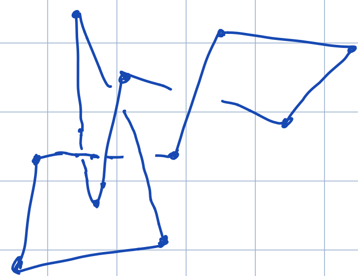
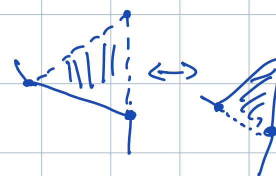
LDT - IUSS 4/4/2020

$\{\text{link DIFF}\}/_{\text{isotopie}} \longleftrightarrow \{\text{diagrammi planari}\}/_{\text{isotopie planari} + \text{mosse } R_I, R_{II}, R_{III}}$.

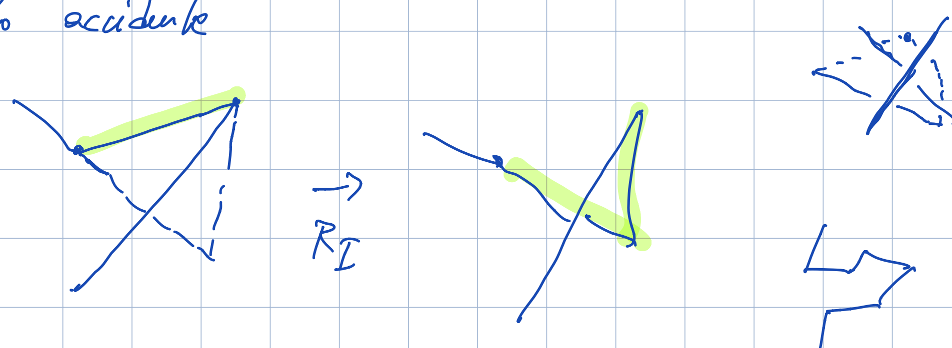
Approccio PL:



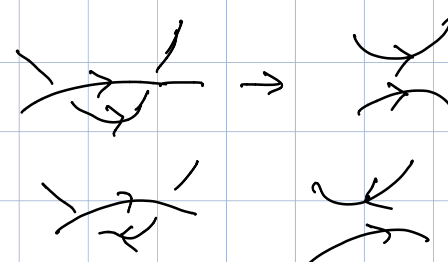
Isotopie

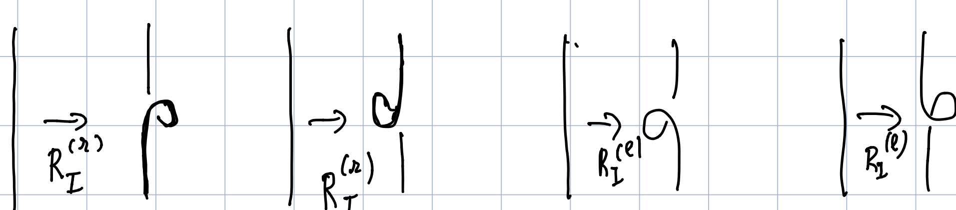


posso suddividere il triangolo in triangoli in cui accade un solo accidente



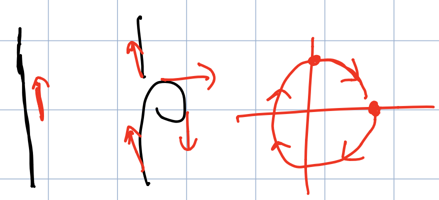
Versioni orientate delle mosse



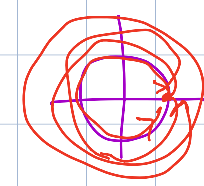


Fatto: $R_I^{(r)} \neq R_I^{(l)}$.

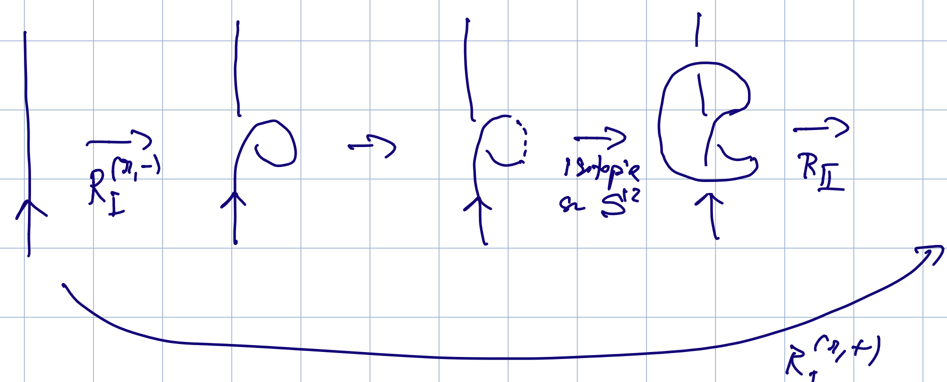
$R_I^{(r,-)} \neq R_I^{(r,+)}$ $\qquad$ $\alpha : S^1 \to \mathbb{R}^2$ $\qquad$ $\frac{\alpha'}{\|\alpha'\|} : S^1 \to S^1$



