Geom - Cir - 22/4/21

$$\mathcal{L} = \left\{ x \in \mathbb{R}^n \ : \ {}^t\begin{pmatrix} x \\ 1 \end{pmatrix} \cdot A \cdot \begin{pmatrix} x \\ 1 \end{pmatrix} = 0 \right\}$$

$$\overline{\mathcal{L}} = \left\{ [x] \in \mathbb{P}^n(\mathbb{R}) \ : \ {}^t x \cdot A \cdot x = 0 \right\}$$ completamento proiettivo

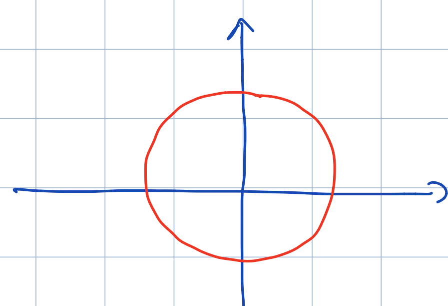
<u>Oss</u>: $\overline{\mathcal{L}} \cap \mathbb{R}^n = \mathcal{L}$

$\mathcal{L}_\infty = \overline{\mathcal{L}} - \mathcal{L} \subset \mathbb{P}^{n-1}(\mathbb{R})$ parte all'infinito di $\mathcal{L}$.

<u>Es</u>: coniche non deg:

<u>ellisse</u>: $\mathcal{L}: x^2+y^2=1$ $\quad \overline{\mathcal{L}}: x^2+y^2=z^2$

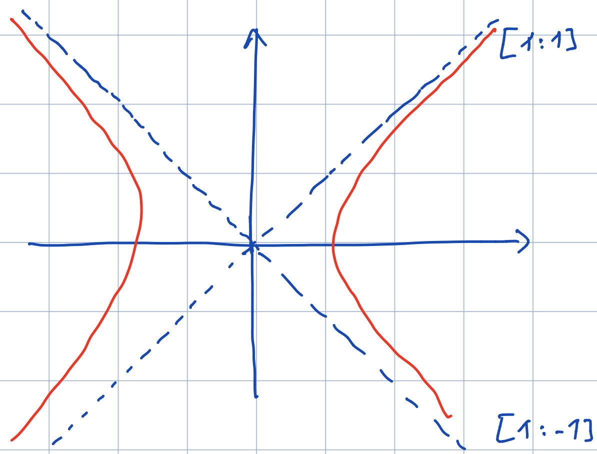
$$\mathcal{L}_\infty = \{ [x:y] \in \mathbb{P}^1(\mathbb{R}) \ : \ x^2+y^2=0 \} = \emptyset$$



ellisse: nessun pto all'∞

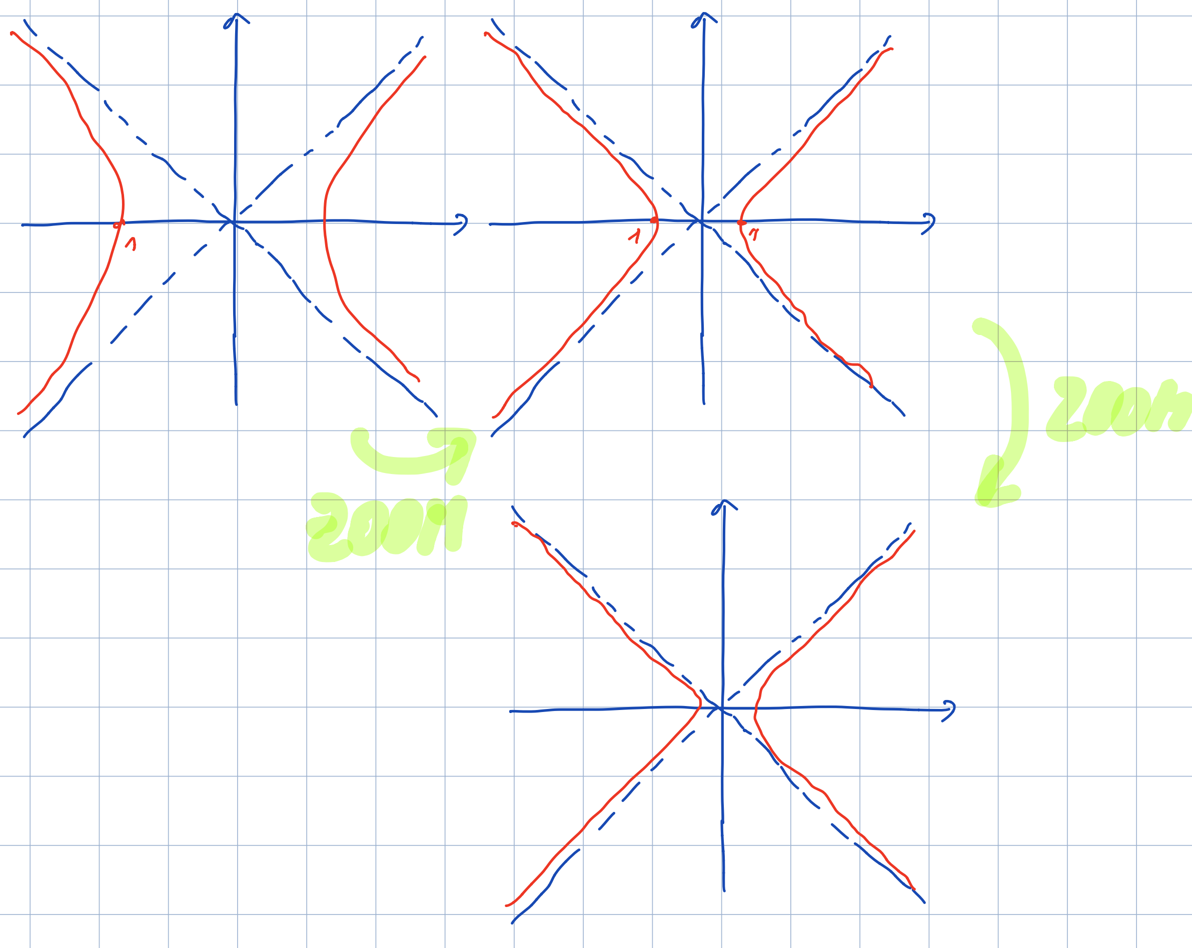
<u>iperbole</u>: $\mathcal{L}: x^2-y^2=1$ $\quad \overline{\mathcal{L}}: x^2-y^2=z^2$

