Geom - Lin - 11/3/21

V sp. vett. su $\mathbb{R}$ con $\langle \cdot | \cdot \rangle$ prod scal, $\|\cdot\|$ norma.

$v_1, \dots, v_k$ è ortog. se $v_i \perp v_j$ $\forall i \neq j$ $(\langle v_i | v_j \rangle = 0)$
ortonormale se inoltre $\|v_i\| = 1$.

- sist. ortog. di vett $\neq 0$ è lin. indip.
- $v_1, \dots, v_n$ base ortog. $\Longrightarrow$ $v = \sum_{i=1}^{n} \frac{\langle v | v_i \rangle}{\underbrace{\|v_i\|^2}_{=1}} \cdot v_i$.

ortonormale

Q: come produrre basi ortogonali.
A: Gram-Schmidt "ortonormalizzazione"

<u>Teo</u>: data $v_1, \dots, v_n$ base di V il seguente procedimento fornisce una base ortonormale $w_1, \dots, w_n$:

- $w_1 = v_1 / \|v_1\|$
- $k \geq 1$ si pone $\tilde{w}_{k+1} = v_{k+1} - \sum_{i=1}^{k} \langle v_{k+1} | w_i \rangle \cdot w_i$

(suppongo $w_1, \dots, w_k$ già fatti)

$$w_{k+1} = \tilde{w}_{k+1} / \|\tilde{w}_{k+1}\|.$$

<u>Es</u>: in $\mathbb{R}^3$ $\underset{v_1}{\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}} \underset{v_2}{\begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix}} \underset{v_3}{\begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}}$

$$w_1 = \begin{pmatrix}1\\2\\-1\end{pmatrix} \Big/ \sqrt{1+4+1} = \frac{1}{\sqrt{6}} \begin{pmatrix}1\\2\\-1\end{pmatrix}$$

$$\tilde{w}_2 = \begin{pmatrix}3\\1\\4\end{pmatrix} - \left\langle \begin{pmatrix}3\\1\\4\end{pmatrix} \Big| \frac{1}{\sqrt{6}} \begin{pmatrix}1\\2\\-1\end{pmatrix} \right\rangle \cdot \frac{1}{\sqrt{6}} \begin{pmatrix}1\\2\\-1\end{pmatrix}$$

$$= \begin{pmatrix}3\\1\\4\end{pmatrix} - \frac{3+2-4}{6} \begin{pmatrix}1\\2\\-1\end{pmatrix} = \begin{pmatrix}3\\1\\4\end{pmatrix} - \frac{1}{6} \begin{pmatrix}1\\2\\-1\end{pmatrix} = \frac{1}{6} \begin{pmatrix}17\\4\\25\end{pmatrix}$$

verifica: vogliamo $\tilde{w}_2 \perp w_1$ cioè $\tilde{w}_2 \perp v_1$

$$\left\langle \begin{pmatrix}1\\2\\-1\end{pmatrix} \Big| \begin{pmatrix}17\\4\\25\end{pmatrix} \right\rangle = 17+8-25=0 \quad \checkmark$$

Oss: $x = \alpha \cdot y \quad \alpha > 0$

$$\frac{x}{\|x\|} = \frac{\alpha \cdot y}{\|\alpha \cdot y\|} = \frac{\alpha \cdot y}{\alpha \cdot \|y\|} = \frac{y}{\|y\|}$$

$$w_2 = \frac{1}{\sqrt{17^2+4^2+25^2}} \cdot \begin{pmatrix}17\\4\\25\end{pmatrix}$$

$$\tilde{w}_3 = \begin{pmatrix}-1\\0\\2\end{pmatrix} - \frac{1}{6} \left\langle \begin{pmatrix}-1\\0\\2\end{pmatrix} \Big| \begin{pmatrix}1\\2\\-1\end{pmatrix} \right\rangle \cdot \begin{pmatrix}1\\2\\-1\end{pmatrix} - \frac{1}{\sqrt{\dots}} \left\langle \begin{pmatrix}-1\\0\\2\end{pmatrix} \Big| \begin{pmatrix}17\\4\\25\end{pmatrix} \right\rangle \cdot \begin{pmatrix}17\\4\\25\end{pmatrix}$$

