Geom - Civ - 27/5/21

8/1/21 ⑤ $A=\begin{pmatrix}0&\alpha&\beta\\-\alpha&0&\gamma\\-\beta&-\gamma&0\end{pmatrix}$ coniugata via ortog. a $\begin{pmatrix}0&k&0\\-k&0&0\\0&0&0\end{pmatrix}$

autoval. di A sono $\pm ik, 0$.

$$p_A(t) = \det\begin{pmatrix} t & -\alpha & -\beta\\ \alpha & t & -\gamma \\ \beta & \gamma & t\end{pmatrix} = t\left(t^2+(\alpha^2+\beta^2+\gamma^2)\right)$$

$$\Rightarrow \quad k = \pm\sqrt{\alpha^2+\beta^2+\gamma^2}$$

⑥ Classificare $(t+3)x^2-4xy+y^2-2x+2y+2=0$
se non degenere

$$A = \begin{pmatrix} t+3 & -2 & -1\\ -2 & \boxed{1} & 1\\ -1 & 1 & 2\end{pmatrix}$$

$d_1 = 1 > 0$

$d_2 = t+3-4 = t-1$

$d_3 = 2t+\cancel{6}\cancel{+2}+2-1\cancel{-8}-t-3 = t-2$; deg. per $t=2$

		1		2	
d_2	$-$	•	$+$		$+$
d_3	$-$		$-$	•	$+$
	ip.	par	ell	deg	$\emptyset$

Es: verificare ciò

$(\ \ )^2+(\ \ )^2=0$

$$(t+3)x^2-4xy+y^2-2x+2y+2=0$$

$$y^2-4yx+(t+3)x^2+2y-2x+2=0$$

$$\begin{cases} X=y \\ Y=x \end{cases} \qquad X^2-4XY+(t+3)Y^2+2X-2Y+2=0$$

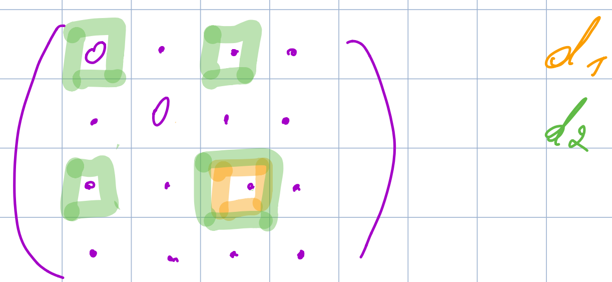
$$\begin{pmatrix} 1 & -2 & 1 \\ -2 & t+3 & -1 \\ 1 & -1 & 2 \end{pmatrix}$$

Per una conica posso scegliere come d_1 il posto $(2,2)$ ma d_2 è fissato.

Per quadrica

$$\begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$

- come d_1 posso scegliere $(1,1), (2,2), (3,3)$
- come d_2 posso scegliere una 2×2 dentro la 3×3 in alto a sinistra che contenga d_1
- d_3 è lui



⑦ Classificare $8x^2+4xy+4xz-2yz-4z=0$.

$$\begin{pmatrix} 8 & 2 & 2 & 0 \\ 2 & 0 & -1 & 0 \\ 2 & -1 & 0 & -2 \\ 0 & 0 & -2 & 0 \end{pmatrix}$$

$d_2<0$

$d_3 = -4-4-8<0$

$$d_4 = 2\cdot\begin{vmatrix} 8 & 2 & 2 \\ 2 & 0 & -1 \\ 0 & 0 & -2 \end{vmatrix} = -4\cdot\begin{vmatrix} 8 & 2 \\ 2 & 0 \end{vmatrix} = 16>0$$

Autoval: $(+-+)- \;=\; (++-)-$

$x^2+y^2=1+z^2$ iperb. iperb. a 1 falda.