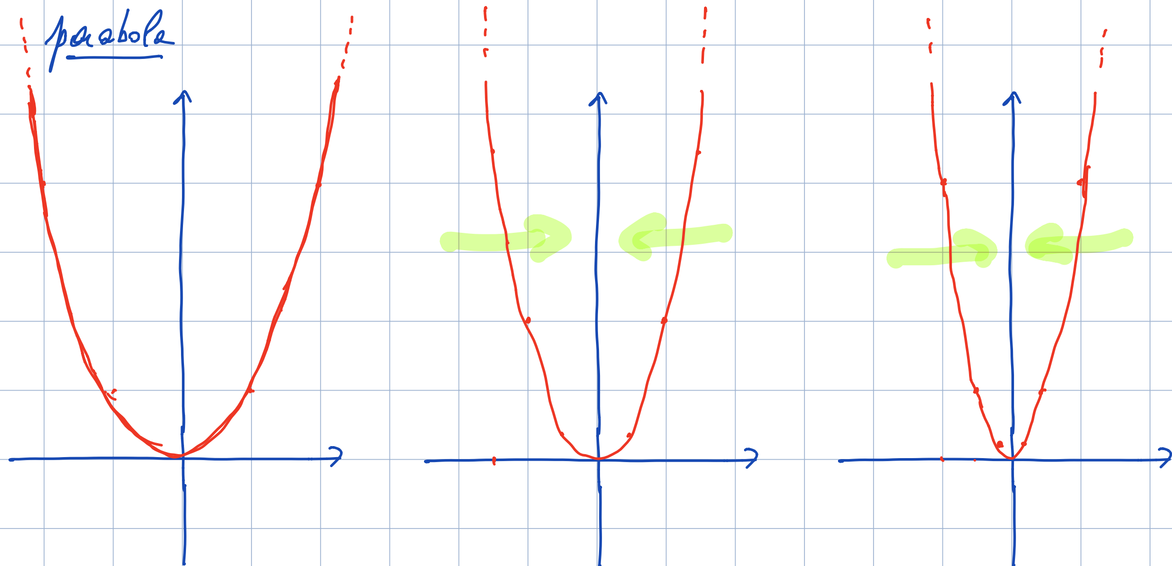
$$\mathcal{L}_\infty = \{ [x:y] \in \mathbb{P}^1(\mathbb{R}) \ : \ x^2-y^2=0 \} = \{ [1:1], [1:-1] \}$$

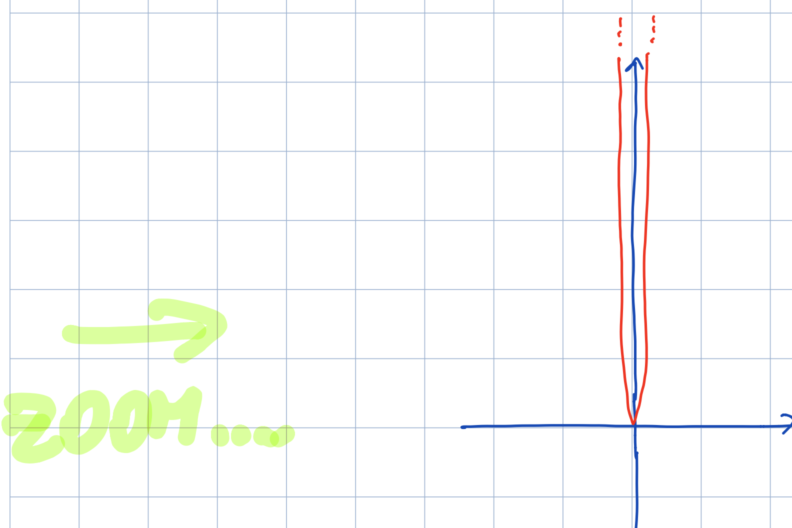


iperbole: due p.ti all'∞

= le direzioni dei suoi asintoti.







