Geom – Civile - 18/3/21

- $f: V \to V$ autoaggiunta $\langle v | f(w) \rangle = \langle f(v) | w \rangle \quad \forall v, w$
- $M \in \mathcal{M}_{n \times n}(\mathbb{R})$ $\langle . | . \rangle_{\mathbb{R}^n}$ autoaggiunta $\Longleftrightarrow$ simmetrica
- p è proiez. ortog. $\Longleftrightarrow$ $p \circ p = p$, p autoaggiunta.

9.3.9 $\quad \mathbb{R}^2 \quad W: 5x_1 + 12x_2 = 0$

$$\boxed{\begin{array}{l} \text{uso questo:} \\ w_1, \dots, w_k \text{ base ortog. di } W \\ p_W(v) = \sum_{i=1}^{k} \frac{\langle v | w_i \rangle}{\|w_i\|^2} \cdot w_i \\ k = 1 \text{ segmento ortog.} \end{array}}$$

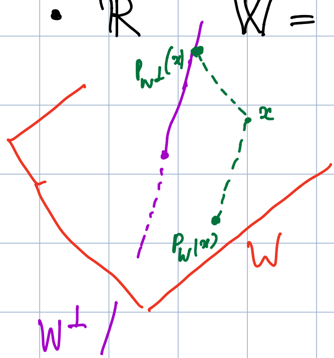
$W = \mathrm{Span}\begin{pmatrix} 12 \\ -5 \end{pmatrix}$

$$p_W(x) = \frac{\left\langle x \middle| \begin{pmatrix} 12 \\ -5 \end{pmatrix} \right\rangle}{13} \begin{pmatrix} 12 \\ -5 \end{pmatrix} = \frac{1}{13}(12x_1 - 5x_2)\begin{pmatrix} 12 \\ -5 \end{pmatrix}$$

$$= \frac{1}{13}\begin{pmatrix} 144x_1 - 60x_2 \\ -60x_1 + 25x_2 \end{pmatrix} = \underbrace{\frac{1}{13}\begin{pmatrix} 144 & -60 \\ -60 & 25 \end{pmatrix}}_{P} \cdot x$$

$${}^tP = P \quad ; \quad P^2 = \frac{1}{13^2}\begin{pmatrix} 144 & -60 \\ -60 & 25 \end{pmatrix}\begin{pmatrix} 144 & -60 \\ -60 & 25 \end{pmatrix} = \dots = P$$

- $\mathbb{R}^3 \quad W = \mathrm{Span}\left(\begin{pmatrix} 2 \\ 1 \\ -5 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}\right) \qquad P_W = \dots \quad P$



$$x = P_W(x) + P_{W^\perp}(x)$$

$$\Rightarrow P_W(x) = x - P_{W^\perp}(x)$$

$$W^\perp = \mathrm{Span}\left(\begin{pmatrix} 2 \\ 1 \\ -5 \end{pmatrix} \wedge \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}\right) = \mathrm{Span}\begin{pmatrix} +7 \\ +11 \\ +5 \end{pmatrix}$$

$14 + 11 - 25$ ✓ $\qquad 7 - 22 + 15$ ✓

$$p_W(x) = x - \frac{\left\langle x \middle| \begin{pmatrix}7\\11\\5\end{pmatrix}\right\rangle}{49+121+25}\begin{pmatrix}7\\11\\5\end{pmatrix} = \begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} - \frac{7x_1+11x_2+5x_3}{195}\begin{pmatrix}7\\11\\5\end{pmatrix}$$

$$= \frac{1}{195}\begin{pmatrix}195x_1 - 49x_1 - 77x_2 - 35x_3\\ 195x_2 - 77x_1 - 121x_2 - 55x_3\\ 195x_3 - 35x_1 - 55x_2 - 25x_3\end{pmatrix}$$

$$= \underbrace{\frac{1}{195}\begin{pmatrix}146 & -77 & -35\\ -77 & 74 & -55\\ -35 & -55 & 170\end{pmatrix}}_{P}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}$$

