Geom - Civili - 17/3/21

V sp. vett. su $\mathbb{R}$ con $\langle\cdot|\cdot\rangle$ prod. scal.

<u>Prop</u> (disug. di Bessel): se $w_1, \dots, w_k \in V$ sono ortog. $\neq 0$, $v \in V$

$$\sum_{i=1}^{k} \frac{\langle v|w_i\rangle^2}{\|w_i\|^2} \leq \|v\|.$$

<u>Dim</u>: $W = \operatorname{Span}(w_1, \dots, w_k)$; $V = W \oplus W^\perp$;

$$v = \underset{\substack{\cap \\ W}}{w} + \underset{\substack{\cap \\ W^\perp}}{u}$$

$$\|v\|^2 = \|w+u\|^2 = \|w\|^2 + 2\underbrace{\langle w|u\rangle}_{0} + \|u\|^2 \geq \|w\|^2$$

$$= \|P_W(v)\|^2 = \Big\| \sum_{i=1}^{k} \frac{\langle v|w_i\rangle}{\|w_i\|^2} \cdot w_i \Big\|^2$$

$$= \sum_{i=1}^{k} \Big\| \frac{\langle v|w_i\rangle}{\|w_i\|^2} \cdot w_i \Big\|^2 = \sum_{i=1}^{k} \frac{\langle v|w_i\rangle^2}{\|w_i\|^{\cancel{4}\,2}} \cancel{\cdot \|w_i\|^2}. \quad \blacksquare$$

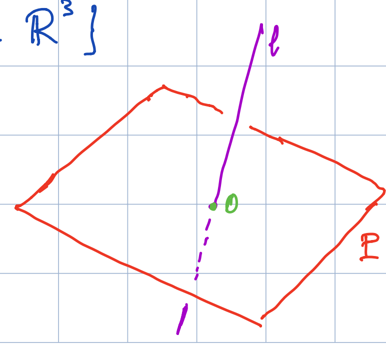
—— o ——

$\mathbb{R}^3$, $\langle\cdot|\cdot\rangle_{\mathbb{R}^3}$;

$$\{\text{piani in } \mathbb{R}^3\} \xleftrightarrow{\perp} \{\text{rette in } \mathbb{R}^3\}$$

$$P \longmapsto \ell = P^\perp$$

$$P = \ell^\perp \longleftarrow \ell$$



P cart $\quad P : \{ax+by+cz=0\}$ cioè $P = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : \left\langle \begin{pmatrix} x \\ y \\ z \end{pmatrix} \middle| \begin{pmatrix} a \\ b \\ c \end{pmatrix} \right\rangle = 0 \right\}$

$\|$ $\qquad$ cioè $P = \left(\mathrm{Span} \begin{pmatrix} a \\ b \\ c \end{pmatrix} \right)^{\perp}$

$l = P^{\perp}$ param $\qquad$ cioè $P^{\perp} = \mathrm{Span} \begin{pmatrix} a \\ b \\ c \end{pmatrix}$

P param $\quad P = \mathrm{Span} \begin{pmatrix} \alpha_1 \\ \beta_1 \\ \gamma_1 \end{pmatrix} \begin{pmatrix} \alpha_2 \\ \beta_2 \\ \gamma_2 \end{pmatrix} \quad P^{\perp} = \left(\mathrm{Span} \begin{pmatrix} \alpha_1 \\ \beta_1 \\ \gamma_1 \end{pmatrix} \begin{pmatrix} \alpha_2 \\ \beta_2 \\ \gamma_2 \end{pmatrix} \right)^{\perp}$

$\|$ $\qquad$ cioè $P^{\perp} = \begin{cases} \alpha_1 x + \beta_1 y + \gamma_1 z = 0 \\ \alpha_2 x + \beta_2 y + \gamma_2 z = 0 \end{cases}$

$l = P^{\perp}$ cart

Passaggi: $\quad P$ param $\longrightarrow P$ cart

$\qquad\qquad\;\; \|$ $\qquad\qquad\qquad\quad \|$

$\qquad\qquad l$ cart $\longrightarrow l$ param $\qquad$ sono fatti con i det 2x2