la parabola ha un solo pto all'∞ (la direz. suo asse)

$\mathcal{L}: \; y = x^2$ $\qquad$ $\overline{\mathcal{L}}: \; zy = x^2$

$$\mathcal{L}_\infty = \{[x:y] \in \mathbb{P}^1(\mathbb{R}) : x^2 = 0\} = \{[0:1]\}$$

$\mathcal{L}$ = qualsiasi luogo def. da equaz. poli in $\mathbb{R}^n$

$\overline{\mathcal{L}} \subset \mathbb{P}^n(\mathbb{R})$: luogo def. dalla equaz. omogeneizzata

$\mathcal{L}_\infty = \overline{\mathcal{L}} \setminus \mathcal{L}$

Es: $\mathcal{L} : \quad 3xy^4 - 17x^2y^3 + 11xy^2 - 7x - 5 = 0$

$\overline{\mathcal{L}} : \quad 3xy^4 - 17x^2y^3 + 11xy^2z^2 - 7xz^4 - 5z^5 = 0$

$\mathcal{L}_\infty : \quad 3xy^4 - 17x^2y^3 = 0$

—— o ——

Teo: una quadrica proiettiva $\mathcal{L} = \{[x] \in \mathbb{P}^n(\mathbb{R}) : {}^t x \cdot A \cdot x = 0\}$
a meno di cambi di coord è

$$\mathcal{L} = \{[x] \in \mathbb{P}^n(\mathbb{R}) : x_1^2 + \dots + x_p^2 = x_{p+1}^2 + \dots + x_{p+q}^2\} \quad \Big| \; p \geq q.$$

Dimo: $\exists M$ ortog t.c. ${}^t M \cdot A \cdot M = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$

$x \mapsto M \cdot x$ da cui $\mathcal{L} = \{[x] : \lambda_1 x_1^2 + \dots + \lambda_n x_n^2 = 0\}$

A meno di riordinare suppongo $\lambda_1, \dots, \lambda_p > 0 \quad \lambda_{p+1}, \dots, \lambda_{p+q} < 0$

$\lambda_{p+q+1} = \dots = \lambda_n = 0.$

$x_j \mapsto \sqrt{\lambda_j}\, x_j \qquad j = 1 \dots p$

$x_j \mapsto \sqrt{-\lambda_j}\, x_j \qquad j = p+1 \dots q$

$\mathcal{L} : \quad x_1^2 + \dots + x_p^2 - x_{p+1}^2 - \dots - x_{p+q}^2 = 0$

Se $p < q$ cambio segno. ▨

Cor: in $\mathbb{P}^2(\mathbb{R})$ le coniche proiettive non dep. $(\det(A) \neq 0)$ sono

$$x^2+y^2+z^2=0 \qquad \emptyset$$

$$\boxed{x^2+y^2-z^2=0}$$ unica conica proiettiva non dep non $\emptyset$.

Dunque per forza ellisse, iperbole, parabola hanno stesso $\overline{\mathcal{L}}$:

ellisse: $\mathcal{L}: x^2+y^2=1$ $\quad \overline{\mathcal{L}}: x^2+y^2=z^2$ cioè $x^2+y^2-z^2=0$

iperbole: $\mathcal{L}: x^2-y^2=1$ $\quad \overline{\mathcal{L}}: x^2-y^2=z^2$ cioè $y^2+z^2-x^2=0$

parabola: $\mathcal{L}: y=x^2$ $\quad \overline{\mathcal{L}}: yz=x^2$

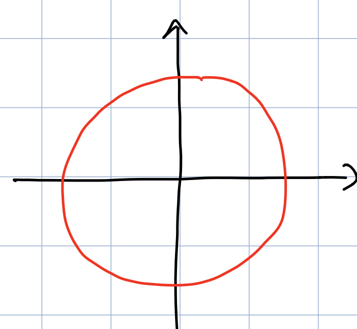
$$(u+v)(u-v)=x^2$$
$$u^2-v^2=x^2$$
$$\boxed{x^2+v^2-u^2=0}$$

cambio coord $\begin{cases} y=u+v \\ z=u-v \\ x=x \end{cases}$

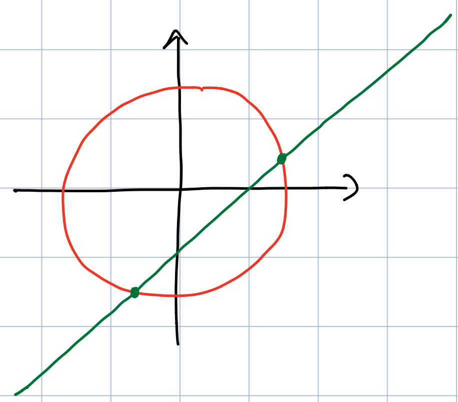
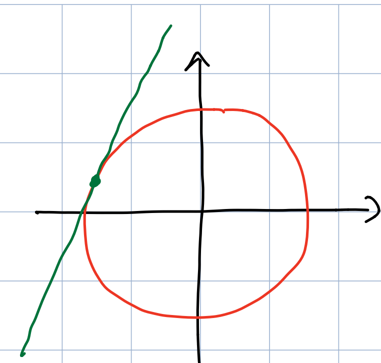
$x^2+y^2-z^2=0$ conica proiett. non dep. non $\emptyset$