(8) $8x^3 - 6x^2y - 11xy^2 + 3y^3 = 0$

$(x+y)(8x^2 - 14xy + 3y^2) = 0$ $\qquad \frac{7 \pm \sqrt{49-24}}{8} = \frac{7 \pm 5}{8}$

$(x+y)(2x-3y)(4x-y) = 0$

$[1:-1], [3:2], [1:4]$

$8x^3 - 6x^2y - 11xy^2 + 3y^3 = 0$

- per $y=0$ trovo $8x^3 = 0 \Rightarrow x = 0 \Rightarrow$ niente : $[0:0] \notin \mathbb{P}^1(\mathbb{R})$
- pongo $t = x/y$, divido equaz per y e trovo

$$\boxed{8t^3 - 6t^2 - 11t + 3 = 0.}$$

Criterio: se $a_0 t^n + a_1 t^{n-1} + \dots + a_{n-1} t + a_n = 0$ con $a_0, a_1, \dots, a_{n-1}, a_n \in \mathbb{Z}$ ha una soluzione $t_0 \in \mathbb{Q}$ allora $t_0 = \frac{k}{h}$ dove k è divisore di a_n e h è un divisore di a_0.

Per la nostra se c'è una soluzione razionale è $\frac{k}{h}$ dove $k = \pm 1, \pm 3$ $h = \pm 1, \pm 2, \pm 4, \pm 8$. La cerco sostituendo

$t = 1$, $t = -1$, $t = 3$, $t = -3$, $t = 1/2$, $t = -1/2$

$t = 3/2$, $t = -3/2$, $t = 1/4$, $t = -1/4$, ...

Appena sostituendo trovo 0 mi fermo, divido e abbasso il grado.

$8t^3-6t^2-11t+3$

$t=1$: $\quad 8-6-11+3 \neq 0 \quad$ No

$t=-1 \quad -8-6+11+3=0 \quad$ Sì

dividido per $t+1$:

$$\begin{array}{c|ccc|c} & 8 & -6 & -11 & 3 \\ -1 & & -8 & 14 & -3 \\ \hline & 8 & -14 & 3 & \checkmark \end{array}$$

$(t+1)(8t^2-14t+3)$

$t_1=-1 \qquad t_{2,3}=\dfrac{7\pm\sqrt{49-24}}{8} = \begin{cases} 3/2 \\ 1/4 \end{cases}$

$\dfrac{x}{y}=-1 \rightsquigarrow [1:-1]$

$\dfrac{x}{y}=\dfrac{3}{2} \rightsquigarrow [3:2]=[3/2:1]$

$\dfrac{x}{y}=\dfrac{1}{4} \rightsquigarrow [1:4]=[-2\sqrt{19}:-8\sqrt{19}]$

(9) $\alpha(t)=\begin{pmatrix} t\cdot e^t \\ 3t-t^2 \end{pmatrix} \qquad \kappa(0) \qquad sgn(\kappa(t))$

$\alpha'(t)=\begin{pmatrix} e^t+t\cdot e^t \\ 3-2t \end{pmatrix} \qquad \alpha''(t)=\begin{pmatrix} 2\cdot e^t+t\cdot e^t \\ -2 \end{pmatrix}$

$\alpha'(0)=\begin{pmatrix} 1 \\ 3 \end{pmatrix} \qquad \alpha''(0)=\begin{pmatrix} 2 \\ -2 \end{pmatrix}$

$$\kappa(0) = \frac{\det\begin{pmatrix} 1 & 2 \\ 3 & -2 \end{pmatrix}}{10^{3/2}} = \dots$$

$$\operatorname{sgn}(\kappa(t)) = \operatorname{sgn} \underbrace{\begin{vmatrix} e^t(t+1) & e^t(t+2) \\ 3-2t & -2 \end{vmatrix}}_{e^t \cdot \begin{vmatrix} t+1 & t+2 \\ 3-2t & -2 \end{vmatrix}}$$

$$= \operatorname{sgn} \begin{vmatrix} t+1 & t+2 \\ 3-2t & -2 \end{vmatrix} = \operatorname{sgn}\left(-2t-2-3t-6+2t^2+4t\right)$$

$$= \operatorname{sgn}\left(2t^2-t-8\right)$$

$$t_{1,2} = \frac{1 \pm \sqrt{65}}{4} \qquad \dots \qquad t \in \left(\frac{1-\sqrt{65}}{4}, \frac{1+\sqrt{65}}{4}\right)$$