$v_1, v_2 \rightsquigarrow u$ perpendicolare a v_1 e v_2 (se lin. indip)

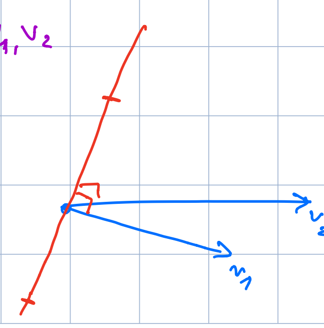
$$\begin{pmatrix} \alpha_1 \\ \beta_1 \\ \gamma_1 \end{pmatrix} \begin{pmatrix} \alpha_2 \\ \beta_2 \\ \gamma_2 \end{pmatrix} \longrightarrow \begin{pmatrix} \alpha_1 \\ \beta_1 \\ \gamma_1 \end{pmatrix} \wedge \begin{pmatrix} \alpha_2 \\ \beta_2 \\ \gamma_2 \end{pmatrix} = \begin{pmatrix} \beta_1 \gamma_2 - \beta_2 \gamma_1 \\ -(\alpha_1 \gamma_2 - \alpha_2 \gamma_1) \\ \alpha_1 \beta_2 - \alpha_2 \beta_1 \end{pmatrix}$$

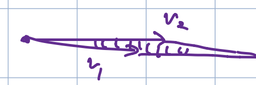
prodotto vettoriale (anche $\times$)

- $v_1 \wedge v_2 = 0$ se lin. dip.
- $v_1 \wedge v_2$ è $\perp$ a entrambi se lin. indip.
- $\|v_1 \wedge v_2\|$ = area parallelogramma con lati v_1, v_2
- $v_1, v_2, v_1 \wedge v_2$ formano "terna levogira"

pollice $\quad$ indice $\quad$ medio

mano destra





Esercizio 9.2.5 (g) ortonormalizzare in $\mathbb{R}^3$

$$\underset{v_1}{\begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix}} \quad \underset{v_2}{\begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix}} \quad \underset{v_3}{\begin{pmatrix} -3e \\ 0 \\ 0 \end{pmatrix}}$$

$$u_1 = \frac{1}{\sqrt{14}} \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} = \frac{v_1}{\|v_1\|}$$

$$\tilde{u}_2 = \begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix} - \frac{\left\langle \begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix} \middle| \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} \right\rangle}{14} \cdot \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix}$$

$$= \begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix} - \frac{-6+2-3}{14} \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix}$$

$$= \underbrace{\begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix}}_{v_2} + \frac{1}{2} \underbrace{\begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix}}_{v_1} = \frac{1}{2} \begin{pmatrix} -1 \\ 4 \\ 5 \end{pmatrix}$$

$$\left\langle \begin{pmatrix} -1 \\ 4 \\ 5 \end{pmatrix} \middle| \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} \right\rangle = -3+8-5 = 0 \checkmark$$

$$u_2 = \frac{1}{\sqrt{42}} \cdot \begin{pmatrix} -1 \\ 4 \\ 5 \end{pmatrix}$$

$$\tilde{u}_3 = v_3 - \langle v_3 | u_1 \rangle \cdot u_1 - \langle v_3 | u_2 \rangle \cdot u_2 = \underline{\underline{\text{conti}}}$$

$$u_3 = \tilde{u}_3 / \|\tilde{u}_3\|$$

Vogliamo $\underbrace{u_1 \quad u_2}_{\mathrm{Span}(\uparrow)}$ u_3 base ortonormale

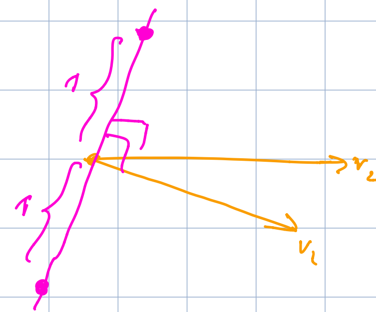
$$\mathrm{Span}(\uparrow) = \mathrm{Span}(v_1, v_2)$$

$\Longrightarrow$ u_3 unitario e ortog. a v_1, v_2

$$\Rightarrow u_3 = \pm \frac{v_1 \wedge v_2}{\|v_1 \wedge v_2\|}$$

matrice cambio da $(v_1 v_2 v_3)$ a $(u_1 u_2 u_3)$

è $\begin{pmatrix} >0 & * & * \\ 0 & >0 & * \\ 0 & 0 & >0 \end{pmatrix}$ ha $\det > 0$



$\Longrightarrow \det(u_1 u_2 u_3)$ è concorde con $\det(v_1, v_2, v_3)$.