$z=1$ parte affine

$z=0$ retta all'∞



Però: scegliendo altre rette all'∞, cambia parte affine

se scelgo una tale retta all'∞ la conica affine che trovo ha 2 pti all'∞
$\Longrightarrow$ è iperbole

se scelgo una tale retta all'∞ la conica affine che trovo ha un pto all'∞
$\Longrightarrow$ è parabola

Quadriche affini non degeneri

$$\mathcal{Q} = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 : \ {}^t\!\begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix} \cdot A \cdot \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix} = 0 \right\}$$

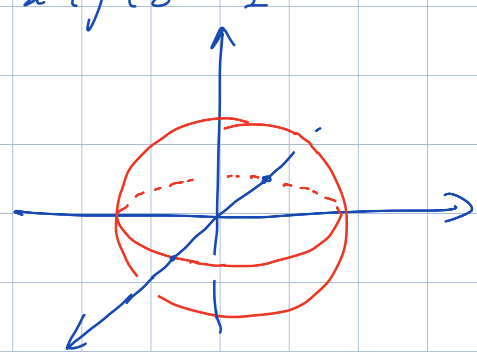
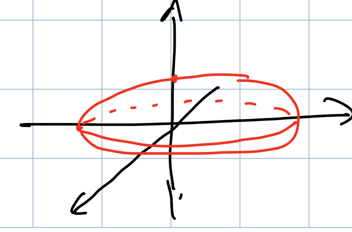
$$A = \begin{pmatrix} Q & \ell \\ {}^t\ell & c \end{pmatrix} \quad \text{simmetrica} \qquad \det(A) \neq 0.$$

Modelli

0. $x^2+y^2+z^2+1=0$ $\emptyset$

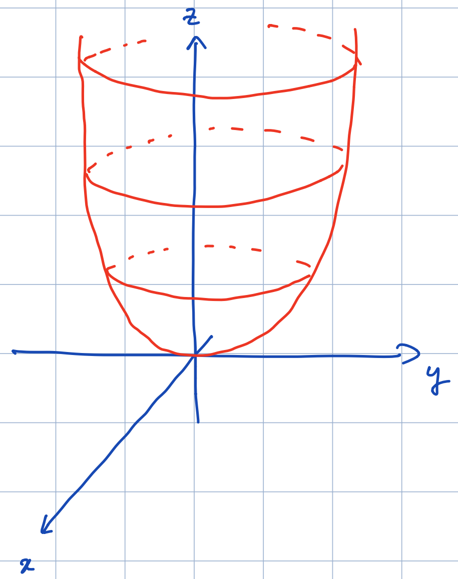
1. $x^2+y^2+z^2=1$ ellissoide

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

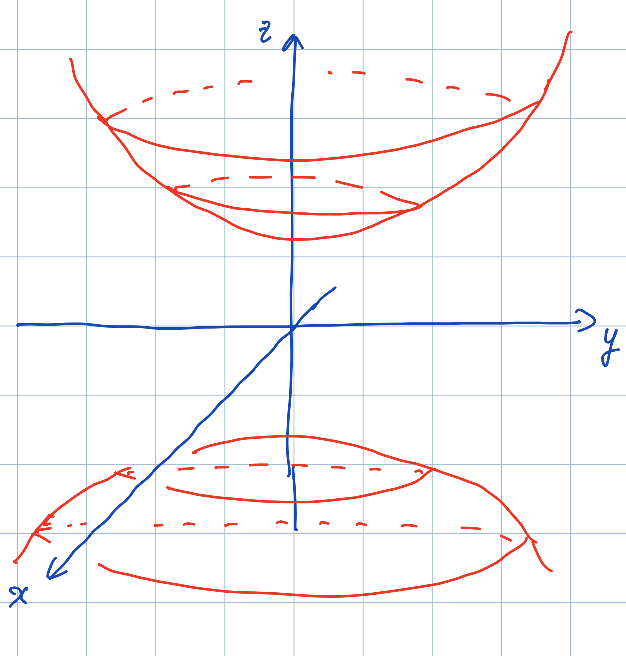
2. $z = x^2 + y^2$

Ricordo: se x, y compaiono solo in x^2+y^2 si tratta di superf. di rotaz. intorno asse z