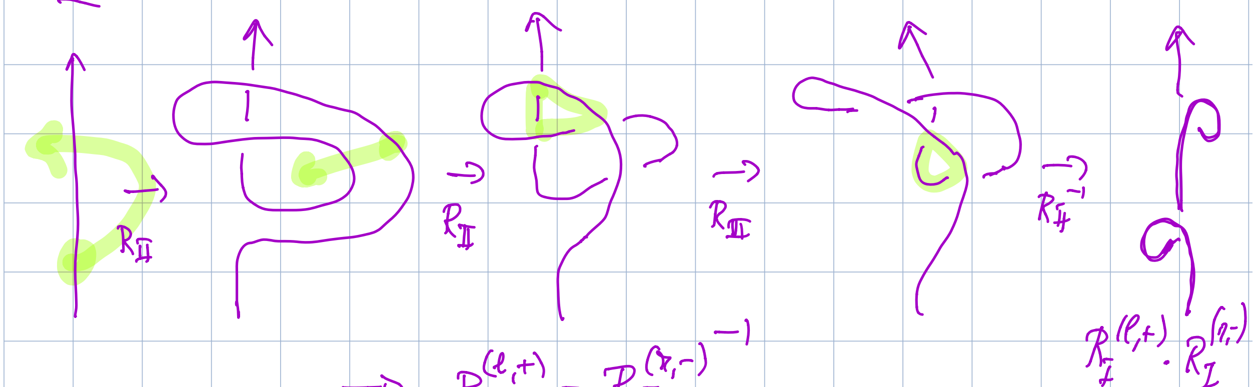
$\pi_1(S^1) = \mathbb{Z}$



<u>Oss</u>: i diagrammi posso vederli in $S^2 = \mathbb{R}^2 \cup \{\infty\}$.

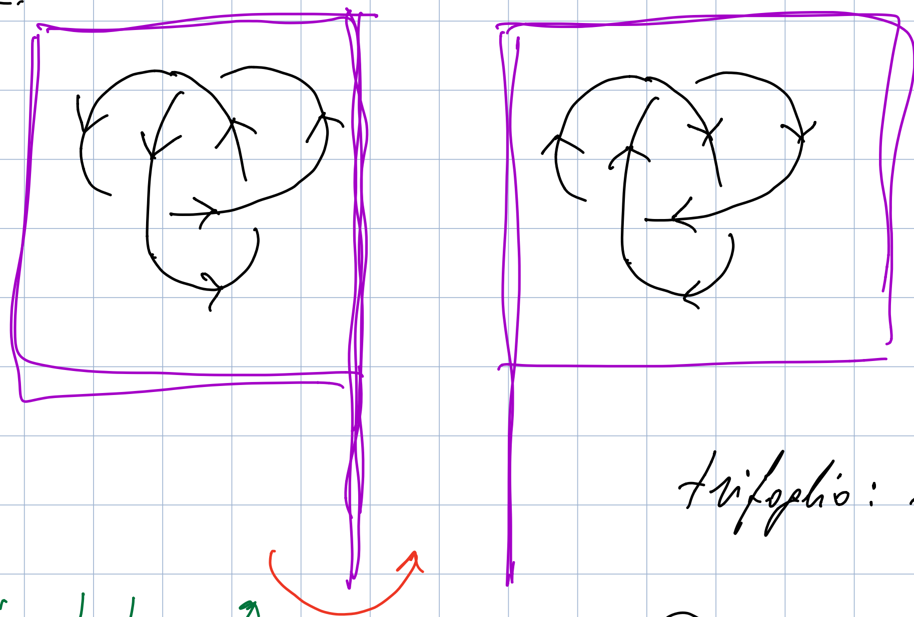


<u>Oss</u>:



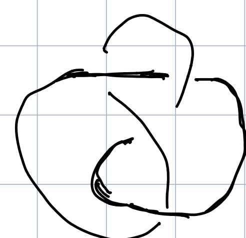
$\Longrightarrow R_I^{(l,+)} = {R_I^{(r,-)}}^{-1}$

invertibile:



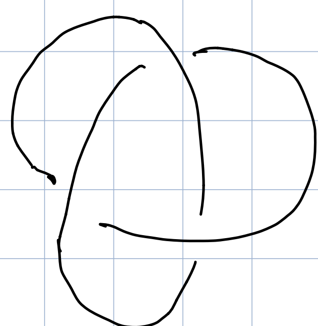
trifoglio: sì

Esercizio: tradurre ↗ in mosse R_I, R_{II}, R_{III}.

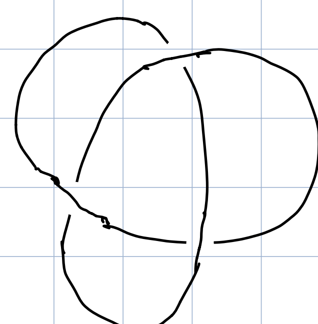


non invertibile

chirale:

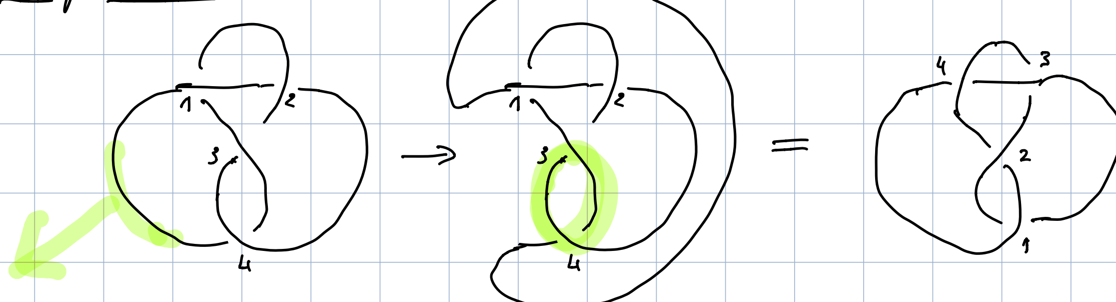


$\neq$ (vedremo)



