 2 - 5t + 2t^2 \|^2 = 4 + 1 + 0 = 5$$



$\lambda \cdot \frac{w}{\|w\|}$

9.2.5 (d) Su $\mathbb{R}^2$ con $\langle . | . \rangle_A$ $A = \begin{pmatrix} 5 & -2 \\ -2 & 1 \end{pmatrix}$

ortonormalizzare $\underbrace{\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}}$

→ già ortonormale rispetto $\langle . | . \rangle_{\mathbb{R}^2}$

$$u_1 = \frac{e_1}{\|e_1\|_A} = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\tilde{u}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \frac{1}{5} \cdot \left\langle \begin{pmatrix} 0 \\ 1 \end{pmatrix} \middle| \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\rangle_A \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \frac{1}{5} \cdot \left(\begin{pmatrix} 0 & 1 \end{pmatrix} \cdot \boxed{\begin{pmatrix} 5 & -2 \\ -2 & 1 \end{pmatrix}} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right) \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \frac{1}{5} (-2) \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{5} \begin{pmatrix} +2 \\ 5 \end{pmatrix}$$

$$u_2 = \frac{\tilde{u}_2}{\|\tilde{u}_2\|_A} = \frac{\frac{1}{5}\begin{pmatrix} 2 \\ 5 \end{pmatrix}}{\left\| \frac{1}{5}\begin{pmatrix} 2 \\ 5 \end{pmatrix} \right\|_A} = \frac{\cancel{\frac{1}{5}} \cdot \begin{pmatrix} 2 \\ 5 \end{pmatrix}}{\cancel{\frac{1}{5}} \cdot \left\| \begin{pmatrix} 2 \\ 5 \end{pmatrix} \right\|_A} = \frac{\begin{pmatrix} 2 \\ 5 \end{pmatrix}}{\sqrt{5 \cdot 2^2 - 4 \cdot 2 \cdot 5 + 5^2}}$$

$u_1 = \frac{1}{\sqrt{5}}\begin{pmatrix}1\\0\end{pmatrix}$ $\qquad$ $u_2 = \frac{1}{\sqrt{5}}\begin{pmatrix}2\\5\end{pmatrix}$

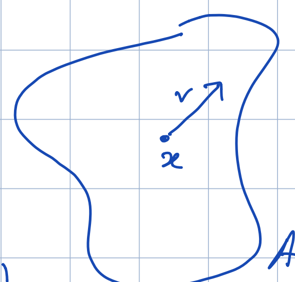
———— o ————

$$\frac{\partial f}{\partial x_i}, \quad \frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial^2 f}{\partial x_j \partial x_i} \qquad \left(f : \underset{\mathbb{R}^m}{\overset{\cup}{A}} \to \mathbb{R}\right)$$

$$(Hf)(x) = \left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right)_{i,j=1,\dots,m}$$ matrice hessiana.

Fatto: se f è decente allora $(Hf)(x)$ è simm.

Approx di Taylor per f del II ordine:



$$f(x+v) = f(x) + \sum_{i=1}^{m} \frac{\partial f}{\partial x_i}(x)\cdot v_i + \frac{1}{2}\cdot\sum_{i,j=1}^{m} \frac{\partial^2 f}{\partial x_i \partial x_j}(x)\cdot v_i v_j + o(\|v\|^2)$$

$$= f(x) + Jf(x)\cdot v + \frac{1}{2}\langle v | v\rangle_{Hf(x)} + o(\|v\|^2)$$

Conseguenze: se x è un pto. di max o min loc. allora $Jf(x) = 0$

Domanda: come capire il segno di $\langle v | v\rangle_{Hf(x)}$?

Idee di dimo: $u = \frac{v}{\|v\|}$

$$g(t) = f(x + t\cdot u) \qquad f(x+v) = g(\|v\|)$$

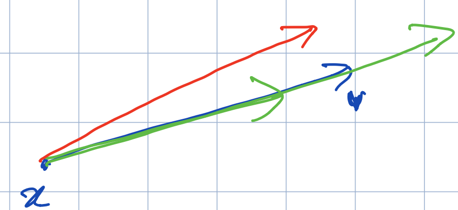
$$g(t) = g(0) + g'(0)\cdot t + \frac{1}{2} g''(0)\cdot t^2 + o(t^2)$$

$$g'(t) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(x+tu)\cdot u_i$$

$$g''(t) = \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial^2 f(x+tu)}{\partial x_j \partial x_i}\cdot u_i \cdot u_j$$

$$f(x+v) = g(\|v\|) = f(x) + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(x)\cdot \underbrace{\|v\|\cdot u_i}_{v_i}$$
$$+ \frac{1}{2}\sum_{i,j=1}^{n} \frac{\partial^2 f}{\partial x_j \partial x_i}(x)\cdot \underbrace{\|v\|^2\cdot u_i \cdot u_j}_{v_i\cdot v_j} + o(\|v\|^2)$$

Non è completo: in realtà non ho fatto variare v ma solo $\|v\|$.