$= \dots$

$$w_3 = \tilde{w}_3 / \|\tilde{w}_3\| .$$

Dimo:
- $w_1 = v_1 / \|v_1\|$
- $k \geq 1$ si pone $\tilde{w}_{k+1} = v_{k+1} - \sum_{i=1}^{k} \langle v_{k+1} | w_i \rangle \cdot w_i$

(suppongo $w_1, \dots, w_k$ già fatti)

$$w_{k+1} = \tilde{w}_{k+1} / \|\tilde{w}_{k+1}\| .$$

Verifico per $k=1,\dots,n$ che w_k si può definire, è unitario, è ortogonale a $w_1,\dots,w_{k-1}$ e $\mathrm{Span}(w_1,\dots,w_k)=\mathrm{Span}(v_1,\dots,v_k)$.

$k=1$: tutto ovvio

Supponiamo vero per k e proviamo per $k+1$:

- $\tilde{w}_{k+1} = v_{k+1} - \underbrace{\sum_{i=1}^{k} \langle v_{k+1} | w_i \rangle \cdot w_i}_{\in \mathrm{Span}(w_1,\dots,w_k) = \mathrm{Span}(v_1,\dots,v_k)}$

$\Longrightarrow \neq 0$ poiché $v_1,\dots,v_{k+1}$ lin. indip.

$\Longrightarrow$ si può definire; ✓

unitario ✓

ortog. ai precedenti : basta vederlo per $\tilde{w}_{k+1}$; $j \leq k$

$$\langle \tilde{w}_{k+1} | w_j \rangle = \left\langle v_{k+1} - \sum_{i=1}^{k} \langle v_{k+1} | w_i \rangle \cdot w_i \;\middle|\; w_j \right\rangle$$

$$= \langle v_{k+1} | w_j \rangle - \sum_{i=1}^{k} \langle v_{k+1} | w_i \rangle \cdot \underbrace{\langle w_i | w_j \rangle}_{= \delta_{ij} = \begin{cases} 1 & \text{se } i=j \\ 0 & \text{se } i \neq j \end{cases}}$$

$$= \langle v_{k+1} | w_j \rangle - \langle v_{k+1} | w_j \rangle = 0.$$ ✓

- $\mathrm{Span}(w_1,\dots,w_{k+1}) = \mathrm{Span}(v_1,\dots,v_{k+1})$

$$\mathrm{Span}(w_1,\dots,w_{k+1}) = \mathrm{Span}(v_1,\dots,v_k,w_{k+1}) = \mathrm{Span}(v_1,\dots,v_k,\tilde{w}_{k+1})$$

$$\tilde{w}_{k+1} = v_{k+1} + u, \quad u \in \mathrm{Span}(v_1,\dots,v_k)$$

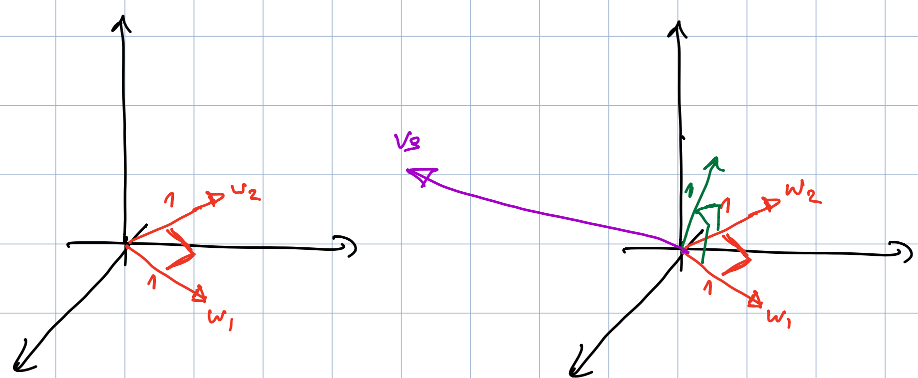
✓ ■

Oss: la matrice di cambio di base da $v_1, \dots, v_n$ a $w_1 \dots w_n$ è

$$\begin{pmatrix} (>0) & * & * & & * \\ 0 & (>0) & * & & \vdots \\ \vdots & 0 & (>0) & & \vdots \\ \vdots & \vdots & 0 & \ddots & * \\ 0 & 0 & 0 \ \cdots & 0 & (>0) \end{pmatrix}$$

Cor: dati $w_1, \dots, w_k$ base ortog. o ortonormale di $W \subset V$ posso sempre completarli a base ortog. o ortonormale di V.