paraboloide ellittico

3. $z^2 = x^2 + y^2 + 1$

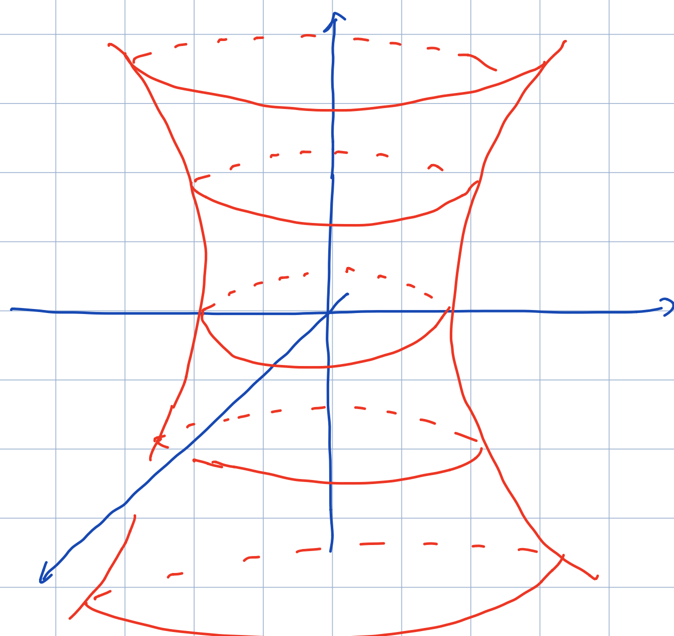


iperboloide ellittico
A DUE FALDE

superficie ottenuta ruotando una iperbole intorno all'asse di simmetria maggiore (che contiene vertici e fuochi)

4. $z^2 = x^2 + y^2 - 1$

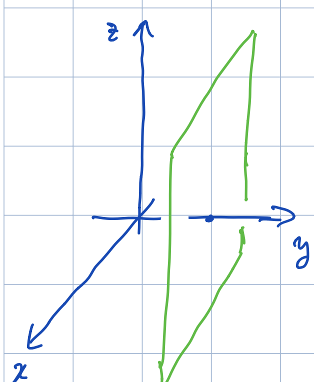
$x^2 + y^2 = z^2 + 1$

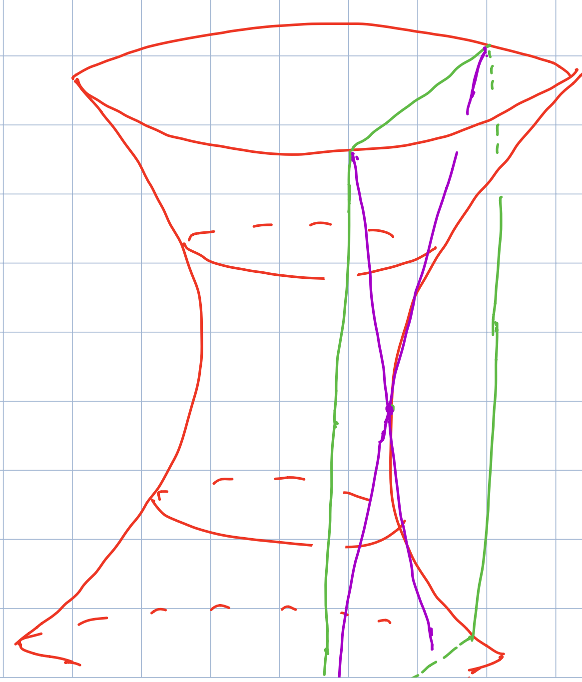


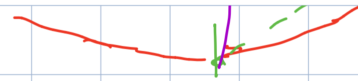
superficie di rotazione ottenuta ruotando iperbole intorno a asse [illegible] (asse del segmento che unisce i vertici)

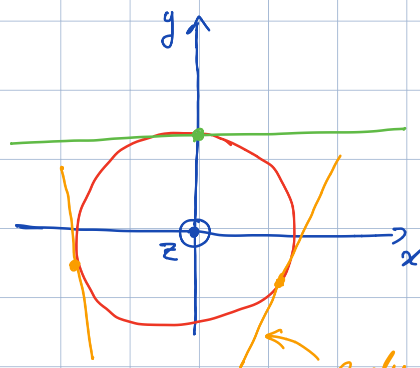
iperboloide iperbolico A UNA FALDA

$z^2 = x^2 + y^2 - 1$; se interseco con piano $y = 1$ trovo $z^2 = x^2$ cioè $z = \pm x$, due rette



