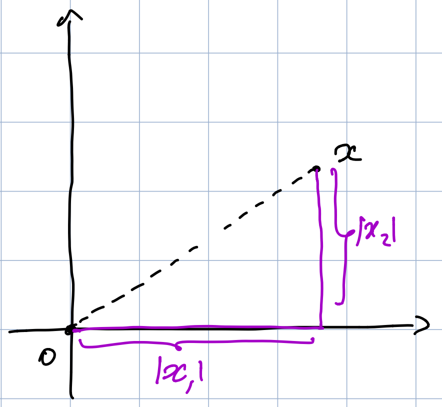
Geometria 9/3/22

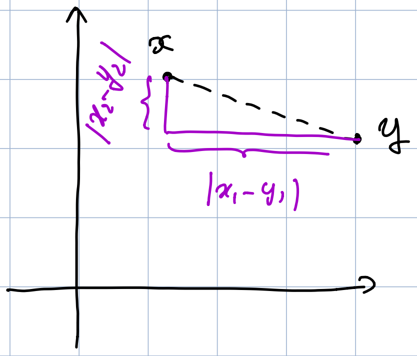
Domani: 8:30 - 9:15 + Lisca.

In $\mathbb{R}^2$



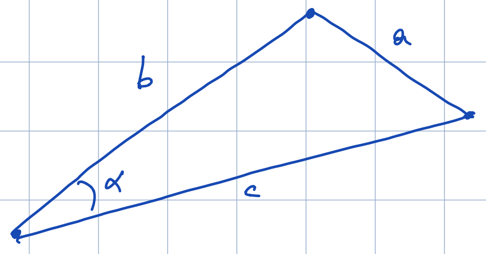
$d(0,x) =$

$= \sqrt{x_1^2 + x_2^2}$



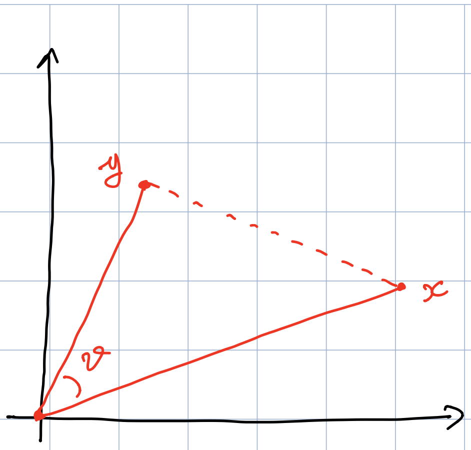
$d(x,y) =$

$= \sqrt{(x_1-y_1)^2 + (x_2-y_2)^2}$



Carnot

$a^2 = b^2 - 2bc \cdot \cos(\alpha) + c^2$



$$\cos(\vartheta) = \frac{d(0,x)^2 + d(0,y)^2 - d(x,y)^2}{2\, d(0,x)\cdot d(0,y)}$$

$$= \frac{x_1^2 + x_2^2 + y_1^2 + y_2^2 - (x_1-y_1)^2 - (x_2-y_2)^2}{2\sqrt{x_1^2+x_2^2}\cdot\sqrt{y_1^2+y_2^2}}$$

$$= \frac{x_1y_1 + x_2y_2}{\sqrt{x_1^2+x_2^2}\cdot\sqrt{y_1^2+y_2^2}}$$

Def: chiamo prodotto scalare standard su $\mathbb{R}^2$ la

$$\langle\cdot|\cdot\rangle_{\mathbb{R}^2} : \mathbb{R}^2\times\mathbb{R}^2 \longrightarrow \mathbb{R}$$

$$(x,y) \longmapsto \langle x|y\rangle_{\mathbb{R}^2} = x_1y_1 + x_2y_2$$

$$= \underset{1\times 2}{(x_1,\ x_2)}\underset{2\times 1}{\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}}$$

$$= {}^t x\cdot y$$

Inoltre norma associata: $\|\cdot\|_{\mathbb{R}^2} : \mathbb{R}^2 \longrightarrow \mathbb{R}$

$$x \longmapsto \|x\|_{\mathbb{R}^2} = \sqrt{\langle x|x\rangle_{\mathbb{R}^2}}$$