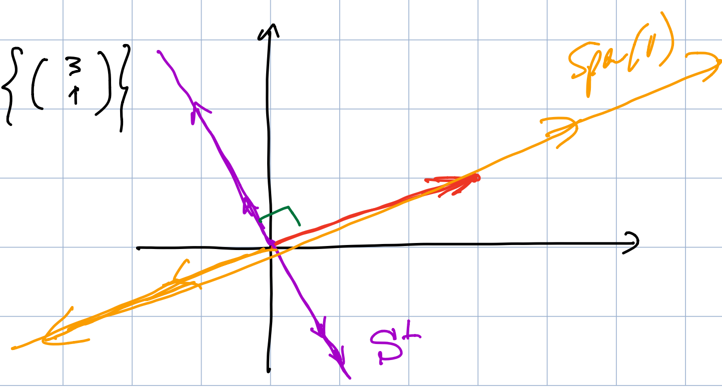
Infatti posso completare a caso e poi ortonormalizzare solo a partire dai vettori aggiunti



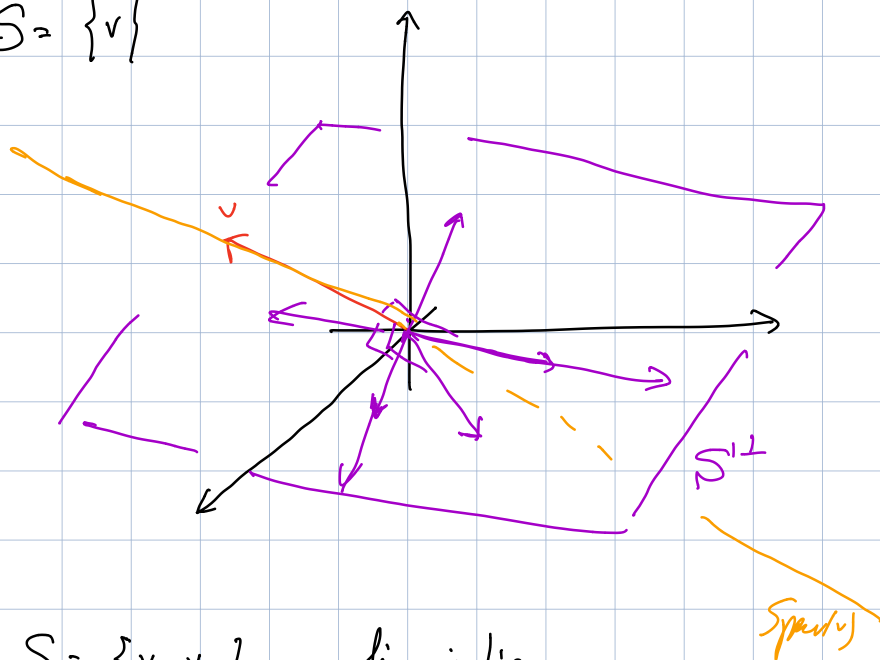
Def: dato $S \subset V$ chiamo ortogonale a S l'insieme:

$$S^\perp = \{ v : \langle v | s \rangle = 0 \quad \forall s \in S \}.$$

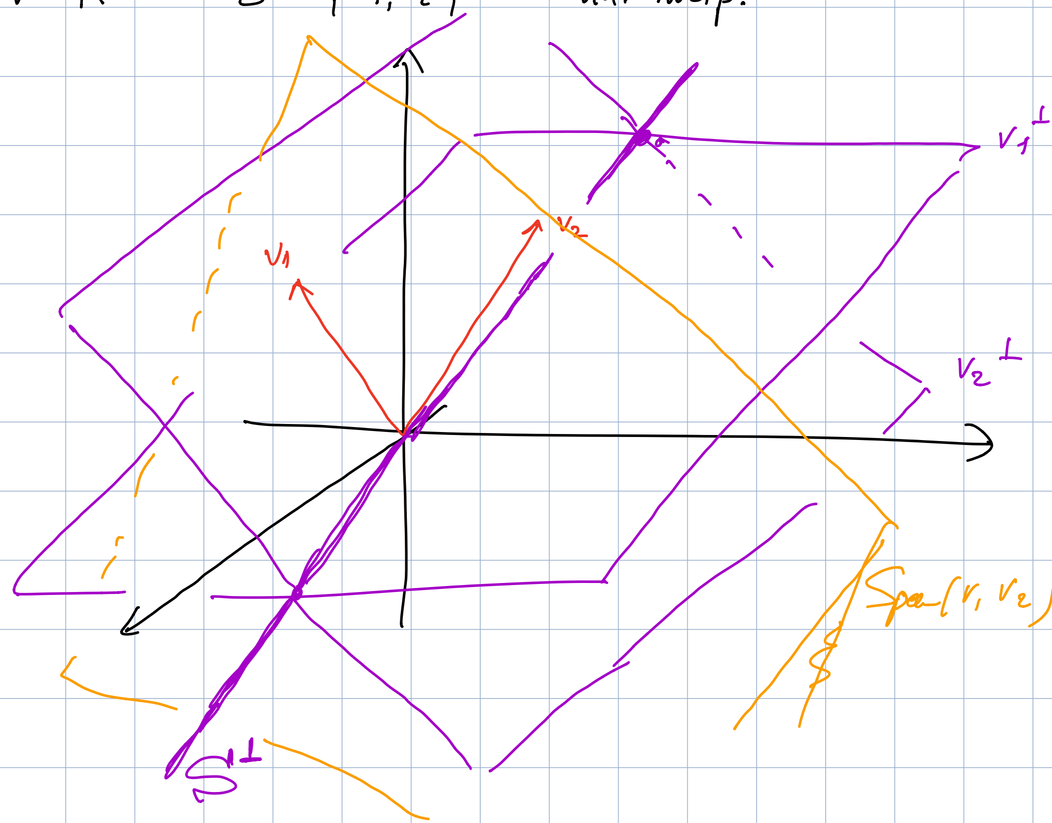
Es: $V = \mathbb{R}^2$, $S = \left\{ \begin{pmatrix} 3 \\ 1 \end{pmatrix} \right\}$



$V = \mathbb{R}^3 \qquad S = \{v\}$



$V = \mathbb{R}^3 \qquad S = \{v_1, v_2\} \qquad$ lin. indip.



Oss: $S^\perp$ è un sottosp vett. di V. Infatti:

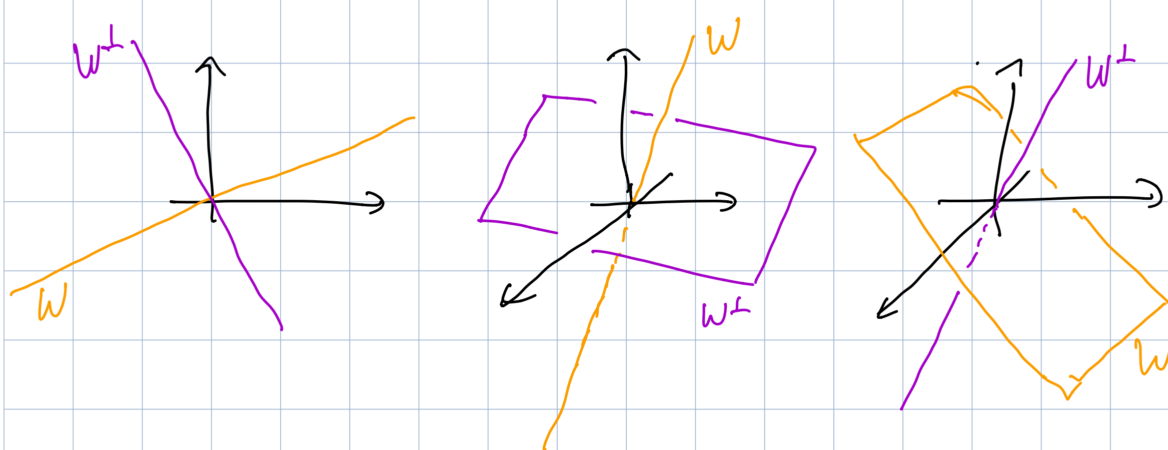
$v_1, v_2 \in S^\perp$ cioè $\langle v_1 | s \rangle = \langle v_2 | s \rangle = 0 \quad \forall s \in S$