(10) $$\int_{\partial D^2} \omega = \int_{D^2} d\omega = \int_{D^2} \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dx\,dy$$

$\omega = f \cdot dx + g \cdot dy$

$$= \int_{D^2} -3\,dx\,dy = -3\pi$$

26/1/21 (1) $\quad p_A(t) \qquad A \in M_{3\times 3}$

$tr(A) = -4 \qquad \det(A) = 5 \qquad p_A(-1) = 9$

Att: $p_A(t) = \det(t \cdot I_n - A) = \boxed{(-1)^n} \cdot \det(A - t \cdot I_n)$

$= t^n - tr(A) \cdot t^{n-1} + \dots + (-1)^n \cdot \det(A)$

$p_A(t) = t^3 + 11 \cdot t^2 + c \cdot t - 5$

$-1 + 4 - c - 5 = 9 \qquad c = -4$

(2) $\begin{pmatrix} t^2-5 & t^2+4t+3 & t^2+3t+2 \\ 0 & 1-t & t+2 \\ 0 & 0 & t^2+t+1 \end{pmatrix}$

$\lambda_1 = t^2-5 \qquad \lambda_2 = 1-t \qquad \lambda_3 = t^2+t+1$

Se $\lambda_1 \neq \lambda_2 \neq \lambda_3 \neq \lambda_1$ ē diago, se no ... m. a/g

$\lambda_1 = \lambda_2 \qquad t^2-5 = 1-t \qquad t^2+t-6=0 \qquad (t+3)(t-2)=0$

$t=-3, \; t=2$

$\lambda_1 = \lambda_3 \qquad t^2-5 = t^2+t+1 \qquad t=-6$

$\lambda_2 = \lambda_3 \qquad 1-t = t^2+t+1 \qquad t^2+2t=0 \qquad t=0, \; t=-2$

Per $t \neq -6, -3, -2, 0, 2$ ē diago.

$\begin{pmatrix} t^2-5 & t^2+4t+3 & t^2+3t+2 \\ 0 & 1-t & t+2 \\ 0 & 0 & t^2+t+1 \end{pmatrix}$

$t=-6$ $\begin{pmatrix} 31 & 15 & 20 \\ 0 & 7 & -4 \\ 0 & 0 & 31 \end{pmatrix}$

$m.a.(31)=2$ $\quad m.g.(31) = 3 - \text{rank}\,(A-31\cdot I)$

$= 3 - \text{rank} \begin{pmatrix} 0 & 15 & 20 \\ 0 & -24 & -4 \\ 0 & 0 & 0 \end{pmatrix} = 3-2=1$

No

$t=-3$ $\begin{pmatrix} 4 & 0 & 2 \\ 0 & 4 & -1 \\ 0 & 0 & 7 \end{pmatrix}$

$m.a.(4)=2$ $\quad m.g.(4) = 3 - \text{rank} \begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & -1 \\ 0 & 0 & 3 \end{pmatrix} = 3-1 = 2$ Sì

$$\begin{pmatrix} t^2-5 & t^2+4t+3 & t^2+3t+2 \\ 0 & 1-t & t+2 \\ 0 & 0 & t^2+t+1 \end{pmatrix}$$

$t=-2$ $\begin{pmatrix} -1 & -1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$ Sì

$t=0$ $\begin{pmatrix} -5 & 3 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$ No

$t=2$ $\begin{pmatrix} -1 & 15 & 12 \\ 0 & -1 & 4 \\ 0 & 0 & 7 \end{pmatrix}$ No

③ $\langle\cdot|\cdot\rangle_{\begin{pmatrix} 2 & -3 \\ -3 & 5 \end{pmatrix}}$ $\quad$ prod. scal: simm, $d_1>0$, $d_2>0$