Scoperto:

$$d(x,y) = \|x-y\|_{\mathbb{R}^2}$$

$$\cos \widehat{x\,y} = \frac{\langle x|y\rangle_{\mathbb{R}^2}}{\|x\|_{\mathbb{R}^2}\cdot\|y\|_{\mathbb{R}^2}}.$$

Prodotto scalare standard di $\mathbb{R}^n$:

$$\langle\cdot|\cdot\rangle_{\mathbb{R}^n} : \mathbb{R}^n\times\mathbb{R}^n \longrightarrow \mathbb{R}$$

$$(x,y) \longmapsto \langle x|y\rangle_{\mathbb{R}^n} = {}^t x\cdot y.$$

Distanze, angoli: come sopra.

Def: dato uno spazio vettoriale V su $\mathbb{R}$ dico che una funzione $f : V \times V \longrightarrow \mathbb{R}$ si dice:

1) bilineare se è
   - lineare a sinistra $\quad f(\lambda_1 v_1 + \lambda_2 v_2, w) = \lambda_1 f(v_1, w) + \lambda_2 f(v_2, w)$
   - lineare a destra $\quad f(w, \lambda_1 v_1 + \lambda_2 v_2) = \lambda_1 f(w, v_1) + \lambda_2 f(w, v_2)$

2) simmetrica se $f(v,w) = f(w,v) \quad \forall v, w$

3) definita positiva se $f(v,v) > 0 \quad \forall v \neq 0$

4) prodotto scalare se è bil. simm. def. positivo.

Prop: $\langle \cdot | \cdot \rangle_{\mathbb{R}^n} \qquad \langle x | y \rangle_{\mathbb{R}^n} = {}^t x \cdot y$ è un prod. scal.

Dim: lin. a sin: $\quad (\alpha x + \beta y | z)_{\mathbb{R}^n} = {}^t(\alpha x + \beta y) \cdot z$

$$= (\alpha\, {}^t x + \beta\, {}^t y) \cdot z = \alpha \cdot {}^t x \cdot z + \beta \cdot {}^t y \cdot z$$

$$= \alpha \cdot \langle x | z \rangle_{\mathbb{R}^n} + \beta \cdot \langle y | z \rangle_{\mathbb{R}^n}.$$

lin. a dx: $\quad \langle x | \alpha y + \beta z \rangle_{\mathbb{R}^n} = {}^t x \cdot (\alpha y + \beta z)$

$$= \alpha\, {}^t x \cdot y + \beta \cdot {}^t x \cdot z = \alpha \langle x | y \rangle_{\mathbb{R}^n} + \beta \langle x | z \rangle_{\mathbb{R}^n}$$

simm $\quad \langle y | x \rangle_{\mathbb{R}^n} = {}^t y \cdot x = {}^t({}^t y \cdot x) \overset{*}{=} {}^t x \cdot {}^{tt} y = {}^t x \cdot y$

$$= \langle x | y \rangle_{\mathbb{R}^n}$$

def. pos. $\quad \langle x | x \rangle_{\mathbb{R}^n} = {}^t x \cdot x = (x_1 \dots x_n) \cdot \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$

$$= x_1^2 + x_2^2 + \dots + x_n^2 > 0 \quad \forall\ x \neq 0. \ ▨$$

* **Prop:** $A \in M_{m\times n}(\mathbb{K}),\ B \in M_{n\times k}$

$$\underbrace{{}^t(\underbrace{\underbrace{A}_{m\times n}\cdot\underbrace{B}_{n\times k}}_{m\times k})}_{k\times m} = \underbrace{\underbrace{{}^tB}_{k\times n}\cdot\underbrace{{}^tA}_{n\times m}}_{k\times m}$$