Come ortonormalizzare v_1, v_2, v_3 base di $\mathbb{R}^3$?

$u_1 = \dots$

$u_2 = \dots$

$u_3 = \pm \dfrac{v_1 \wedge v_2}{\|v_1 \wedge v_2\|}$ segno t.c. $\det(u_1 u_2 u_3)$ concorde con $\det(v_1 v_2 v_3)$

$$\underset{v_1}{\begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix}} \quad \underset{v_2}{\begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix}} \quad \underset{v_3}{\begin{pmatrix} -3e \\ 0 \\ 0 \end{pmatrix}} \qquad \det(v_1 v_2 v_3) = \begin{vmatrix} 3 & -2 & -3e \\ 2 & 1 & 0 \\ -1 & 3 & 0 \end{vmatrix} < 0$$

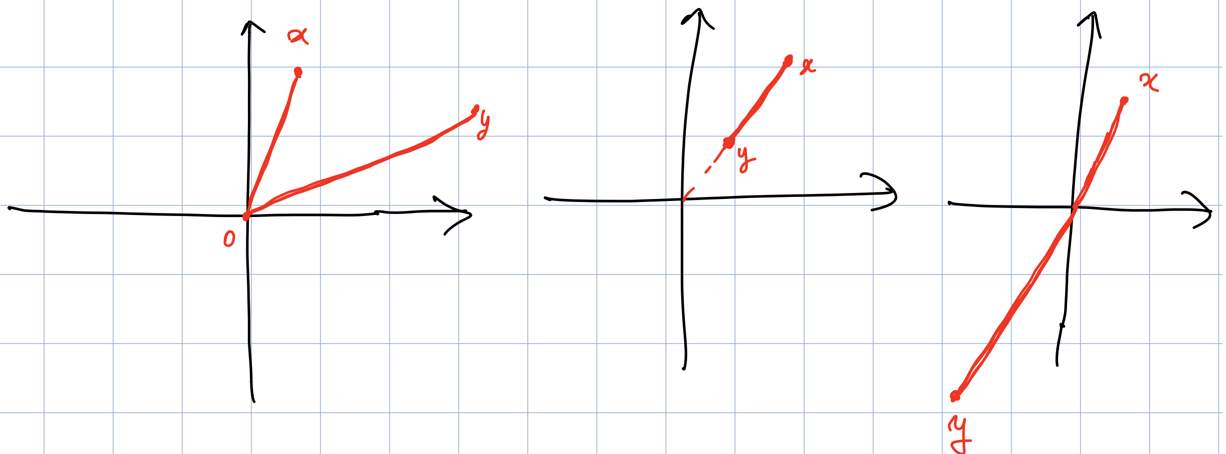
$$u_3 = - \frac{v_1 \wedge v_2}{\|v_1 \wedge v_2\|} = - \frac{1}{\sqrt{3}} \cancel{\begin{pmatrix} 7 \\ -7 \\ 7 \end{pmatrix}} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

$$\left\langle \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \middle| \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} \right\rangle = 3 - 2 - 1 = 0 \checkmark$$

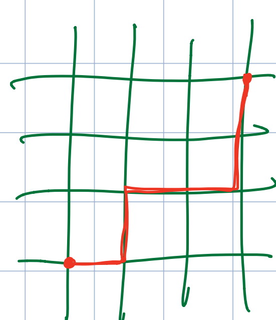
$$\left\langle \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \middle| \begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix} \right\rangle = -2 - 1 + 3 = 0 \checkmark$$