$\begin{pmatrix} x \\ y \end{pmatrix} \perp_A \begin{pmatrix} 2 \\ -1 \end{pmatrix}$ $\qquad \left\| \begin{pmatrix} x \\ y \end{pmatrix} \right\|_A = 1$

$$(x\ y)\begin{pmatrix}2 & -3\\ -3 & 5\end{pmatrix}\begin{pmatrix}2\\ -1\end{pmatrix}=0$$

$$(x\ y)\cdot\begin{pmatrix}7\\ -11\end{pmatrix}=0 \qquad 7x-11y=0$$

$$v=\begin{pmatrix}11\\ 7\end{pmatrix} \qquad \text{Risposta: } \pm\frac{v}{\|v\|_A}$$

$$\|v\|_A^2=(11\ 7)\begin{pmatrix}2 & -3\\ -3 & 5\end{pmatrix}\begin{pmatrix}11\\ 7\end{pmatrix}=(11\ 7)\cdot\begin{pmatrix}1\\ 2\end{pmatrix}$$

$$=11+14=25$$

$$\text{Risp} = \pm\frac{1}{5}\begin{pmatrix}11\\ 7\end{pmatrix}$$

(4) $A=\begin{pmatrix}5 & iz-i\\ iz+i & z+2i\end{pmatrix}$ autoval. reali diago via unitarie per $z=?$

Teo spettrale su $\mathbb{C}$

A diago da unitarie $\Longleftrightarrow$ A normale

$\|$ $\qquad\qquad\qquad\qquad$ $\|$

$\exists U\in M_{n\times n}(\mathbb{C})$ $\qquad$ $A^*\cdot A=A\cdot A^*$

$U^*=U^{-1}$, $U^{-1}\cdot A\cdot U=\begin{pmatrix}\lambda_1 & & 0\\ & \ddots & \\ 0 & & \lambda_n\end{pmatrix}$

Visto: A hermitiana $\Longrightarrow$ (normale) e ha autoval. reali

$$\begin{array}{c} \| \\ A^* = A \end{array}$$

Oss: vale anche viceversa:

A normale autoval. reali $\Longrightarrow U^* \cdot A \cdot U = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} \quad \lambda_j \in \mathbb{R}$

$$\Longrightarrow A = U \cdot \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} \cdot U^*$$

$$= A^* = {}^t\overline{A} = {}^t\overline{\left(U \cdot \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} \cdot U^* \right)}$$

$$= {}^t\overline{U^* \cdot \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}} \cdot {}^tU$$

$$= U^{**} \cdot \underbrace{\begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}}_{\text{poiché } \lambda_j \in \mathbb{R}} \cdot U^*$$

$$= U \cdot \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} \cdot U^* = A$$

Ricordare: A diag. via unitarie con autoval. reali $\Longleftrightarrow A$ hermitiana.

$$A = \begin{pmatrix} 5 & iz - i \\ iz + i & z + 2i \end{pmatrix}$$

$\in \mathbb{R}$ $\quad$ $\in \mathbb{R}$

complessi coniugati

$z = t - 2i \qquad t \in \mathbb{R}$

$$i(t-2i) - i = \overline{i(t-2i) + i}$$

$$i\cdot t + 2 - i = \overline{i\cdot t + 2 + i}$$

$$it \cancel{+2} \cancel{-i} = -it \cancel{+2} \cancel{-i}$$

$$t = 0$$

Risp. $z = -2i$

$$\begin{pmatrix} 5 & iz - i \\ iz + i & z + 2i \end{pmatrix} \qquad z = -2i \qquad \begin{pmatrix} 5 & 2-i \\ 2+i & 0 \end{pmatrix}$$

⑤ Che isom. di $\mathbb{R}^2$ è $\frac{1}{17}\begin{pmatrix} 8 & 15 \\ 15 & -8 \end{pmatrix}$