Dimo:

$$\left({}^t(A\cdot B)\right)_{ij} = (A\cdot B)_{ji}$$

$$= \sum_{l=1}^{n} (A)_{jl}\cdot(B)_{li}$$

$$\left(({}^tB)\cdot({}^tA)\right)_{ij} = \sum_{l=1}^{n} ({}^tB)_{il}\cdot({}^tA)_{lj}$$

$$= \sum_{l=1}^{n} (B)_{li}\cdot(A)_{jl} \qquad ▨$$

Esempi: (1) $V = M_{n\times n}(\mathbb{R})$

$$\langle A | B\rangle = \mathrm{tr}(\underbrace{\underbrace{{}^tA}_{n\times n}\cdot\underbrace{B}_{n\times n}}_{n\times n})$$

E' prod. scal: lin. a sin

$$\mathrm{tr}\left({}^t(\lambda A+\mu B)\cdot C\right) =$$

$$= \mathrm{tr}\left((\lambda\,{}^tA + \mu\,{}^tB)\cdot C\right)$$

$$= \mathrm{tr}\left(\lambda\,{}^tAC + \mu\,{}^tB\cdot C\right)$$

$$= \lambda\,\mathrm{tr}({}^tA\cdot C) + \mu\cdot\mathrm{tr}({}^tB\cdot C)$$

lin a dx ...

simm. $$\mathrm{tr}({}^tB\cdot A) = \mathrm{tr}({}^t({}^tB\cdot A))$$
$$= \mathrm{tr}({}^tA\cdot {}^{tt}B) = \mathrm{tr}({}^tA\cdot B)$$

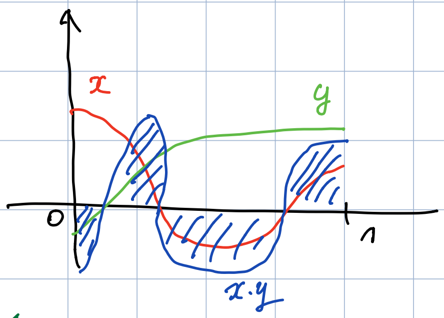
def. pos. $$\mathrm{tr}({}^tA\cdot A) = \sum_{i=1}^{n} ({}^tA\cdot A)_{ii}$$
$$= \sum_{i=1}^{n}\sum_{j=1}^{m} ({}^tA)_{ij}\cdot (A)_{ji}$$
$$= \sum_{i=1}^{n}\sum_{j=1}^{m} (A)_{ji}^{2}$$

= somma dei quadrati di tutti i coeff. di A

> 0 se $A \neq 0$.

(2) $$V = C^0([0,1],\mathbb{R})$$

$$\langle x|y\rangle = \int_0^1 x(t)\cdot y(t)\,dt$$



lin. a sin. $$\langle(\alpha x+\beta y)|z\rangle = \int_0^1 (\alpha x+\beta y)(t)\cdot z(t)\,dt$$
$$= \int_0^1 (\alpha x(t)+\beta y(t))\cdot z(t)\,dt$$
$$= \int_0^1 (\alpha x(t)z(t)+\beta y(t)z(t))\,dt$$

$$= \alpha \int_0^1 x(t)\cdot z(t)\,dt + \beta \int_0^1 y(t)\cdot z(t)\,dt$$

$$= \alpha \langle x|z\rangle + \beta \langle y|z\rangle$$

Oss: se $f: V\times V \to \mathbb{R}$ è lin. a sin. e simm, è lin a dx.