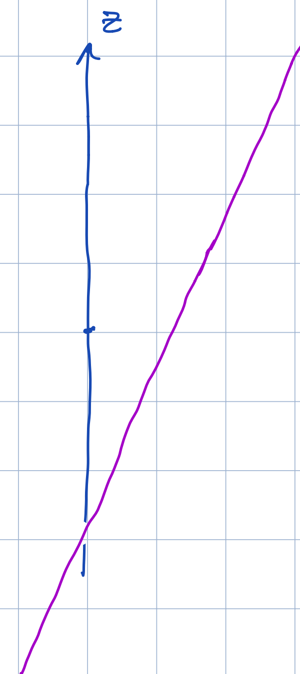






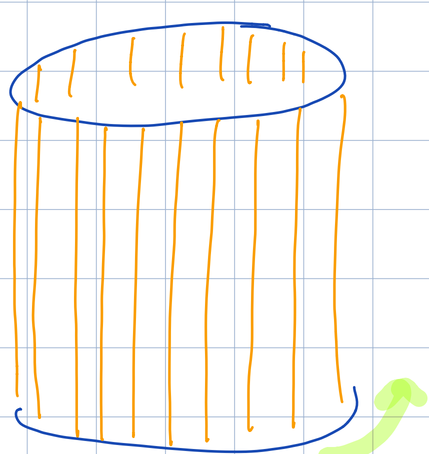
visto dall'alto

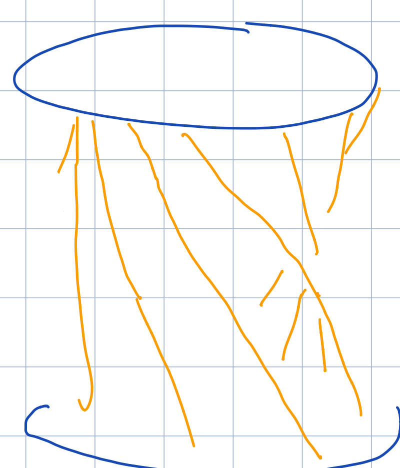
anche con questi piani l'intersez. è due rette



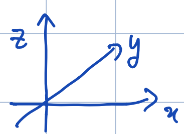
l'iperboloide è la superficie generata dalla rotazione di una retta (viola) intorno a un asse ad essa sghemba (asse z)

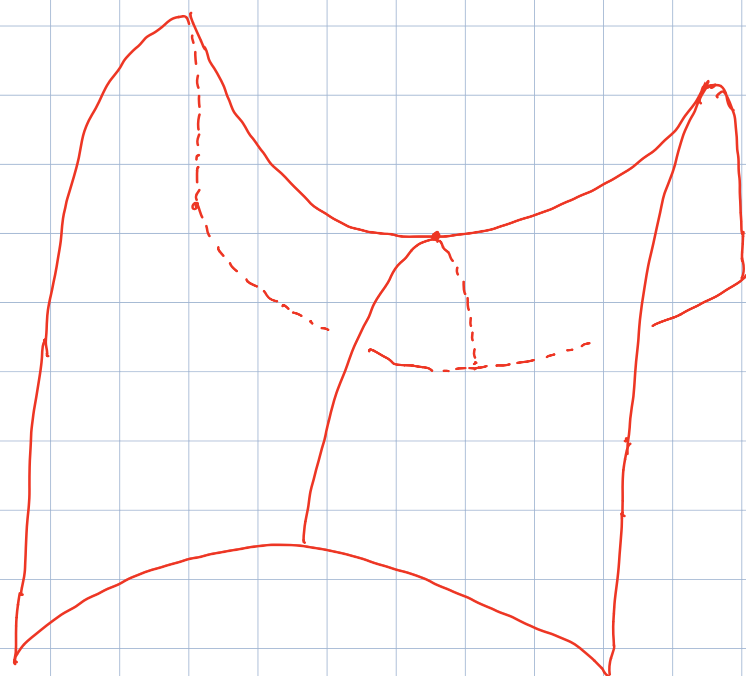
Supp: Google : iperboloide + rigato (tensostrutture)