$\lambda_1, \lambda_2 \in \mathbb{R} \Rightarrow \langle \lambda_1 v_1 + \lambda_2 v_2 | s \rangle = \lambda_1 \langle v_1 | s \rangle + \lambda_2 \langle v_2 | s \rangle$

$= \lambda_1 \cdot 0 + \lambda_2 \cdot 0 = 0$

Oss (esercizio): $S^\perp = (\mathrm{Span}(S))^\perp$.

Prop: se $\dim_{\mathbb{R}}(V) < +\infty$ e W è stsp. vett. di V allora $V = W \oplus W^\perp$.



(**Falso** in $\dim = +\infty$).

Dimo: devo vedere che $W \cap W^\perp = \{0\}$ e $W + W^\perp = V$.

$w \in W \cap W^\perp$;

$\langle w | w \rangle = 0 \Rightarrow \|w\| = 0$

$\underset{W}{\overset{\cap}{w}} \quad \underset{W^\perp}{\overset{\cap}{w}} \Rightarrow w = 0$

(non uso $\dim < +\infty$)

Sia $W = \mathrm{Span}(w_1, \dots, w_k)$ base ortonormale; completo a base ortonormale $w_1, \dots, w_n$ di V.

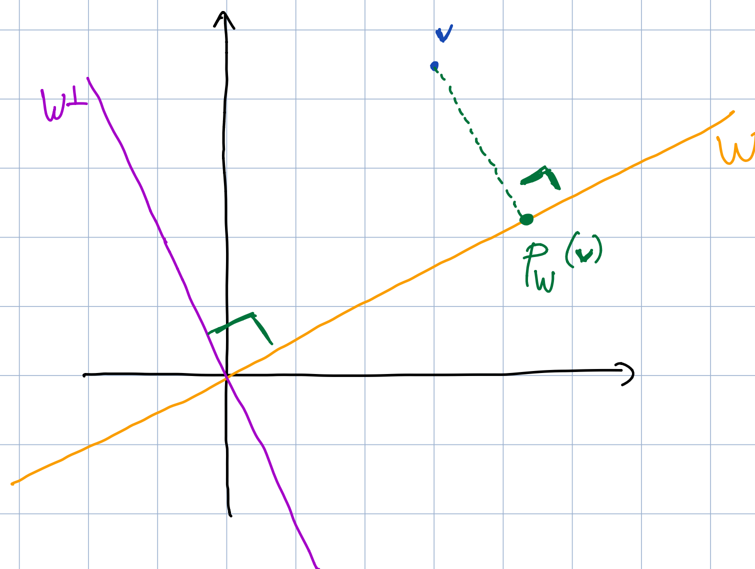
Per $j > k$ ho $\langle w_j | w_i \rangle = 0$ $i = 1, \dots, k$

$\Longrightarrow w_j \in W^\perp$; ora $u \in V$

$$u = \underbrace{\alpha_1 w_1 + \dots + \alpha_k w_k}_{\substack{\cap \\ W}} + \underbrace{\alpha_{k+1} w_{k+1} + \dots + \alpha_n w_n}_{\substack{\cap \\ W^\perp}}$$

$\blacksquare$

Def: la proiezione associata alla decomposizione $V = W \oplus W^\perp$ è detta proiezione ortogonale su W, detta p_W.



Oss (esercizio): Se $\dim(V) < +\infty$ e W è sttsp. ho $(W^\perp)^\perp = W$.

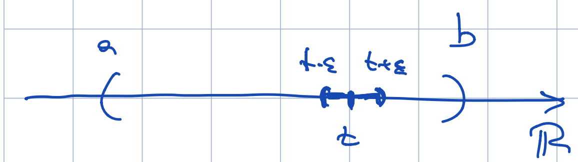
Calcolo differenziale in più variabili.