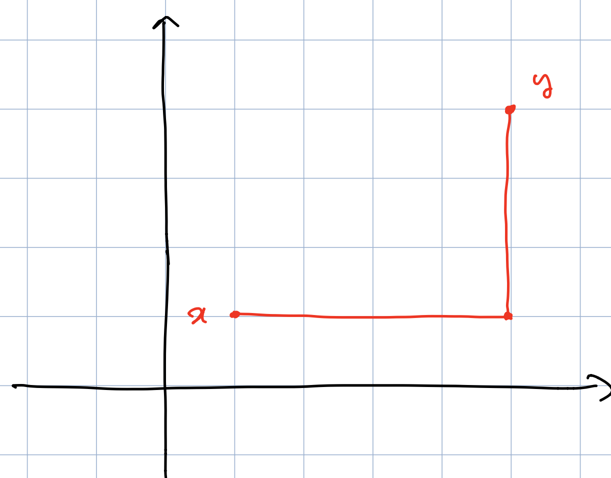
9.2.4. Verificare che soddisfano le 3 proprietà della distanza in $\mathbb{R}^n$

(a) $\mathrm{SNCF}(x,y) = \begin{cases} \|x\| + \|y\| & \text{se lin. indip} \\ \|x-y\| & \text{se lin. dip.} \end{cases}$



(b) $\mathrm{NYC}(x,y) = \sum_{i=1}^{n} |x_i - y_i|$



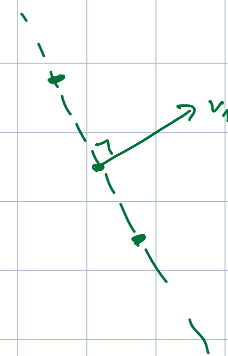


$d(x,x) = 0$ ✓ $\quad d(x,y) > 0 \quad y \neq x$ ✓

$d(x,y) = d(y,x)$ ✓ $\quad d(x,y) < d(x,z) + d(z,y)$ ✓

9.2.5 ortonormalizzare

(b) $\langle\cdot|\cdot\rangle_{\mathbb{R}^2}$ $\quad \underset{v_1}{\begin{pmatrix}-2\\5\end{pmatrix}}, \quad \underset{v_2}{\begin{pmatrix}-e^2\\1\end{pmatrix}}$



$u_1 = \frac{1}{\sqrt{29}}\begin{pmatrix}-2\\5\end{pmatrix}$

u_2 : $\perp u_1$ $\quad \begin{pmatrix}5\\2\end{pmatrix}$

unitario $\quad \frac{1}{\sqrt{29}}\begin{pmatrix}5\\2\end{pmatrix}$

$\det(u_1,u_2)$ concorde $\det(v_1,v_2)$

$\qquad\qquad\qquad\qquad\qquad ||$

$\det\begin{pmatrix}-2 & -e^2\\5 & 1\end{pmatrix} = -2+5e^2 > 0$

$\det\begin{pmatrix}-2 & 5\\5 & 2\end{pmatrix} = -29 < 0$

$-\frac{1}{\sqrt{29}}\begin{pmatrix}5\\2\end{pmatrix}$

(d) $\langle\cdot|\cdot\rangle_{\begin{pmatrix}5 & -2\\-2 & 1\end{pmatrix}}$ $\quad \underset{v_1}{\begin{pmatrix}1\\0\end{pmatrix}} \underset{v_2}{\begin{pmatrix}0\\1\end{pmatrix}}$

$\left\|\begin{pmatrix}1\\0\end{pmatrix}\right\|^2 = (1,0)\begin{pmatrix}5 & -2\\-2 & 1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = 5 \qquad u_1 = \frac{1}{\sqrt5}\begin{pmatrix}1\\0\end{pmatrix}$

$\tilde{u}_2 = \begin{pmatrix}0\\1\end{pmatrix} - \frac{(0,1)\begin{pmatrix}5 & -2\\-2 & 1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}}{5}\cdot\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}0\\1\end{pmatrix} - \frac{-2}{5}\begin{pmatrix}1\\0\end{pmatrix}$