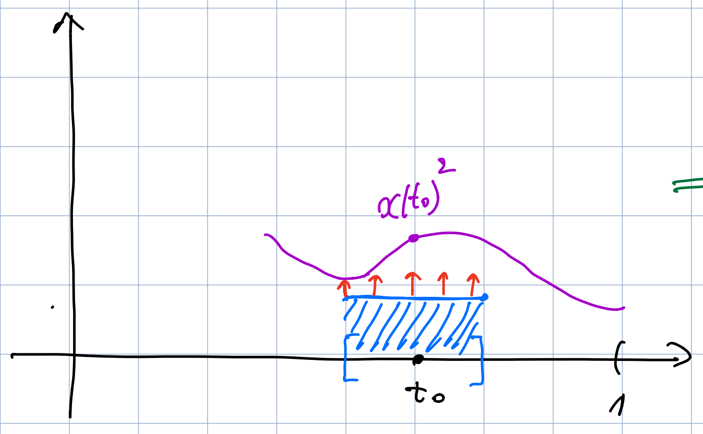
simm. $\langle y|x\rangle = \int_0^1 y(t)\cdot x(t)\,dt = \int_0^1 x(t)\cdot y(t)\,dt = \langle x|y\rangle$

def. pos. $\langle x|x\rangle = \int_0^1 x(t)^2\cdot dt \geq 0$ poiché $x(t)^2 \geq 0$ $\forall t$.

Se $x \neq 0$, cioè $\exists t_0$ t.c. $x(t_0)\neq 0 \Rightarrow x(t_0)^2 > 0$

ma x^2 è continua, dunque esiste $\varepsilon > 0$ t.c.

$x(t)^2 \geq \frac{1}{2} x(t_0)^2$ su $[t_0-\varepsilon, t_0+\varepsilon]$



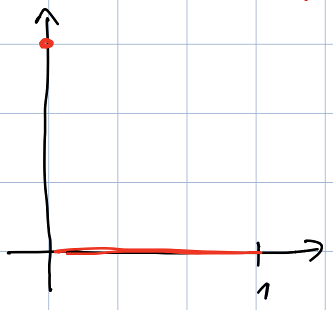
$$\Rightarrow \int_0^1 x(t)^2\,dt \geq \frac{1}{2} x(t_0)^2 \cdot 2\varepsilon > 0.$$

Es: su $\mathcal{I}([0,1]\to\mathbb{R})$ = le funzioni $[0,1]\to\mathbb{R}$ integrabili

$\langle x|y\rangle = \int_0^1 x(t)\cdot y(t)\,dt$ è bil. simm.

ma non è def. pos.

Infatti: $x(t) = \begin{cases} 1 & \text{per } t=0 \\ 0 & \text{per } t\in (0,1] \end{cases}$ $\qquad x \neq 0$



$$\int_0^1 x(t)^2\, dt = 0$$

Prod. scal. su V è $V \times V \to \mathbb{R}$ bil. simm. def. pos.

$\langle x|y\rangle_{\mathbb{R}^n} = {}^t x \cdot y$ lo è.

Q: ce ne sono altri su $\mathbb{R}^n$?

A: sì, ad. es. se $\lambda_1, \dots, \lambda_n > 0$

$$\langle x|y\rangle = \lambda_1 x_1 y_1 + \lambda_2 x_2 y_2 + \dots + \lambda_n x_n y_n$$
$$= (x_1, \dots, x_n) \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$$

Def: date $A \in M_{n\times n}(\mathbb{R})$ definisco

$$\langle \cdot | \cdot \rangle_A : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$$
$$(x, y) \longmapsto \langle x|y\rangle_A = \underbrace{\overset{1\times n}{{}^t x} \cdot \overset{n\times n}{A} \cdot \overset{n\times 1}{y}}_{1\times 1}.$$

Oss: $\langle \cdot|\cdot \rangle_{\mathbb{R}^n} = \langle \cdot | \cdot \rangle_{I_n}$

Prop: •) ogni $\langle \cdot | \cdot \rangle_A$ è bilineare

••) tutte le bilineari su $\mathbb{R}^n$ sono così

Dimo: •) lin. a sinistra

$$\begin{aligned}\langle \alpha x + \beta y | z \rangle_A &= {}^t(\alpha x + \beta y) \cdot A \cdot z \\ &= (\alpha\, {}^t x + \beta \cdot {}^t y) \cdot A \cdot z \\ &= \alpha\, {}^t x \cdot A \cdot z + \beta \cdot {}^t y\, A\, z \\ &= \alpha \langle x | z \rangle_A + \beta \cdot \langle y | z \rangle_A\end{aligned}$$

a dx ...