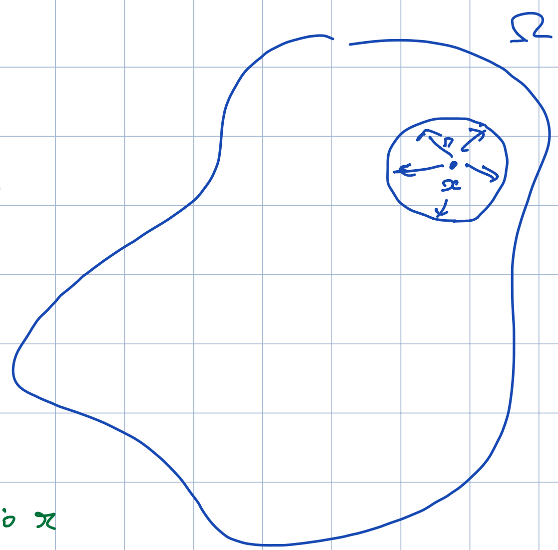
dominio: intervallo aperto.

In una variabile si considerano $f: (a,b) \to \mathbb{R}$
per calcolare $f'(t)$, $f''(t)$, ...

In n variabili analogamente $f: \underset{\mathbb{R}^n}{\overset{\cap}{\Omega}} \longrightarrow \mathbb{R}$ oppure $f: \Omega \to \mathbb{R}^k$

con $\Omega \subset \mathbb{R}^n$ è aperto:





$\forall x \in \Omega \;\; \exists r > 0$ t.c.
$\{y \in \mathbb{R}^n : \|y - x\| < r\} \subset \Omega$

palle di centro x e raggio r

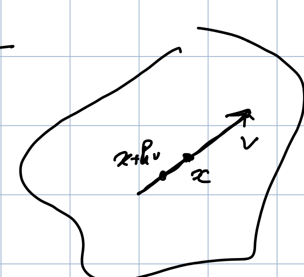
Il calcolo diff. in n variabili si fa su aperti di $\mathbb{R}^n$
ma qui prendo solo $\Omega = \mathbb{R}^n$.

$n = 1$ $\qquad f'(t) = \lim_{h \to 0} \dfrac{f(t+h) - f(t)}{h}$

$n > 1$ $\qquad$ • derivata in una direzione $v \in \mathbb{R}^n$, $v \neq 0$:

$f: \mathbb{R}^n \to \mathbb{R}$, $\quad x \in \mathbb{R}^n$

$$\frac{\partial f}{\partial v}(x) = \lim_{h \to 0} \frac{f(x + h \cdot v) - f(x)}{h}$$



In particolare ho derivata parziale di f in x
nella variabile x_i :

$$\frac{\partial f}{\partial x_i}(x) = \frac{\partial f}{\partial e_i}(x) = \lim_{h \to 0} \frac{f\left(\begin{pmatrix} x_1 \\ \vdots \\ x_i + h \\ \vdots \\ x_n \end{pmatrix}\right) - f(x)}{h}$$

cioè: derivo rispetto a x_i considerando le altre variabili fisse.

Es: $f(x,y) = x^2 \cdot \sin(3x + 7y^2) + e^{5y}$

$$\frac{\partial f}{\partial x}(x,y) = 2x \cdot \sin(3x + 7y^2) + x^2 \cdot 3 \cdot \cos(x + 7y^2)$$

$$\frac{\partial f}{\partial y}(x,y) = x^2 \cdot 14y \cdot \cos(3x + 7y^2) + 5 \cdot e^{5y}$$

Fatto: in ipotesi decenti $\frac{\partial f}{\partial v}(x) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(x) \cdot v_i$.

Derivazione di composizione:

$n = 1$ $\quad (a,b) \xrightarrow{f} (c,d) \xrightarrow{g} \mathbb{R}$

$$(g \circ f)'(x) = g'(f(x)) \cdot f'(x)$$

$n > 1$ $\quad f : \mathbb{R}^n \to \mathbb{R}^k \qquad f = \begin{pmatrix} f_1 \\ \vdots \\ f_k \end{pmatrix}$

$$Jf(x) = \left(\frac{\partial f_i}{\partial x_j}(x) \right)_{\substack{i = 1 \dots k \\ j = 1 \dots n}} \in \mathcal{M}_{k \times n}(\mathbb{R})$$

matrice jacobiana

$$\mathbb{R}^n \xrightarrow{f} \mathbb{R}^k \xrightarrow{g} \mathbb{R}^p$$

$$\underset{p\times n}{J(g\circ f)(x)} = \underset{p\times k}{(Jg)(f(x))} \cdot \underset{k\times n}{(Jf)(x)}$$

$$\frac{\partial(g\circ f)}{\partial x_i}(x) = \sum_{j=1}^{k} \frac{\partial g}{\partial y_j}(f(x)) \cdot \frac{\partial f_j}{\partial x_i}(x)$$