$${}^tP = P$$

$$P\cdot P = \frac{1}{195^2}\begin{pmatrix}146 & -77 & -35\\ & & \\ & & \end{pmatrix}\begin{pmatrix}146 & & \\ -77 & & \\ -35 & & \end{pmatrix}$$

$$= \frac{1}{195^2}\begin{pmatrix}28470 & * & *\\ * & * & *\\ * & * & *\end{pmatrix} = \frac{1}{195}\begin{pmatrix}146 & * & *\\ * & * & *\\ * & * & *\end{pmatrix}$$

$$= P$$

Oss: M autoaggiunta rispetto a $\langle\cdot|\cdot\rangle_A$ su $\mathbb{R}^4$

$\iff \langle Mx|y\rangle_A = \langle x|My\rangle_A \quad \forall x,y$

$\iff {}^t(Mx)\cdot A\cdot y = {}^tx\cdot A\cdot(My) \quad \forall x,y$

$\iff {}^tx\cdot{}^tM\cdot A\cdot y = {}^tx\cdot A\cdot M\cdot y \quad \forall x,y \iff {}^tM\cdot A = A\cdot M$

9.3.11 $\mathbb{R}^2$ con $\langle\cdot|\cdot\rangle_A$ $\quad A = \begin{pmatrix} 5 & -2 \\ -2 & 1 \end{pmatrix}$

$W: 4x_1 - 3x_2 = 0.$

$W = \mathrm{Span}\begin{pmatrix} 3 \\ 4 \end{pmatrix}$ $\quad \left\|\begin{pmatrix} 3 \\ 4 \end{pmatrix}\right\|_A^2 = 5\cdot 3^2 - 4\cdot 3\cdot 4 + 1\cdot 4^2$

$= 45 - 48 + 16 = 13$

$$P_W(x) = \frac{\langle x | \begin{pmatrix} 3 \\ 4 \end{pmatrix}\rangle_A}{13}\cdot\begin{pmatrix} 3 \\ 4 \end{pmatrix} = \frac{(x_1\ x_2)\begin{pmatrix} 5 & -2 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} 3 \\ 4 \end{pmatrix}}{13}\begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

$$= \frac{(x_1, x_2)\cdot\begin{pmatrix} 7 \\ -2 \end{pmatrix}}{13}\begin{pmatrix} 3 \\ 4 \end{pmatrix} = \frac{1}{13}\begin{pmatrix} 21x_1 - 6x_2 \\ 28x_1 - 8x_2 \end{pmatrix}$$

$$= \underbrace{\frac{1}{13}\begin{pmatrix} 21 & -6 \\ 28 & -8 \end{pmatrix}}_{P}\cdot\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

${}^tP\cdot A = A\cdot P$: $\quad \frac{1}{13}\begin{pmatrix} 21 & 28 \\ -6 & -8 \end{pmatrix}\cdot\begin{pmatrix} 5 & -2 \\ -2 & 1 \end{pmatrix} = \frac{1}{13}\begin{pmatrix} * & -14 \\ * & * \end{pmatrix}$

$\begin{pmatrix} 5 & -2 \\ -2 & 1 \end{pmatrix}\cdot\frac{1}{13}\begin{pmatrix} 21 & -6 \\ 28 & -8 \end{pmatrix} = \frac{1}{13}\begin{pmatrix} * & -14 \\ * & * \end{pmatrix}$

OK

$$P\cdot P = \frac{1}{13^2}\begin{pmatrix} 21 & -6 \\ 28 & -8 \end{pmatrix}\begin{pmatrix} 21 & -6 \\ 28 & -8 \end{pmatrix} = \frac{1}{13^2}\begin{pmatrix} * & * \\ 13\cdot 28 & * \end{pmatrix}$$

$$= \frac{1}{13}\begin{pmatrix} * & * \\ 28 & * \end{pmatrix}$$

OK

9.3.13 $\quad W: \begin{cases} 3x_1 - 2x_2 = 0 \\ 3x_2 - 5x_3 = 0 \end{cases}$ $\quad \mathbb{R}^3$

$$W = \mathrm{Span}\begin{pmatrix} 3 \\ -2 \\ 0 \end{pmatrix}\wedge\begin{pmatrix} 0 \\ 3 \\ -5 \end{pmatrix} = \mathrm{Span}\begin{pmatrix} 10 \\ 15 \\ 9 \end{pmatrix}$$