Prop: dato V con $\langle\cdot|\cdot\rangle$, W sottosp.
$P_W(v)$ è il pto di W più vicino a v.

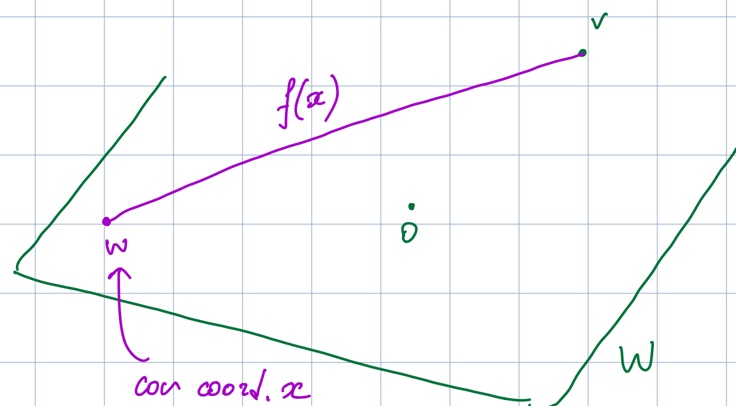
Dimo: sappiamo che se $w_1,\dots,w_k$ è base ortog. di W

Allora $P_W(v) = \sum_{i=1}^{k} \frac{\langle v | w_i \rangle}{\|w_i\|^2} \cdot w_i$ _

Definisco $f : \mathbb{R}^k \longrightarrow \mathbb{R}$

$$f(x) = \underbrace{\Big\| v - \underbrace{\sum_{i=1}^{k} x_i w_i}_{\text{generico pto di } W \text{ con coord. } x} \Big\|^2}_{d(v, \dots)^2}$$

Devo vedere: min si ha con $x_i = \frac{\langle v | w_i \rangle}{\|w_i\|^2}$.



Oss: per $\|x\| \to +\infty$ ho $f(x) \longrightarrow +\infty$.

Dunque basta vedere che l'unico pto con $Jf(x) = 0$ è quello con $x_i = \frac{\langle v | w_i \rangle}{\|w_i\|^2}$.

$$f(x) = \left\| v - \sum_{i=1}^{k} x_i w_i \right\|^2$$

$$= \|v\|^2 - 2 \left\langle v \,\middle|\, \sum_{i=1}^{k} x_i w_i \right\rangle + \underbrace{\left\| \sum_{i=1}^{k} x_i w_i \right\|^2}_{\text{ta le: } x_i x_j \underbrace{\langle w_i | w_j \rangle}_{\overset{\shortparallel}{0} \text{ per } i \neq j}}$$

$$= \|v\|^2 - 2 \sum_{i=1}^{k} \langle v | w_i \rangle \cdot x_i +$$

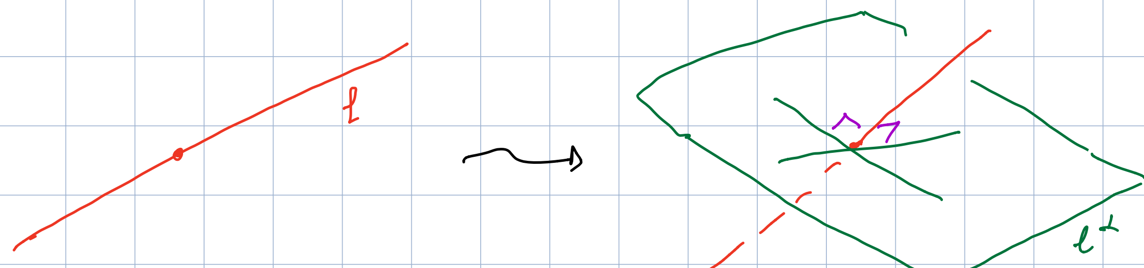
$$+ \sum_{i=1}^{k} \|w_i\|^2 \cdot x_i^2$$

$$\frac{\partial f}{\partial x_j}(x) = 0 - 2 \langle v | w_j \rangle \cdot 1 + \|w_j\|^2 \cdot 2 x_j$$

nullo solo se $\quad x_j = \dfrac{\langle v | w_j \rangle}{\|w_j\|^2}$. $\quad \boxtimes$

In $\mathbb{R}^3$ con $\langle \cdot | \cdot \rangle_{\mathbb{R}^3}$ canonico.