$= \begin{pmatrix}0\\1\end{pmatrix} + \frac{2}{5}\begin{pmatrix}1\\0\end{pmatrix} = \frac{1}{5}\begin{pmatrix}2\\5\end{pmatrix}$

$\begin{pmatrix}2\\5\end{pmatrix}\cdot\begin{pmatrix}5 & -2\\-2 & 1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = 0$ ✓

$u_2 = \frac{1}{\sqrt{5\cdot2^2-4\cdot2\cdot5+1\cdot5^2}}\begin{pmatrix}2\\5\end{pmatrix} = \dots$

(e) $\langle\cdot|\cdot\rangle_{\begin{pmatrix}3&-1\\-1&2\end{pmatrix}}$ $\qquad \underset{v_1}{\begin{pmatrix}2\\3\end{pmatrix}} \underset{v_2}{\begin{pmatrix}-1\\5\end{pmatrix}}$

$$u_1 = \frac{1}{\sqrt{3\cdot 2^2 - 2\cdot 2\cdot 3 + 2\cdot 3^2}} \cdot \begin{pmatrix}2\\3\end{pmatrix} = \frac{1}{\sqrt{18}}\begin{pmatrix}2\\3\end{pmatrix}$$

$$\tilde{u}_2 = \begin{pmatrix}-1\\5\end{pmatrix} - \frac{1}{18}\cdot\Big(\underset{(-1,5)\cdot\begin{pmatrix}3\\4\end{pmatrix}}{(-1,5)\cdot\begin{pmatrix}3&-1\\-1&2\end{pmatrix}\cdot\begin{pmatrix}2\\3\end{pmatrix}} \Big) \cdot \begin{pmatrix}2\\3\end{pmatrix}$$

$$= \begin{pmatrix}-1\\5\end{pmatrix} - \frac{17}{18}\begin{pmatrix}2\\3\end{pmatrix} = \frac{1}{18}\begin{pmatrix}-52\\39\end{pmatrix} =$$

$$(-52,39)\begin{pmatrix}3&-1\\-1&2\end{pmatrix}\cdot\begin{pmatrix}2\\3\end{pmatrix} =$$
$$(-52,39)\begin{pmatrix}3\\4\end{pmatrix} = -156+156 = 0 \checkmark$$

$$u_2 = \frac{1}{\sqrt{3\cdot 52^2 - 2\cdot(-52)\cdot 39 + 2\cdot 39^2}}\cdot\begin{pmatrix}-52\\39\end{pmatrix} = \ldots$$

(i) $\langle\cdot|\cdot\rangle_{\begin{pmatrix}1&2&0\\2&5&-1\\0&-1&3\end{pmatrix}}$ $\qquad \underset{v_1}{\begin{pmatrix}2\\1\\3\end{pmatrix}} \underset{v_2}{\begin{pmatrix}3\\-2\\2\end{pmatrix}}$

$$u_1 = \frac{1}{\sqrt{1\cdot 2^2 + 5\cdot 1^2 + 3\cdot 3^2 + 4\cdot 2\cdot 1 + 0\cdot 2\cdot 3 - 2\cdot 1\cdot 3}}\begin{pmatrix}2\\1\\3\end{pmatrix}$$

$$= \frac{1}{\sqrt{38}}\begin{pmatrix}2\\1\\3\end{pmatrix}$$

$$\tilde{u}_2 = \begin{pmatrix}3\\-2\\2\end{pmatrix} - \frac{1}{38}\cdot\left(\left(3,-2,2\right)\begin{pmatrix}1&2&0\\2&5&-1\\0&-1&3\end{pmatrix}\begin{pmatrix}2\\1\\3\end{pmatrix}\right)\cdot\begin{pmatrix}2\\1\\3\end{pmatrix} = \ldots$$

9.3.2. Trovare base ortogonale di

$$\left(\mathrm{Span}\left(\begin{pmatrix}3\\1\\-5\\2\end{pmatrix}\begin{pmatrix}-2\\4\\3\\-1\end{pmatrix}\right)\right)^{\perp}$$

Cerco base: due vett. lin. indip.

$$\bullet\quad\begin{cases}3x+y-5z+2w=0\\-2x+4y+3z-w=0\end{cases}\qquad \dots\ \underline{\underline{\text{ok}}}$$

Ricordo: in $\mathbb{R}^3$ $\quad v_1, v_2 \rightsquigarrow v_1\wedge v_2$ è perp. a loro.