30-30 ✓ 45-45 ✓

$$P_W(x) = \frac{10x_1 + 15x_2 + 9x_3}{100 + 225 + 49} \begin{pmatrix} 10 \\ 15 \\ 9 \end{pmatrix} = \underbrace{\frac{1}{\cdots} \begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}}_{P} \cdot x$$

$${}^tP = P^2 = P.$$ finire calcoli

Caso complesso

$$\mathbb{C}^n = \mathbb{R}^{2n} \qquad z = x + iy \qquad \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix} = \begin{pmatrix} x_1 + iy_1 \\ \vdots \\ x_n + iy_n \end{pmatrix}$$

$$\mathbb{C}^n \ni z \Leftrightarrow \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^{2n}$$

$$\begin{aligned} \|z\|^2 = \left\| \begin{pmatrix} x \\ y \end{pmatrix} \right\| &= x_1^2 + \cdots + x_n^2 + y_1^2 + \cdots + y_n^2 \\ &= (x_1^2 + y_1^2) + \cdots + (x_n^2 + y_n^2) \\ &= |z_1|^2 + \cdots + |z_n|^2 \\ &= z_1 \cdot \overline{z_1} + \cdots + z_n \cdot \overline{z_n} \end{aligned}$$

$$\|x\|^2 = x_1 \cdot x_1 + \cdots + x_n \cdot x_n$$

Def: prodotto scalare hermitiano canonico su $\mathbb{C}^n$:

$$\begin{aligned} \langle z | w \rangle_{\mathbb{C}^n} &= z_1 \cdot \overline{w_1} + \cdots + z_n \cdot \overline{w_n} \\ &= (\overline{w_1}, \ldots, \overline{w_n}) \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix} = {}^t\overline{w} \cdot z \end{aligned}$$

In generale per $A \in M_{m \times n}(\mathbb{C})$

$$A^* = {}^t\bar{A} \in M_{n \times m}(\mathbb{C})$$ <u>coniugata</u>.

<u>Def</u>: se V è sp. rett. su $\mathbb{C}$ chiamo $f: V \times V \to \mathbb{C}$

- sesquilineare se è
  - * lineare a sinistra $f(\lambda_1 v_1 + \lambda_2 v_2 | w) = \lambda_1 f(v_1, w) + \lambda_2 f(v_2, w)$
  - * antilineare a destra $\boxed{f(v, \lambda_1 w_1 + \lambda_2 w_2) = \bar{\lambda}_1 \cdot f(v, w_1) + \bar{\lambda}_2 \cdot f(v, w_2)}$

  $\forall \ldots$

- hermitiano se $f(w, v) = \overline{f(v, w)}$
- def. pos. se $f(v, v) > 0 \quad \forall v \neq 0$
- prodotto scalare hermitiano se sesquilin, hermitiano, def. pos.

<u>Oss</u>: $\alpha \in \mathbb{C}$, "$\alpha > 0$" non ha senso; ma se f è hermitiano $f(v, v) = \overline{f(v, v)} \Rightarrow f(v, v) \in \mathbb{R}$

$\Rightarrow$ ha senso "$f(v, v) > 0$".