••) data $f : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$ cerco A t.c.

$f = \langle \cdot | \cdot \rangle_A$ cioè $f(x,y) = \langle x | y \rangle_A$ $\forall x, y$.

Oss: $\langle e_i | e_j \rangle_A = {}^t e_i \cdot A \cdot e_j$

$$= \begin{pmatrix} 0 & \cdots & 0 & \underset{\underset{i}{\uparrow}}{1} & 0 & \cdots & 0 \end{pmatrix} \begin{array}{c|cccc} & & & j & \\ \hline & a_{11} & \cdots & & a_{1n} \\ & \vdots & & & \vdots \\ i & & & \boxed{\;} & \\ & \vdots & & & \vdots \\ & a_{n1} & \cdots & & a_{nn} \end{array} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \leftarrow j$$

$$= a_{ij}$$

Dunque poniamo $a_{ij} = f(e_i, e_j)$ e $A = (a_{ij})_{i,j=1,\dots,n}$

Resta da vedere che $f(x,y) = \langle x | y \rangle_A$ $\forall x, y$

Infatti:

$$f(x,y) = f\left(\sum_{i=1}^{n} x_i e_i \,,\, \sum_{j=1}^{n} y_j e_j\right)$$

$$\underset{\text{lin a sin}}{=} \sum_{i=1}^{n} x_i \, f\Big(e_i, \sum_{j=1}^{n} y_j e_j\Big)$$

$$= \sum_{i=1}^{n} x_i \sum_{j=1}^{n} y_j \underbrace{f(e_i, e_j)}_{a_{ij}} \quad \text{per def.}$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} x_i \cdot a_{ij} \cdot y_j = {}^t x \cdot A \cdot y \,. \qquad ▨$$

Prop: $\langle \cdot | \cdot \rangle_A$ è simm. $\Longleftrightarrow$ A è simm.

Dim: $\langle \cdot | \cdot \rangle_A$ simm $\Longleftrightarrow \langle y | x \rangle_A = \langle x | y \rangle_A \quad \forall x,y$

$$\Longleftrightarrow \underbrace{{}^t y \cdot A \cdot x}_{1 \times 1} = {}^t x \cdot A \cdot y \quad \forall x,y$$

$$\Longleftrightarrow {}^t\left({}^t y \cdot A \cdot x\right) = {}^t x \cdot A \cdot y \quad \forall x,y$$

$$\Longleftrightarrow {}^t x \cdot {}^t A \cdot {}^{tt} y = {}^t x \cdot A \cdot y \quad \forall x,y$$

$$\Longleftrightarrow {}^t x \cdot {}^t A \cdot y - {}^t x \cdot A \cdot y = 0 \quad \forall x,y$$

$$\Longleftrightarrow \ {}^t x \cdot ({}^t A \cdot y - A \cdot y) = 0 \qquad \forall x,y$$

$$\Longleftrightarrow \ {}^t x \cdot ({}^t A - A) \cdot y = 0 \qquad \forall x,y$$

Se ${}^tA = A$ ciò è vero.
Viceversa se ciò è vero vale in particolare per $x = e_i, y = e_j$ $i,j \in \{1,\dots,n\}$ dunque

$$({}^tA - A)_{ij} = 0 \qquad \forall i,j$$

$$\Longrightarrow {}^tA = A .$$ ▨

Per quali matrici A simm la $\langle \cdot | \cdot \rangle_A$ è prod. scal?
Risposta difficile nel seguito.