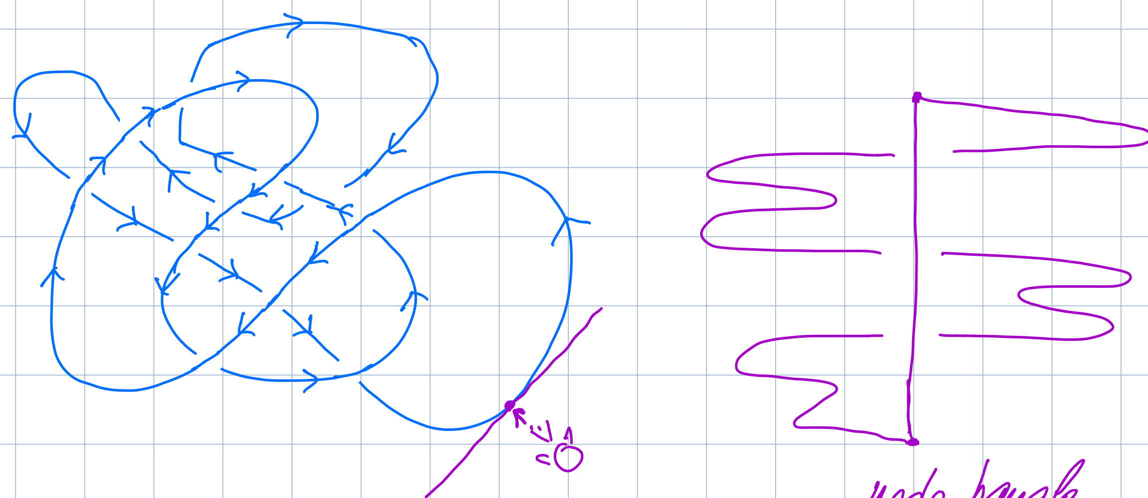
trifoglio sinistro   trifoglio destro

amphichiral



Prop: dato D qualsiasi diagramma di link esistono scambi di incroci che rendono D diagramma del link banale ○○…○.

Dim: per un nodo:

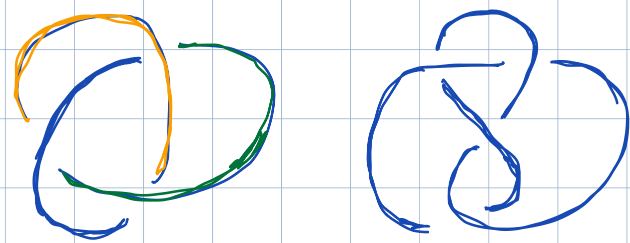


nodo banale

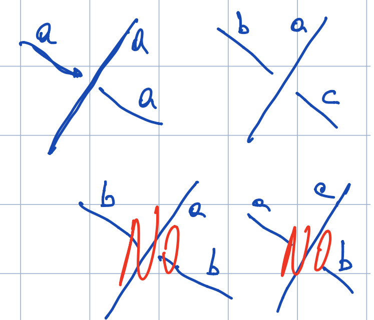
link $K_1 \sqcup \dots \sqcup K_m$ $\quad$ K_1 sempre sopra gli altri, lo sposto … ■

Invariante per nodi: $I : \{\text{diagrammi}\} \to S$
t.c. se due diagrammi sono legati da mosse R_I, R_{II}, R_{III} allora I ha lo stesso valore.

Tricolorazioni: $\{a,b,c\}$ tricolorazione di diag. D
un colore per gli overarc (gli archi con cui D è disegnato) t.c. a ogni incrocio vedo o 3 volte lo stesso colore o



tutti e tre i colori:

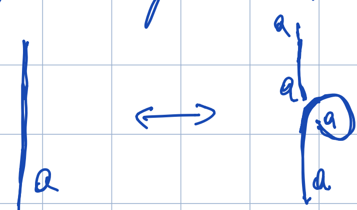


Def: $c_3(D) = \#\{\text{tricolorazioni}\}/\mathfrak{S}_{\{a,b,c\}}$

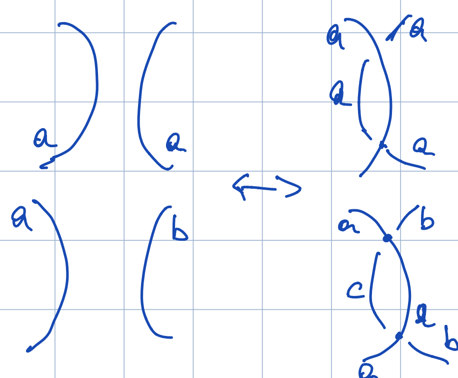
Teo: c_3 è un invariante.

Dim: provo che per ogni mossa R_I, R_{II}, R_{III} c'è una naturale bijezione tra le tricolorazioni prima e dopo.