Ortogonale:
$$\left\| \frac{1}{17}\begin{pmatrix} 8 \\ 15 \end{pmatrix} \right\| = \frac{1}{17}\cdot\sqrt{64+225} = \frac{1}{17}\sqrt{289} = 1$$

$$\left\| \frac{1}{17}\begin{pmatrix} 15 \\ 8 \end{pmatrix} \right\| = 1$$

$$\left\langle \begin{pmatrix} 8 \\ 15 \end{pmatrix} \middle| \begin{pmatrix} 15 \\ -8 \end{pmatrix} \right\rangle = 0$$

Fatto: 2×2 ortog. è coniugata a
- $\begin{pmatrix} \cos\vartheta & -\sin\vartheta \\ \sin\vartheta & \cos\vartheta \end{pmatrix}$ rotaz. $\det = +1$
- $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ rifl. $\det = -1$

$\det = -1 \Longrightarrow$ rifl.

⑥ $[t : t+2 : 2-3t] = [2-2t : t-13 : 4t-1] \in \mathbb{P}^2(\mathbb{R})$

per $t = \dots ?$

$$[\alpha:\beta:\gamma] = [\alpha':\beta':\gamma']$$

$$\iff \begin{pmatrix}\alpha'\\ \beta'\\ \gamma'\end{pmatrix} = k\cdot\begin{pmatrix}\alpha\\ \beta\\ \gamma\end{pmatrix}$$

$$\iff \mathrm{Span}\begin{pmatrix}\alpha\\ \beta\\ \gamma\end{pmatrix} = \mathrm{Span}\begin{pmatrix}\alpha'\\ \beta'\\ \gamma'\end{pmatrix}$$

$$\iff \mathrm{rank}\begin{pmatrix}\alpha & \alpha'\\ \beta & \beta'\\ \gamma & \gamma'\end{pmatrix} = 1$$

$\iff$ tutti e tre i sottodet 2×2 di $\begin{pmatrix}\alpha & \alpha'\\ \beta & \beta'\\ \gamma & \gamma'\end{pmatrix}$ sono 0

$\begin{pmatrix} t & 2-2t\\ t+2 & t-13\\ 2-3t & 4t-1\end{pmatrix}$: imponiamo tutti e tre i sottodet 2×2 nulli

I+III : $\underline{4t^2} - \underline{t} - 4 + \underline{4t} + \underline{6t} - \underline{6t^2} = 0$

$2t^2 - 9t + 4 = 0$

$(2t-1)(t-4) = 0$

$t = 1/2 \qquad t = 4$

$t = 1/2$: $\begin{pmatrix} 1/2 & 1\\ 5/2 & -25/2\\ 1/2 & 1\end{pmatrix}$ No

$t = 4$: $\begin{pmatrix} 4 & -6\\ 6 & -9\\ -10 & 15\end{pmatrix}$ Sì

$$\begin{matrix} \| & \| \\ 2\cdot\begin{pmatrix}2\\3\\-5\end{pmatrix} & -3\cdot\begin{pmatrix}2\\3\\-5\end{pmatrix}\end{matrix}$$

⑦ Classificare $\underline{2x^2} + \underline{4xz} + \underline{2x} + \underline{3y^2} + \underline{4y} + \underline{8z^2} - \underline{12z} + \underline{11} = 0$

$$\begin{pmatrix} 2 & 0 & 2 & 1\\ 0 & 3 & 0 & 2\\ 2 & 0 & 8 & -6\\ 1 & 2 & -6 & 11\end{pmatrix}$$

$d_1 > 0 \qquad d_2 > 0$

$d_3 = 48 - 12 > 0$

$$d_4 = \begin{vmatrix} 0 & 0 & 0 & 1 \\ -4 & 3 & -4 & 2 \\ 14 & 0 & 20 & -6 \\ -21 & 2 & -28 & 11 \end{vmatrix} = \dots \begin{cases} > 0 \longrightarrow \emptyset \\ < 0 \longrightarrow \text{ellissoide} \end{cases}$$