Cercando vett. $\perp$ a $\begin{pmatrix}3\\1\\-5\\2\end{pmatrix},\begin{pmatrix}-2\\4\\3\\-1\end{pmatrix}$, se uno lo cerco

$\begin{pmatrix}x\\y\\z\\0\end{pmatrix}$ la condizione è che $\begin{pmatrix}x\\y\\z\end{pmatrix}$ sia $\perp$ $\boxed{\begin{pmatrix}3\\1\\-5\end{pmatrix}\begin{pmatrix}-2\\4\\3\end{pmatrix}}$ in $\mathbb{R}^3$

$$\begin{pmatrix}23\\1\\14\\0\end{pmatrix}\qquad\begin{matrix}69+1-70\ \checkmark\\-46+4+42\ \checkmark\end{matrix}$$

se uno lo cerco $\begin{pmatrix}0\\y\\z\\w\end{pmatrix}$ la condizione è che

$\begin{pmatrix}y\\z\\w\end{pmatrix}$ sia $\perp$ a $\begin{pmatrix}1\\-5\\2\end{pmatrix}\begin{pmatrix}4\\3\\-1\end{pmatrix}$ in $\mathbb{R}^3$: uso $\wedge$

$$\begin{pmatrix}0\\-1\\9\\23\end{pmatrix}\qquad\begin{matrix}-1-45+46\ \checkmark\\-4+27-23\ \checkmark\end{matrix}$$

Base: $\underset{v_1}{\begin{pmatrix}23\\1\\14\\0\end{pmatrix}}\ \underset{v_2}{\begin{pmatrix}0\\-1\\9\\23\end{pmatrix}}$

$$v_2' = \begin{pmatrix} 0 \\ -1 \\ 3 \\ 23 \end{pmatrix} - \frac{1}{23^2+1^2+14^2} \cdot (-1+3\cdot 14) \cdot \begin{pmatrix} 23 \\ 1 \\ 14 \\ 0 \end{pmatrix} = \dots$$

V sp. vett. su $\mathbb{R}$ con $\langle \cdot | \cdot \rangle$; $f : V \to V$ lineare.

Def: f è autoaggiunta se $\langle f(v) | u \rangle = \langle v | f(u) \rangle \quad \forall u, v$;
f è ortogonale se $\langle f(v) | f(u) \rangle = \langle v | u \rangle \quad \forall v, u$.

Oss: f è ortogonale $\iff$ f conserva le distanze (isometria)

- f ortogonale $\Rightarrow \|f(v)\| = \sqrt{\langle f(v) | f(v) \rangle} = \sqrt{\langle v | v \rangle} = \|v\|$
  $\Rightarrow f$ conserva distanza
- f isometria $\Rightarrow$ f conserva $\|\cdot\|$ $\underset{(\text{su } \mathbb{R})}{\Longrightarrow}$ conserva $\langle \cdot | \cdot \rangle$

Oss: $\forall M \in M_{n\times n}(\mathbb{R})$; $\langle M\cdot x | y \rangle_{\mathbb{R}^n} = \langle x | {}^tM \cdot y \rangle_{\mathbb{R}^n}$.

$$\langle M\cdot x | y \rangle_{\mathbb{R}^n} = {}^t(M\cdot x)\cdot y = {}^tx \cdot {}^tM \cdot y = \langle x | {}^tM\cdot y \rangle_{\mathbb{R}^n}.$$

Fatto: $M \in M_{n\times n}(\mathbb{R})$

- autoaggiunta $\iff$ ${}^tM = M$
- ortogonale $\iff$ ${}^tM = M^{-1}$.