<u>Oss</u>: la perpendicolarità dà una corrispondenza tra rette e piani (per O).



eq. cart. di piano P :

$$P = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : \alpha x + \beta y + \gamma z = 0 \right\}$$
$$= \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : \left\langle \begin{pmatrix} x \\ y \\ z \end{pmatrix} \middle| \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} \right\rangle = 0 \right\}$$
$$= \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}^{\perp}$$

cioè $P^{\perp} = \mathrm{Span}\begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}$ cioè eq. par. della retta ortog.

eq. cart. retta:

$$\ell = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : \begin{cases} \alpha_1 x + \beta_1 y + \gamma_1 z = 0 \\ \alpha_2 x + \beta_2 y + \gamma_2 z = 0 \end{cases} \right\}$$
$$= \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : \left\langle \begin{pmatrix} \alpha_1 \\ \beta_1 \\ \gamma_1 \end{pmatrix} \middle| \begin{pmatrix} x \\ y \\ z \end{pmatrix} \right\rangle = \left\langle \begin{pmatrix} \alpha_2 \\ \beta_2 \\ \gamma_2 \end{pmatrix} \middle| \begin{pmatrix} x \\ y \\ z \end{pmatrix} \right\rangle = 0 \right\}$$
$$= \left\{ \begin{pmatrix} \alpha_1 \\ \beta_1 \\ \gamma_1 \end{pmatrix}, \begin{pmatrix} \alpha_2 \\ \beta_2 \\ \gamma_2 \end{pmatrix} \right\}^{\perp}$$

cioè $\ell^{\perp} = \mathrm{Span}\left(\begin{pmatrix} \alpha_1 \\ \beta_1 \\ \gamma_1 \end{pmatrix} \begin{pmatrix} \alpha_2 \\ \beta_2 \\ \gamma_2 \end{pmatrix} \right)$

cioè eq. param. del piano ortog.

Dunque i problemi

piano param $\rightsquigarrow$ piano cart

cart. retta $\longrightarrow$ retta param

sono uguali e significano:

dati due vettori di $\mathbb{R}^3$ (lin. indip.)
trovarne uno perpendicolare a loro due:

$\mathbb{P}$ param $\longrightarrow$ $\mathbb{P}^\perp$ param $=$ $\mathbb{P}$ cart

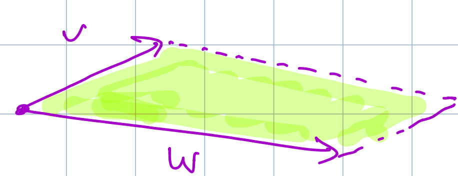
<u>Def</u>: chiamo prodotto vettoriale

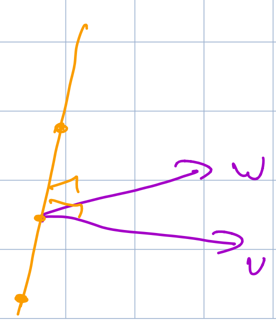
$$\wedge : \mathbb{R}^3 \times \mathbb{R}^3 \longrightarrow \mathbb{R}^3 \qquad \text{(oppure } \times \text{)}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \wedge \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} yc - zb \\ -(xc - za) \\ xb - ya \end{pmatrix}$$

<u>Fatto</u>:

- $v \wedge w$ è $\perp$ sia a v sia a w (calcolo facile)
- $\|v \wedge w\| = \text{area}$





• $v, w, v \wedge w$ soddisfano regola mano destra

pollice, indice, medio

mano dx

<u>Esercizio</u>: $\{x_1, \dots, x_k\}^{\perp} = \mathrm{Span}(x_1, \dots, x_k)^{\perp}$

$\subset$ Sapendo che $\langle v | x_i \rangle = 0$ $i = 1 \dots k$ devo vedere che $\langle v | \alpha_1 x_1 + \dots + \alpha_k x_k \rangle = 0$ $\forall \alpha_1, \dots, \alpha_k$. Usare linearità a dx.

$\supset$ Sapendo che $\langle v | \alpha_1 x_1 + \dots + \alpha_k x_k \rangle = 0$ $\forall \alpha_1, \dots, \alpha_k$ devo vedere che $\langle v | \alpha_i \rangle = 0$ $i = 1 \dots k$. Basta prendere $\alpha_i = 1$, $\alpha_j = 0$ per $j \neq i$.