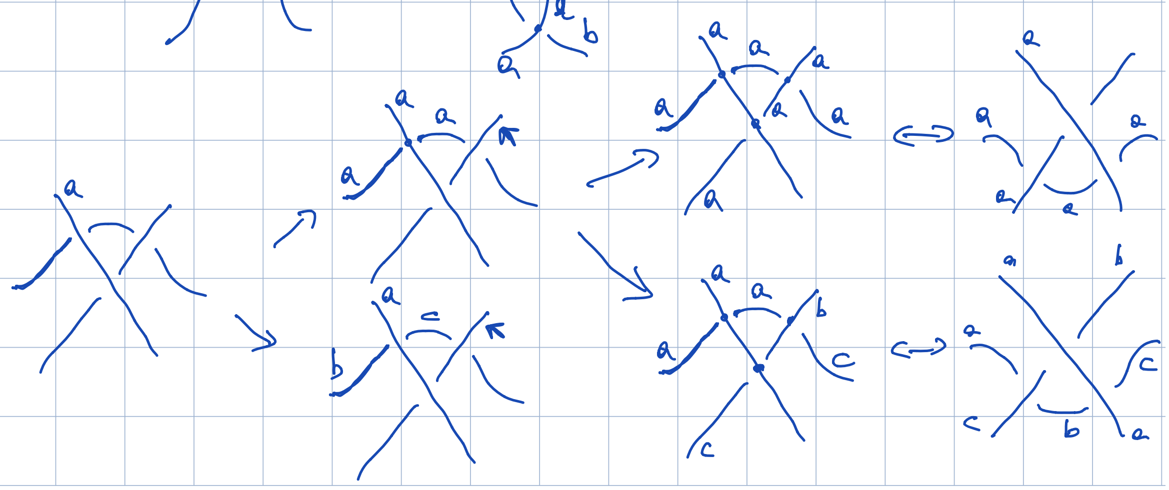
R_I:

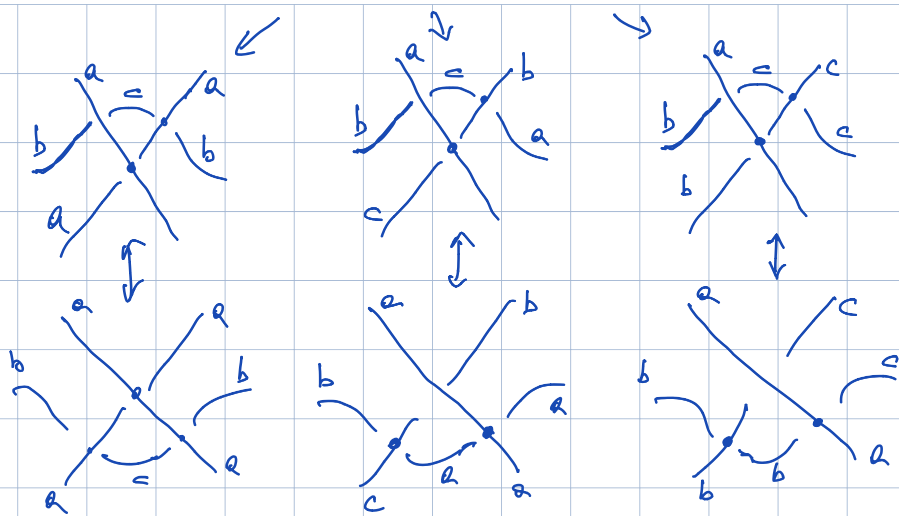


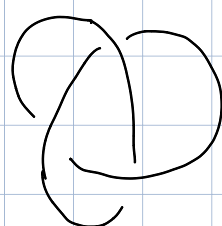
R_{II}

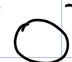


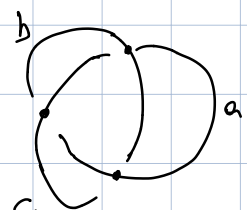
R_{III}



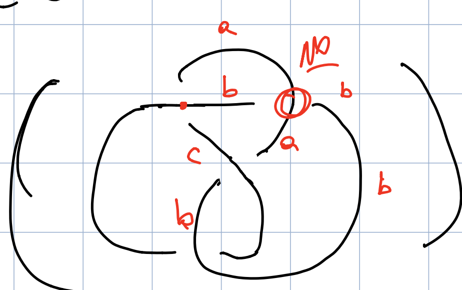


Cor:  $\neq$ 

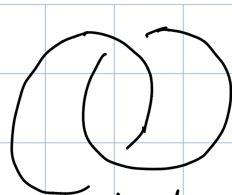
Dim: $c_3($  $) = 1$



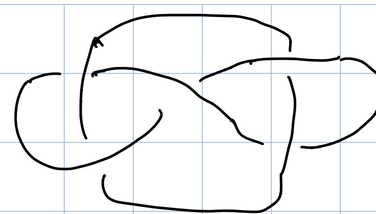
$c_3(\text{trif}) = 2.$

Oss: c_3  $= 1$

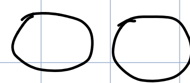
Con:



Hopf

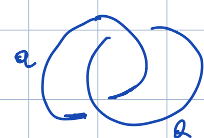


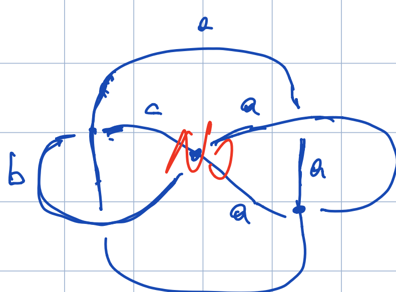
Whitehead

diversi da 

(Hopf $\neq \bigcirc\bigcirc$ facile
Hopf $\neq$ Whitehead facile)

$c_3(\bigcirc\bigcirc) = 2$

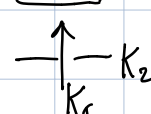
C_3 () $= 1$

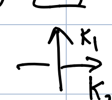


$c_3(\text{Whitehead}) = 1.$

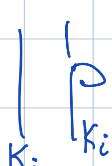
Linking number per coppie di nodi orientati

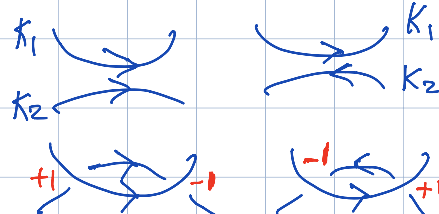
$\vec{K}_1, \vec{K}_2$ $\qquad lk(\vec{K}_1, \vec{K}_2) = \sum \pm 1 = \sum (+1) + \sum (-1)$

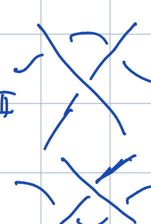


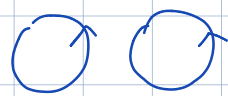


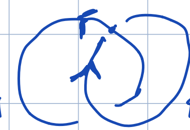
Prop: è ben def (inv. per R_I, R_{II}, R_{III}):

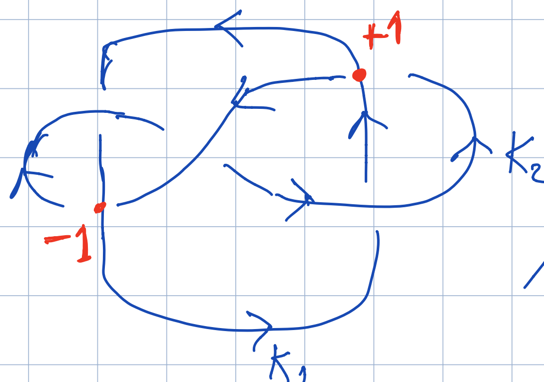
R_I: 

R_{II} 

R_{III} 

Es: $lk\Big($  $\Big) = 0$

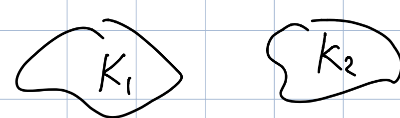
$\Big|lk\Big(K_1$  $K_2\Big)\Big| = |-1| = 1$

$lk\Big($  $\Big) = 0$

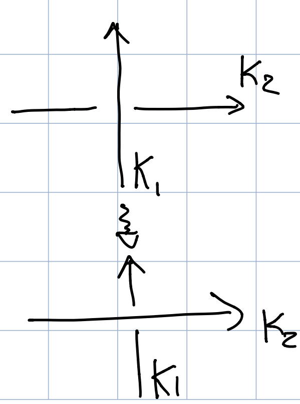
$\Rightarrow$ Whitehead $\neq$ Hopf.

Prop: $lk(\vec{K}_2, \vec{K}_1) = lk(\vec{K}_1, \vec{K}_2)$

Dimo: se sono lontani sono entrambi 0.



Tramite scambi incroci diventano isotopi e lontani $\Rightarrow$ basta vedere che $lk(\vec{K}_2, \vec{K}_1) - lk(\vec{K}_1, \vec{K}_2)$ non cambia con scambi incroci:



$lk(K_2, K_1) - lk(K_1, K_2)$

$0 - (-1) = +1$

$+1 - 0 = +1$

∎

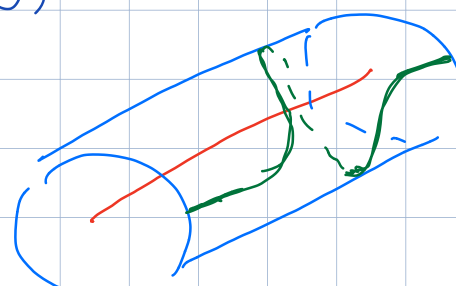
Oss: $lk(\vec{K}_1, \vec{K}_2) = \frac{1}{2} \left(\sum_{\substack{\text{incrocio positivo } K_i \text{ su } K_j \\ \{i,j\}=\{1,2\}}} (+1) + \sum_{\substack{\text{incrocio negativo } K_i \text{ su } K_j \\ \{i,j\}=\{1,2\}}} (-1) \right)$

Def: chiamo nodo con framing un nodo con una scelta di una sua longitudine / isotopia:

K, $U(K) \cong D^2 \times S^1$, $\lambda \subset \partial U(K)$

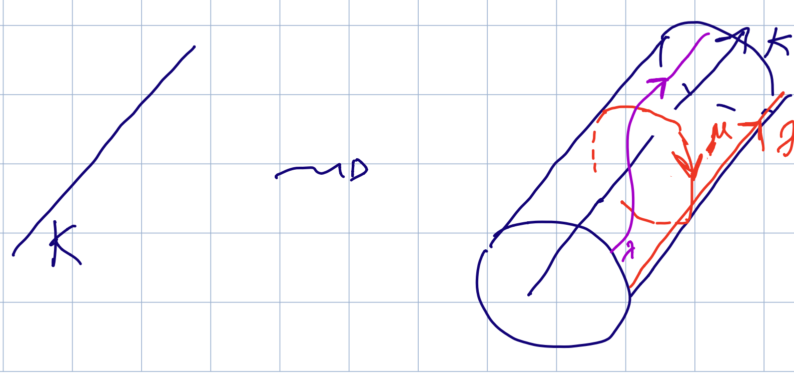
$[\lambda] =$ generatore di $H_1(U)$

visto a meno di isotopie di U e isotopia delle longitudini su ∂U.