M autoaggiunta $\Longleftrightarrow \langle M\cdot x | y\rangle = \langle x | M\cdot y\rangle \quad \forall x,y$

$$\langle x | {}^tM\cdot y\rangle \underset{\text{ESE}}{\Longleftrightarrow} {}^tM = M.$$

M ortog. $\Longleftrightarrow \langle Mx | My\rangle = \langle x|y\rangle \quad \forall x,y$

$$\langle x | {}^tM\cdot M\cdot y\rangle \underset{\text{ESE}}{\Longleftrightarrow} {}^tM\cdot M = I_n.$$

Oss: $M = (u_1, \dots, u_n) \in \mathcal{M}_{n\times n}(\mathbb{R})$.

$\boxed{M \text{ ortogonale} \Longleftrightarrow} \ {}^tM\cdot M = I_n$

$$\Longleftrightarrow {}^t(u_1,\dots,u_n)\cdot(u_1,\dots,u_n) = I_n$$

$$\Longleftrightarrow \begin{pmatrix} {}^tu_1 \\ \vdots \\ {}^tu_n \end{pmatrix}\cdot(u_1,\dots,u_n) = I_n$$

$$\Longleftrightarrow {}^tu_i\cdot u_j = \delta_{ij} \quad \forall i,j$$

$$\Longleftrightarrow \langle u_i | u_j\rangle_{\mathbb{R}^n} = \delta_{ij} \quad \forall i,j$$

$\boxed{\Longleftrightarrow \text{le sue colonne sono base ortonormale}}$

V sp. vett. su $\mathbb{R}$ con $\langle \cdot | \cdot \rangle$.

Ricordo: $p: V\to V$ lin. è proiezione rispetto a qualche $V = W\oplus Z$
$\Longleftrightarrow p\circ p = p$; in tal caso $W = \mathrm{Im}(p)$, $Z = \ker(p)$.

Teorema: p proiez. ortog. $\Longleftrightarrow p\circ p = p$ e p è autoaggiunta.

Dim: $\Longrightarrow$ $\quad V = W \oplus W^\perp \quad p$ proiez. su W

$p \circ p = p$ ✓ $\quad v_1, v_2 \in V$; dobbiamo provare

$$\langle p(v_1) | v_2 \rangle = \langle v_1 | p(v_2) \rangle$$

$$\underset{W}{\overset{\cap}{v_1 = w_1}} + \underset{W^\perp}{\overset{\cap}{u_1}} \qquad v_2 = \underset{W}{\overset{\cap}{w_2}} + \underset{W^\perp}{\overset{\cap}{u_2}}$$

$$\langle p(v_1) | v_2 \rangle = \langle w_1 | w_2 + u_2 \rangle = \langle w_1 | w_2 \rangle + \cancel{\langle w_1 | u_2 \rangle}^{0}$$

$$\langle v_1 | p(v_2) \rangle = \langle w_1 + u_1 | w_2 \rangle = \langle w_1 + w_2 \rangle + \cancel{\langle u_1 | w_2 \rangle}^{0}$$

ok

$\Longleftarrow$ $\quad p \circ p = p$, p autoaggiunto.

Sappiamo p è proiez. su $W = \mathrm{Im}(p)$ rispetto a $V = W \oplus \mathrm{Ker}(p)$.

Devo vedere che $\mathrm{Ker}(p) = W^\perp$.

Chiamo $k = \dim(W)$, $n = \dim(V)$; so che $\dim W^\perp = \dim \mathrm{Ker}(p) = n - k$

Per concludere $\mathrm{Ker}(p) = W^\perp$ basta vedere che $\mathrm{Ker}(p) \subset W^\perp$.

Prendo $u \in \mathrm{Ker}(p)$; devo vedere che $\langle w | u \rangle = 0 \ \forall w \in W$.

$$\langle w | u \rangle = \langle p(w) | u \rangle \underset{p \text{ autoaggiunto}}{=} \langle w | p(u) \rangle = \langle w | 0 \rangle = 0.$$

$\blacksquare$

Cor (fare esercizi): in $\mathbb{R}^n$ con $\langle \cdot | \cdot \rangle_{\mathbb{R}^n}$

$M \in M_{n \times n}(\mathbb{R})$ è proiez. ortog. $\iff M^2 = M = {}^tM$.