Esempio: $A = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$. La $\langle \cdot | \cdot \rangle_A$ è def. pos.

$$\Longleftrightarrow \left\langle \begin{pmatrix} x \\ y \end{pmatrix} \middle| \begin{pmatrix} x \\ y \end{pmatrix} \right\rangle_A > 0 \qquad \forall \begin{pmatrix} x \\ y \end{pmatrix} \neq 0$$

$$\Longleftrightarrow (x \ \ y) \begin{pmatrix} a & b \\ b & c \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} > 0 \qquad \forall \begin{pmatrix} x \\ y \end{pmatrix} \neq 0$$

$$\Longleftrightarrow ax^2 + 2bxy + cy^2 > 0 \qquad \forall \begin{pmatrix} x \\ y \end{pmatrix} \neq 0.$$

$$\Longleftrightarrow a > 0, \quad at^2 + 2bt + c > 0 \qquad \forall t$$

(separando caso $y=0$ diviso per y^2 e posto $t = x/y$)

$\Longleftrightarrow a>0 \ , \ \Delta/4 < 0$

$\Longleftrightarrow a>0 \quad b^2-ac<0$

$\Longleftrightarrow a>0 \quad ac-b^2>0$

$\Longleftrightarrow a>0 \ , \ \det(A)>0.$

Vedremo le $n \times n$.

———— ∘ ————

Fissiamo V sp. vett. su $\mathbb{R}$ con prod. scal. $\langle \cdot | \cdot \rangle$.

Def: chiamo norma associata $\|\cdot\| : V \to \mathbb{R}$

$$\|v\| = \sqrt{\langle v|v\rangle} \geq 0.$$

Oss:
$$\begin{aligned}\|x+y\|^2 &= \langle x+y|x+y\rangle \\ &= \langle x|x+y\rangle + \langle y|x+y\rangle \\ &= \langle x|x\rangle + \langle x|y\rangle + \langle y|x\rangle + \langle y|y\rangle \\ &= \|x\|^2 + 2\langle x|y\rangle + \|y\|^2\end{aligned}$$

$$\Longrightarrow \langle x|y\rangle = \frac{1}{2}\left(\|x+y\|^2 - \|x\|^2 - \|y\|^2\right)$$

Oss: $\|v\| > 0 \quad \forall v \neq 0 \quad , \quad \|0\| = 0 \quad ;$

$$\|\lambda v\| = \sqrt{\langle \lambda v | \lambda v\rangle} = \sqrt{\lambda^2 \cdot \langle v|v\rangle} = |\lambda| \cdot \|v\|$$

Prop (disug. Cauchy-Schwatz):

$$|\langle v|w\rangle| \leq \|v\| \cdot \|w\|$$

e l'uguaglianza vale se e solo se sono proporzionali.

Dimo: proporzionali: $w = \lambda v$

$$|\langle v|\lambda v\rangle| = |\lambda \cdot \langle v|v\rangle| = |\lambda| \cdot \|v\|^2$$

$$\|v\| \cdot \|\lambda . v\| = |\lambda| . \|v\|^2$$

OK

non proporzionali: $tv + w \neq 0 \quad \forall t \in \mathbb{R}$

$$\Rightarrow \quad \|tv+w\|^2 > 0 \quad \forall t$$

$$\Rightarrow \quad t^2 \cdot \|v\|^2 + 2t \langle v|w\rangle + \|w\|^2 > 0 \quad \forall t$$

$\Rightarrow \quad \Delta/4 < 0$ cioè $\langle v|w\rangle^2 - \|v\|^2 \cdot \|w\| < 0$

$$\rightarrow \quad \langle v|w\rangle^2 < \|v\|^2 \cdot \|w\|^2,$$

$$\Rightarrow \quad |\langle v|w\rangle| < \|v\| \cdot \|w\| .$$

▨