<u>Oss</u>: $\langle \cdot | \cdot \rangle_{\mathbb{C}^n}$ soddisfa

- $\langle \lambda_1 z_1 + \lambda_2 z_2 | w \rangle_{\mathbb{C}^n} = \lambda_1 \langle z_1 | w \rangle_{\mathbb{C}^n} + \lambda_2 \langle z_2 | w \rangle_{\mathbb{C}^n}$

  $$\langle \lambda_1 z_1 + \lambda_2 z_2 | w \rangle_{\mathbb{C}^n} = w^* \cdot (\lambda_1 z_1 + \lambda_2 z_2) = \lambda_1 w^* z_1 + \lambda_2 w^* z_2 = \lambda_1 \langle z_1 | w \rangle_{\mathbb{C}^n} + \lambda_2 \langle z_2 | w \rangle_{\mathbb{C}^n}$$

  <u>OK</u>

$$\langle z | \lambda_1 w_1 + \lambda_2 w_2 \rangle_{\mathbb{C}^n} = \bar{\lambda}_1 \langle z | w_1 \rangle_{\mathbb{C}^n} + \bar{\lambda}_2 \langle z | w_2 \rangle$$

$$\langle z | \lambda_1 w_1 + \lambda_2 w_2 \rangle_{\mathbb{C}^n} = (\lambda_1 w_1 + \lambda_2 w_2)^* \cdot z = {}^t\overline{(\lambda_1 w_1 + \lambda_2 w_2)} \cdot z$$

$$= {}^t\left(\bar{\lambda}_1 \bar{w}_1 + \bar{\lambda}_2 \cdot \bar{w}_2\right) \cdot z$$

$$= \left(\bar{\lambda}_1 \cdot w_1^* + \bar{\lambda}_2 \cdot w_2^*\right) \cdot z = \bar{\lambda}_1 \cdot w_1^* \cdot z + \bar{\lambda}_2 \cdot w_2^* \cdot z$$

OK

- $\overline{\langle w|z\rangle}_{\mathbb{C}^n} = \overline{z^* \cdot w} = \overline{{}^t\bar{z} \cdot w} = {}^t z \cdot \bar{w}$
$= {}^t\left({}^t z \cdot \bar{w}\right)$
$= {}^t\bar{w} \cdot z = w^* \cdot z$
$= \langle z|w\rangle_{\mathbb{C}^n}$

- $\langle z|z\rangle_{\mathbb{C}^n} = z^* \cdot z = \left(\bar{z}_1 \cdots \bar{z}_n\right)\begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix}$

$= |z_1|^2 + \cdots + |z_n|^2 > 0$ se $z \neq 0$.

Oss: qualche libro usa lin a dx e antilin a sin.

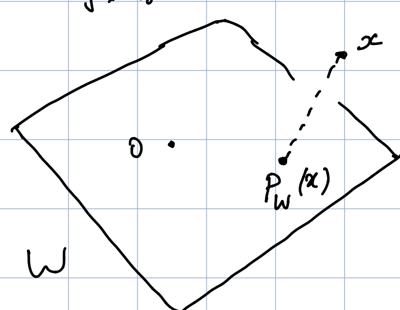
Però: nel caso $\mathbb{C}$ che una delle due parti è antilineare.

ANTILINEARE A DX:

$$f(v, \alpha u + \beta w) = \bar{\alpha} \cdot f(v, u) + \bar{\beta} \cdot f(v, w)$$

$W \subset V$ sttsp. vett. $\quad P_W =$ proiez. ortog. su W

$P_W(x) =$ proiezione $\perp$ su W di x



La teoria si ripete quasi identica salvo ricordare che c'è sesquilin. e non bilin., hermit. non simmetria

Es: $\mathcal{M}_{m\times n}(\mathbb{C})$ $\quad \langle A | B \rangle = tr(B^* \cdot A)$

verifiche per esercizio

Es: $C^0([0,1],\mathbb{C})$ $\quad z = x + iy \quad x,y \in C^0([0,1],\mathbb{R})$

$$\langle z | w \rangle = \int_0^1 z(t) \cdot \overline{w(t)} \cdot dt$$

Su $\mathbb{C}^n$:

- sesquilineari sono tutte e sole $\langle \cdot | \cdot \rangle_A$ $\quad A \in M_{n\times n}(\mathbb{C})$
  dove $\langle z | w \rangle_A = w^* \cdot A \cdot z$

- sesquilin. hermitiane sono tutte e sole $\langle \cdot | \cdot \rangle_A$
  con $A^* = A^*$ (dirò: hermitiana)

oss: A hermitiana se $A^* = A$ cioè

$$\begin{pmatrix} x_{11} & x_{12}+iy_{12} & \cdot & \\ x_{12}-iy_{12} & x_{22} & \cdot & \\ \cdot & & x_{33} & \\ \cdot & & & \ddots \end{pmatrix}$$

→ reali su diag. princ.