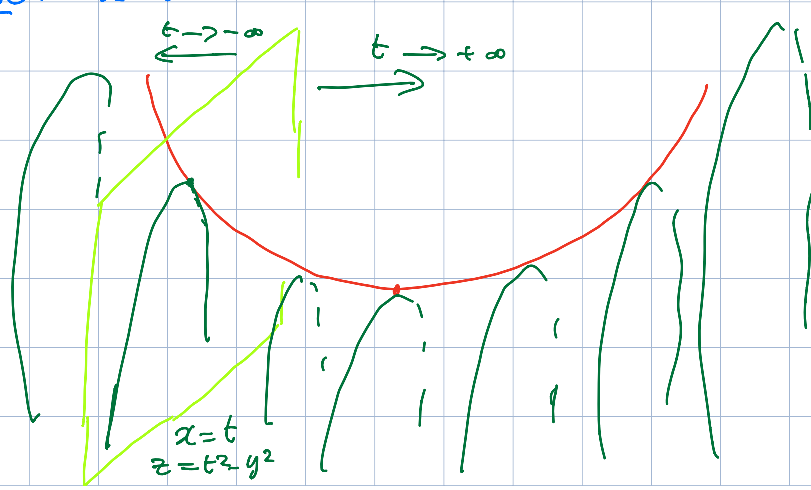
5. $z = x^2 - y^2$



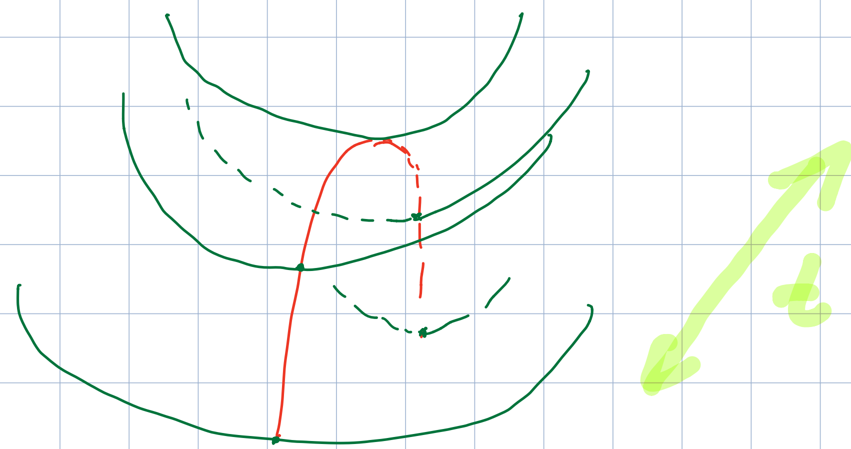


paraboloide a sella
iperbolico

Figure animate: $x = t \quad -\infty < t < +\infty$

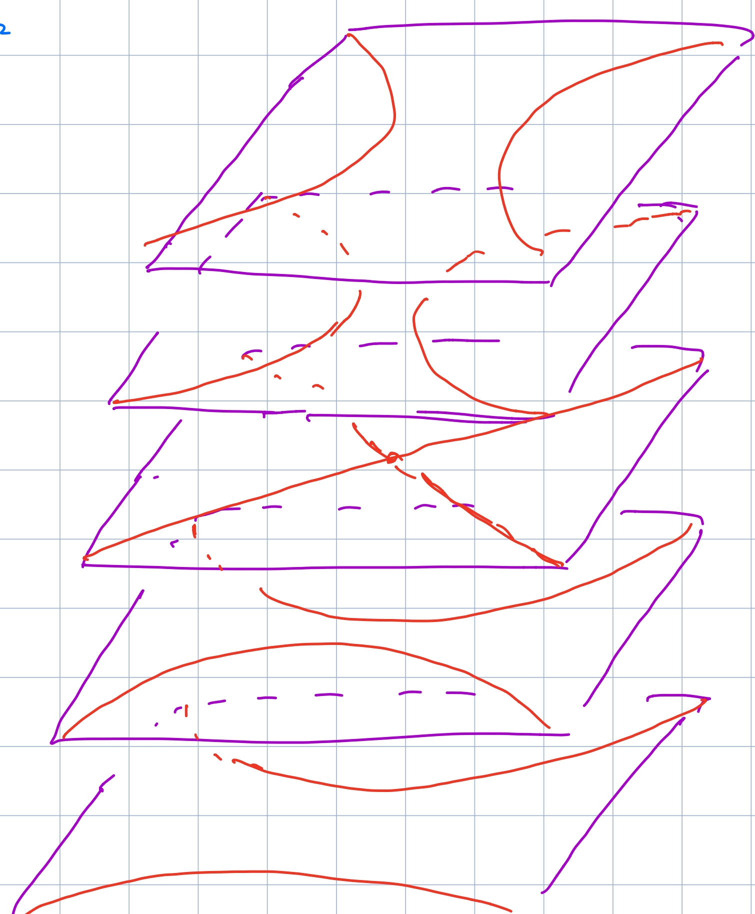


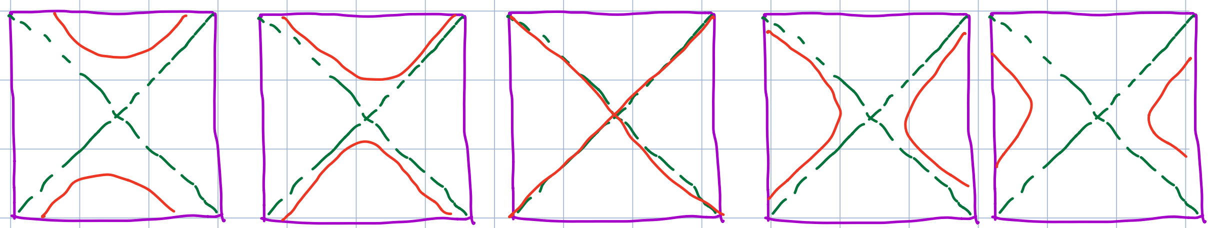
$y = t \quad -\infty < t < +\infty$



$z = t \quad -\infty < t < +\infty$

$z = x^2 - y^2$





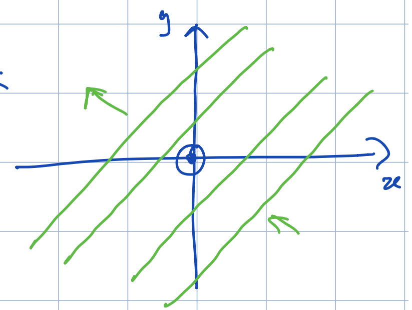
$z = x^2 - y^2$

$z = (x+y)(x-y)$

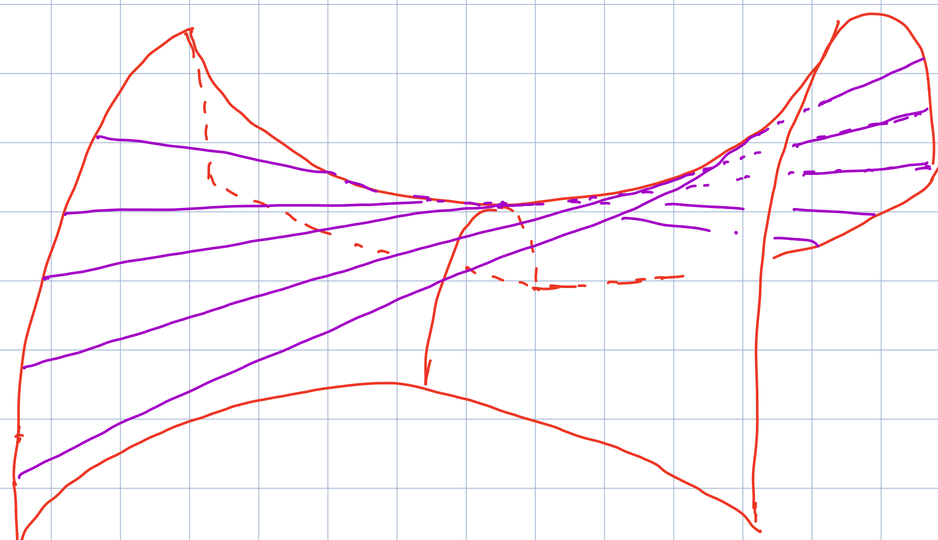
Sul piano $x - y = k$

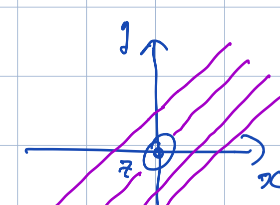
ho $z = k \cdot (x-y)$

una retta



Analog. su $x+y=k$ ho $z = k(x+y)$





Google: paraboloide rigato

ruled

0. $x^2+y^2+z^2+1=0$ $\quad \emptyset$

1. $x^2+y^2+z^2=1$ ellissoide

2. $z=x^2+y^2$ parab. ell.

3. $z^2=x^2+y^2+1$ iperb. ell.

4. $z^2=x^2+y^2-1$ iperb. iperb

5. $z=x^2-y^2$ parab. iperb

perché

Quadriche proiettive non deg.:

$x^2+y^2+z^2+w^2=0$ $\quad \emptyset$