Formula Taylor II ordine per $n=1$:

$$f(x+h) = f(x) + f'(x)\cdot h + \frac{1}{2} f''(x)\cdot h^2 + o(h^2)$$

trascurabile risp. ad h^2

x è di min. loc. se $f(x+h) \geq f(x)$ $\forall h \in (-\varepsilon, \varepsilon)$

- x min loc $\Longrightarrow f'(x) = 0$, $f''(x) \geq 0$
- $f'(x)=0$, $f''(x) > 0 \Longrightarrow x$ min. loc.

Fissiamo $f: \mathbb{R}^n \to \mathbb{R}$. Chiamo gradiente di f in x

$$\mathrm{grad}_x(f) = \left(\frac{\partial f}{\partial x_i}(x)\right)_{i=1,\dots,n} \in \mathbb{R}^n.$$

Pongo ora $\displaystyle \frac{\partial^2 f}{\partial x_i \partial x_j}(x) = \frac{\partial}{\partial x_i}\left(\frac{\partial f}{\partial x_j}\right)(x)$

(scrivo $\frac{\partial^2 f}{\partial x_i \partial x_i} = \frac{\partial^2 f}{\partial x_i^2}$).

Chiamo matrice hessiana di f in x

$$H_x(f) = \left(\frac{\partial^2 f}{\partial x_i \partial x_j}(x)\right)_{i,j=1\dots n} \in M_{n\times n}(\mathbb{R}).$$

Teo: se f è decente $\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial^2 f}{\partial x_j \partial x_i}$

cioè $H_x(f)$ è simmetrica.

Es: $f(x,y) = \cos(x^2+3y^8)+e^{x^5 \cdot y^3}$

$$\frac{\partial f}{\partial x} = -2x \cdot \sin(x^2+3y^8) + 5x^4 \cdot e^{x^5 \cdot y^3}$$

$$\frac{\partial f}{\partial y} = -24y^7 \cdot \sin(x^2+3y^8) + 3y^2 \cdot e^{x^5 \cdot y^3}$$

$$\frac{\partial^2 f}{\partial x^2} = -2 \cdot \sin(*) - 4x^2 \cdot \cos(*) + 20x^3 \cdot e^* + 25x^8 \cdot e^*$$

$$\frac{\partial^2 f}{\partial y \partial x} = -48xy^7 \cdot \cos(*) + 15x^4 \cdot y^2 \cdot e^*$$

$$\frac{\partial^2 f}{\partial x \partial y} = -48xy^7 \cdot \cos(*) + 15x^4 y^2 \cdot e^*$$

$$\frac{\partial^2 f}{\partial y^2} = \dots$$

Prop: se $w_1,\dots,w_k$ è base ortog. di $W \subset V$ stsp rett

$$P_W(v) = \sum_{i=1}^{k} \frac{\langle v|w_i\rangle}{\|w_i\|^2}\cdot w_i .$$

Dimo: possiamo completare $w_1,\dots,w_k$ a base ortog. $w_1 \dots w_n$ di V; inoltre $w_{k+1},\dots,w_n \in W^\perp$. Ora:

$$v = \sum_{i=1}^{n} \frac{\langle v/w_i\rangle}{\|w_i\|^2}\cdot w_i = \underbrace{\sum_{i=1}^{k} \frac{\langle v/w_i\rangle}{\|w_i\|^2}\cdot w_i}_{\in W} + \underbrace{\sum_{i=k+1}^{n} \frac{\langle v/w_i\rangle}{\|w_i\|^2}\cdot w_i}_{\in W^\perp}$$

$$\Longrightarrow P_W(v) = \sum_{i=1}^{k} \frac{\langle v/w_i\rangle}{\|w_i\|^2}\cdot w_i . \qquad \blacksquare$$

Prop: $P_W(v)$ è il punto di W più vicino a v.