→ coniugati quelli simm. risp. a diag. princ.

Oss: $\{A \in M_{n\times n}(\mathbb{C}) : A \text{ hermitiana}\}$ è sttsp. reale ma non un sttsp. complesso.

• def. pos $\langle \cdot | \cdot \rangle_A$??

$\|\cdot\|$ associata $\langle \cdot | \cdot \rangle$ prod. scal. hermitiano su V

- $\|\lambda \cdot v\| = |\lambda| \cdot \|v\|$
- $\|v+w\| \leq \|v\| + \|w\|$ triang.
- $|\langle v|w\rangle| \leq \|v\| \cdot \|w\|$ C-S. (dimo diversa)

$$\|v+w\|^2 = \langle v+w|v+w\rangle = \langle v|v+w\rangle + \langle w|v+w\rangle$$

$$= \underbrace{\langle v|v\rangle}_{\|v\|^2} + \underbrace{\langle v|w\rangle + \underbrace{\langle w|v\rangle}_{\overline{\langle v,w\rangle}}}_{2\cdot \mathrm{Re}(\langle v|w\rangle)} + \underbrace{\langle w|w\rangle}_{\|w\|^2}$$

Oss: su $\mathbb{C}$, $\|\cdot\|$ non determina $\langle \cdot | \cdot \rangle$

ortonormalizzazione G-S: $v_1, \dots, v_k \rightsquigarrow u_1, \dots, u_k$

$$u_1 = v_1 / \|v_1\|$$

$$\tilde{u}_{j+1} = v_{j+1} - \sum_{i=1}^{j} \langle v_{j+1} | u_i \rangle \cdot u_i$$

NON $\langle u_i | v_{j+1} \rangle$

a sin perché $\langle \cdot | \cdot \rangle$ è lin a sin.

$$u_{j+1} = \frac{\tilde{u}_{j+1}}{\|\tilde{u}_{j+1}\|}$$

$$W \rightsquigarrow W^\perp = \{ u \in V : \langle u|w\rangle = 0 \}$$

$$V = W \oplus W^\perp \quad (\text{se } \dim < +\infty)$$

P_W = proiez. ortog.

Se $w_1, \dots, w_k$ è base ortog. di W

NON $\langle w_i | v \rangle$

$$P_W(v) = \sum_{i=1}^{k} \frac{\langle v | w_i \rangle}{\|w_i\|^2} \cdot w_i$$

a sin. perché $\langle \cdot | \cdot \rangle$ è lin. a sin.

lin. in v.

Bessel $\displaystyle\sum_{i=1}^{k} \frac{|\langle v | w_i \rangle|^2}{\|w_i\|^2} \leq \|v\|^2$

$A \in M_{n \times n}(\mathbb{C})$ autoaggiunta risp. a $\langle \cdot | \cdot \rangle_{\mathbb{C}^n} \iff A^* = A$

unitaria se isometria $\iff A^* = A^{-1}$

$\langle z | w \rangle_{\mathbb{C}^n} = \langle Az | Aw \rangle_{\mathbb{C}^n} \ \forall z, w$

Oss: A unitaria $\iff$ colonne base ortonormale di $\mathbb{C}^n$.

Fatto: $M \in M_{n \times n}(\mathbb{C})$ è proiez. ortog.
$\iff M \cdot M = M^* = M$.