$\boxed{x^2+y^2+z^2=w^2}$ ellissoide proiettivo (w = 1 : ellissoide)

$\boxed{x^2+y^2=z^2+w^2}$ iperboloide proiettivo (w = 1 : iperboloide)

2. $\mathcal{L}: z=x^2+y^2$ $\quad \overline{\mathcal{L}}: zw=x^2+y^2 \rightsquigarrow u^2-v^2=x^2+y^2$

$\boxed{x^2+y^2+v^2=u^2}$ ellittico

3. $\mathcal{L}: z^2=x^2+y^2+1$ $\quad \overline{\mathcal{L}}: \boxed{z^2=x^2+y^2+w^2}$ ellittico

4. $\mathcal{L}: z^2=x^2+y^2-1$ $\quad \overline{\mathcal{L}}: z^2=x^2+y^2-w^2$ $\quad \boxed{x^2+y^2=z^2+w^2}$ iperbolico

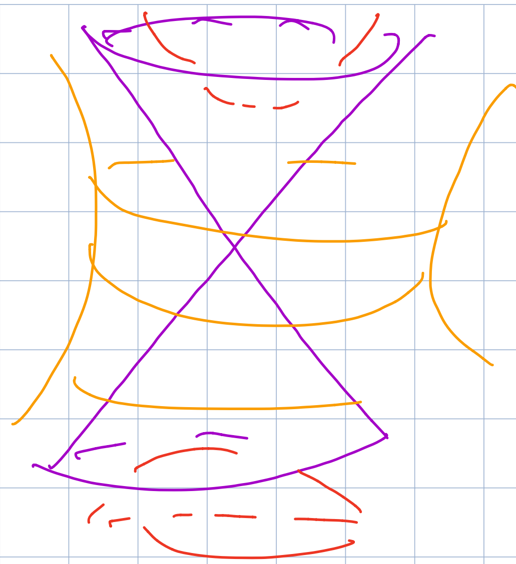
5. $\mathcal{L}: z=x^2-y^2$ $\quad \overline{\mathcal{L}}: zw=x^2-y^2$

$\rightsquigarrow u^2-v^2=x^2-y^2$ $\quad \boxed{x^2+v^2=u^2+y^2}$ iperbolico

Pti all'∞: 0. 1. $\rightsquigarrow \emptyset$

2. $z=x^2+y^2$ $\quad \mathcal{L}_\infty: x^2+y^2=0$ $\quad \mathcal{L}_\infty=\{[0:0:1]\}$

3.4 $z^2=x^2+y^2$ conica non deg. (circonf)



5. $z = x^2 - y^2$ $\qquad x^2 - y^2 = 0$ $\qquad x = \pm y$ $\qquad z$ liber [unc. prox]