Es: $\mathbb{C}^2$ con $\langle \cdot | \cdot \rangle_{\mathbb{C}^2}$ $\quad W = \mathrm{Span}\begin{pmatrix} 1-3i \\ 2+i \end{pmatrix}$

$$P_W(z) = \frac{\langle z \mid \binom{1-3i}{2+i} \rangle}{1+9+4+1} \cdot \begin{pmatrix} 1-3i \\ 2+i \end{pmatrix}$$

$$= \frac{(1+3i)\cdot z_1 + (2-i)\cdot z_2}{15} \cdot \begin{pmatrix} 1-3i \\ 2+i \end{pmatrix}$$

$(2-i)(1-3i)$

$(1+3i)(2+i)$

$$= \underbrace{\frac{1}{15} \begin{pmatrix} 10 & -1-7i \\ -1+7i & 5 \end{pmatrix}}_{P} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$

$P^* = P$ ✓

$$P \cdot P = \frac{1}{15^2} \begin{pmatrix} 10 & -1-7i \\ -1+7i & 5 \end{pmatrix} \cdot \begin{pmatrix} 10 & -1-7i \\ -1+7i & 5 \end{pmatrix}$$

$$= \frac{1}{15^2} \begin{pmatrix} 100+1+49 & -10-70i-5-35i \\ -10+70i-5+35i & 1+49+25 \end{pmatrix}$$

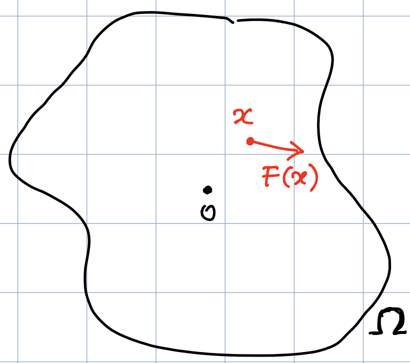
$$= \frac{1}{15^2} \begin{pmatrix} 150 & -15-105i \\ -15+105i & 75 \end{pmatrix} = \frac{1}{15} \begin{pmatrix} 10 & -1-7i \\ -1+7i & 5 \end{pmatrix} = P$$

OK

Teorema spettrale

Prendiamo $\Omega \subset \mathbb{R}^n \quad (n = 2, 3)$

corpo rigido soggetto a forze ma in equilibrio.



$F : \Omega \to \mathbb{R}^n$

$F(0) = 0$

- supponiamo F campo continuo
- operiamo approssimazione lineare

$$F(x) = M \cdot x \qquad M \in \mathcal{M}_{n\times n}(\mathbb{R})$$

- $M = \frac{1}{2}(M + {}^tM) + \frac{1}{2}(M - {}^tM)$

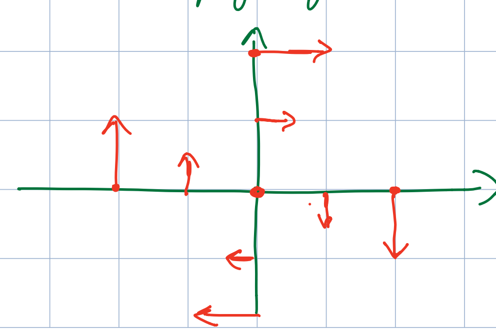
o = baricentro

$$\mathcal{M}_{n\times n}(\mathbb{R}) = \{\text{simmetriche}\} \oplus \{\text{antisimmetriche}\}$$

$$\begin{pmatrix} 4 & 2 \\ -5 & 1 \end{pmatrix} = \begin{pmatrix} 4 & -3/2 \\ -3/2 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 7/2 \\ -7/2 & 0 \end{pmatrix}$$

$$M = \underbrace{\underset{\text{simm}}{T}}_{\substack{\text{tensore degli sforzi} \\ \text{responsabile della deformazione}}} + \underbrace{\underset{\text{antisimmetrica}}{A}}_{\substack{\text{responsabile} \\ \text{di movimenti} \\ \text{rigidi}}}$$

spiego per $n=2$



$$\begin{pmatrix} 0 & \lambda \\ -\lambda & 0 \end{pmatrix} \quad \lambda > 0$$

$T \in M_{n\times n}(\mathbb{R})$ simmetrica $\begin{pmatrix} 3 & 2 & -1 \\ 2 & 5 & 4 \\ -1 & 4 & -7 \end{pmatrix}$