Visto: c'è una longitudine privilegiata.

Oss: se λ_0 è la long. privilegiata le altre sono parametrizzate da $\mathbb{Z}$ così:

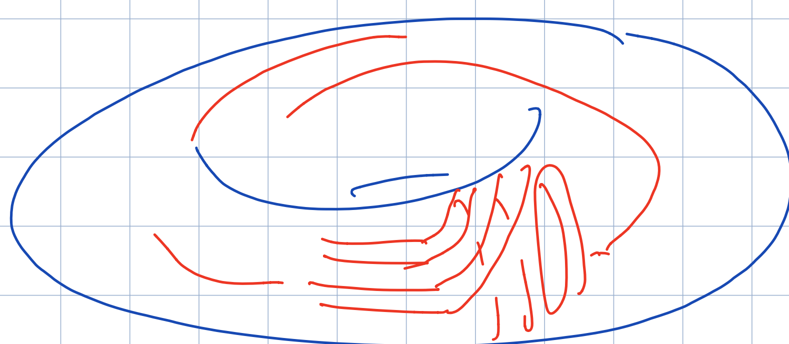


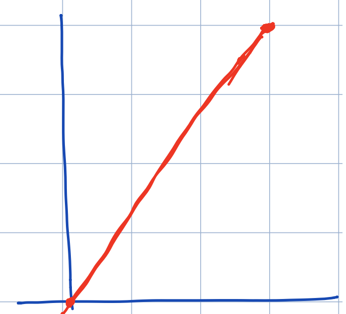
$[\lambda] = [\lambda_0 + p\mu] \in H_1(\partial U)$

$\lambda \longleftrightarrow p \in \mathbb{Z}$

Oss: non dipende da orientazione.

Oss: su ∂U ogni qualsiasi curva semplice chiusa non banale α ($\alpha \neq \partial$ ●) si scrive come $[\vec{\alpha}] = [a\mu + b\lambda_0]$ con $(a,b) = 1$.





Conseguenze: $\left\{\text{curve semplici chiuse non banali in } \partial U\right\} / \text{isotopia} \longleftrightarrow \mathbb{Q} \cup \{\infty\}$

$[\alpha] \longleftrightarrow a/b \in \mathbb{Q} \cup \{\infty\}$ se $[\vec{\alpha}] = [a\mu + b\lambda_0]$ in $H_1(\partial U)$

$\cup$

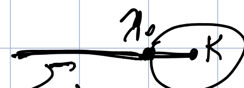
$\{\text{longitudini}\} \longleftrightarrow \mathbb{Z}$

Prop: dato $\vec{K} \subset S^3$ $\exists\, \Sigma$ sup. orientata $\subset S^3$ t.c. $\partial\Sigma = \vec{K}$.

Def Ogni tale Σ è detta sup. di Seifert per K.

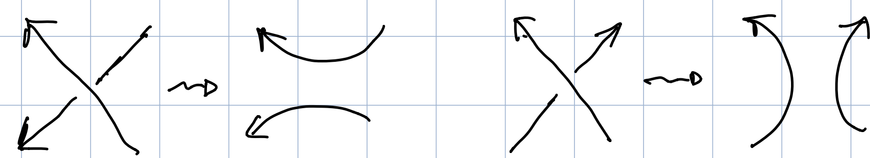
Dim: (1) $[\lambda_0] = 0 \in H_1(E)$ $\quad (E = S^3 \setminus \mathring{U})$

"$\Longrightarrow$" λ_0 borda superficie orientata in E

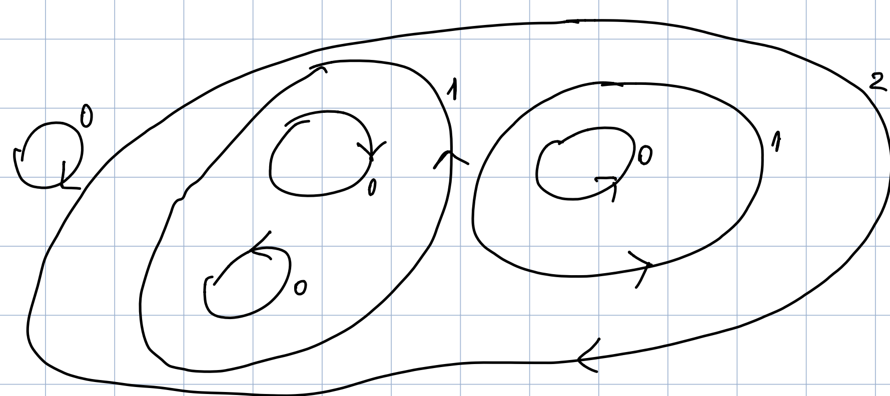


(2) algoritmo di Seifert:

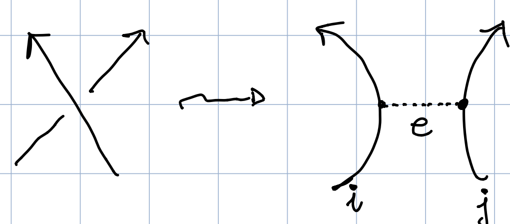
- parto da diagramma D
- semplifico incroci usando orientazione:



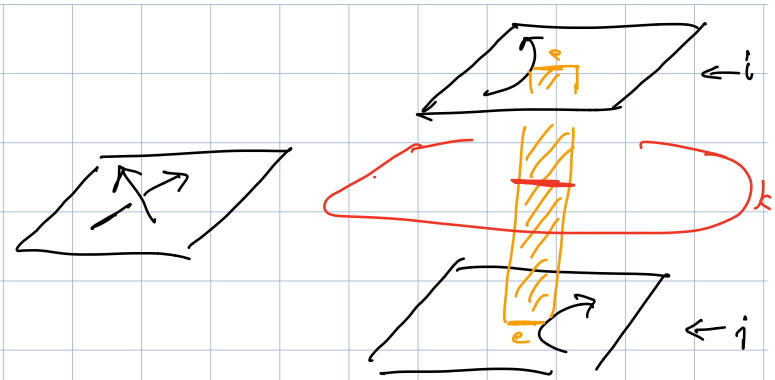
- trovo circonferenze in $\mathbb{R}^2$: le ordino dichiarando livello 0 quelle che non circondano altre, livello 1 quelle che circondano solo livello 0 etc.



- metto il livello l ad altezza l e in $\mathbb{R}^2\times\{l\}$ considero dischi orientati bordati positivamente dalle circonf. a livello l.

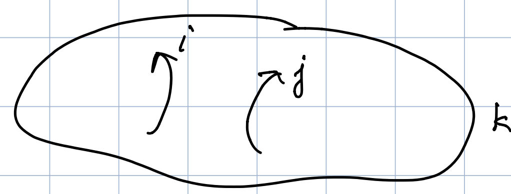
- affermo: 

ex[i,j] non incontra alcun disco



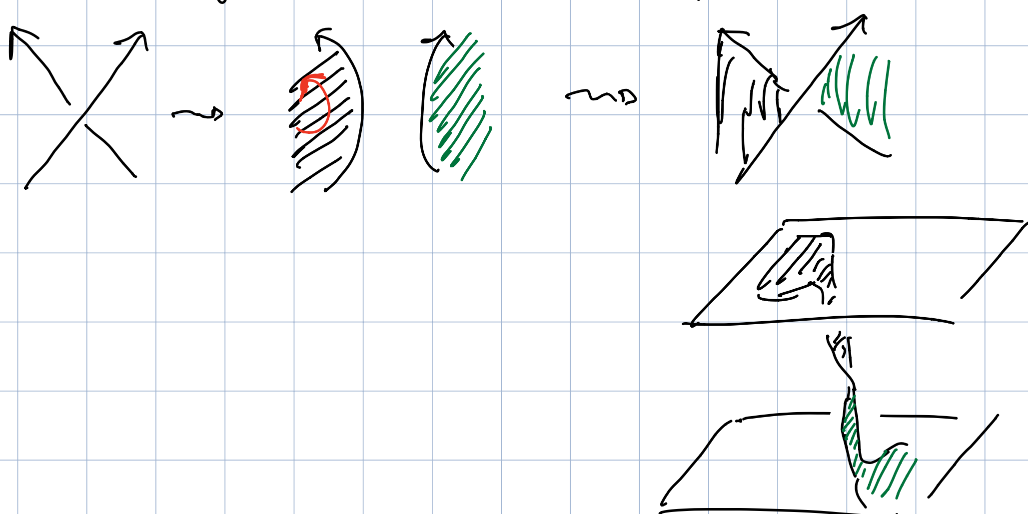
altrimenti:

$j < k < i$

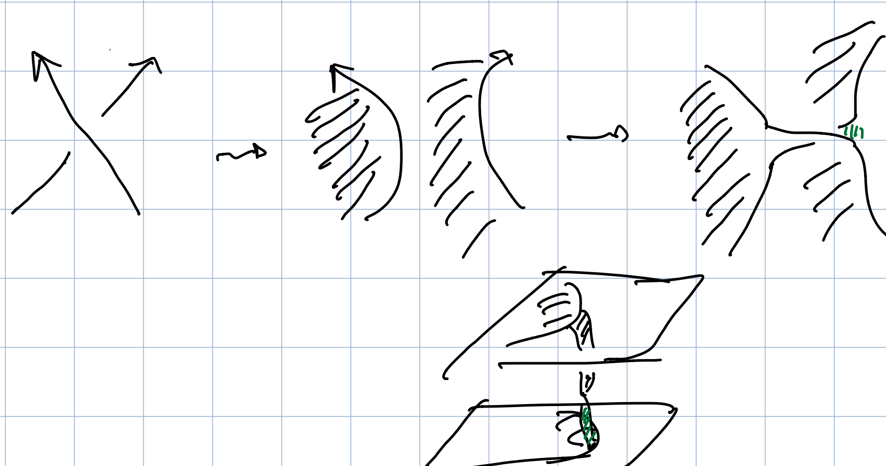


$\Longrightarrow$ k circonda i
ASSURDO

- ripristino gli incroci incollando pezzi di disco:



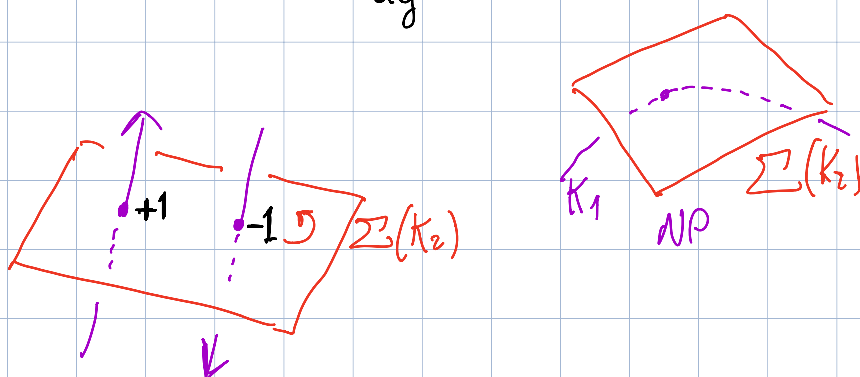
Oppure:



∎

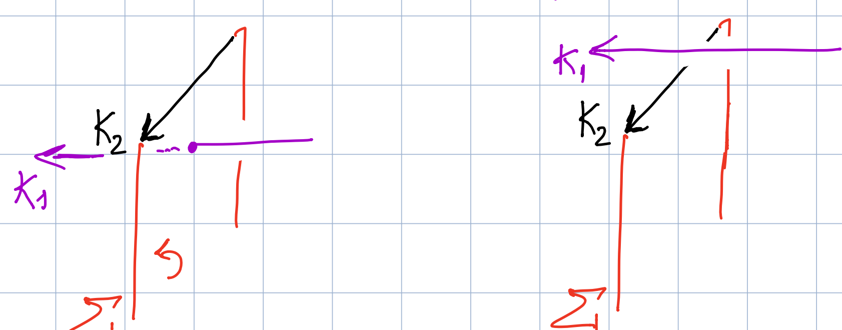
Es:

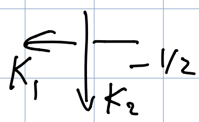
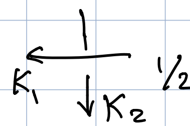
Prop: $lk(K_1, K_2) = \#_{alg}\left(K_1 \cap \Sigma(K_2)\right)$



Dim: se K_1 è lontano da $\Sigma(K_2)$ entrambi 0.
Provo che spostando K_1 lontano da $\Sigma(K_2)$ le differenze non cambia:

• 

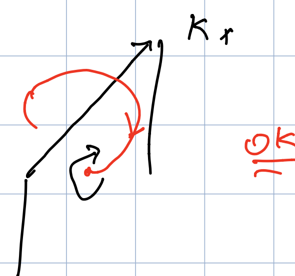
• 

+1	0	$\#_{alg}\, K_1 \pitchfork \Sigma(K_2)$
		$lk(K_1, K_2)$

$\blacksquare$

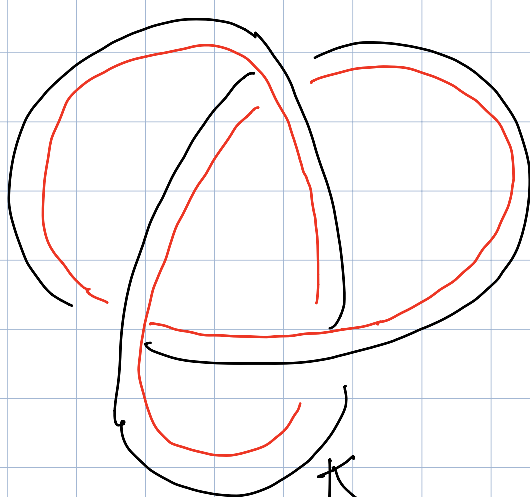
<u>Cor</u>: $lk(\vec{K}_1, \vec{K}_2) = p$ se $[\vec{K}_2] = p \cdot [\vec{\mu}_1] \in H_1(E(K_1))$

<u>Dim</u>: basta vedere che $\#_{alg}\, \vec{\mu}_1 \pitchfork \Sigma(K_1) = +1$



$\blacksquare$

<u>Def</u>: chiamo framing diagrammatico di un diagramma D di un nodo K quello in cui la longitudine è parallela a D sul piano:

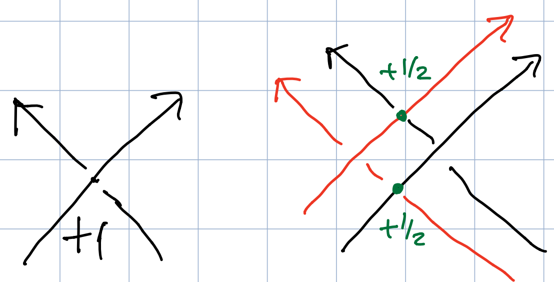


Def: $slk(D) = \sum_{\leftarrow\uparrow-} (+1) + \sum_{-\uparrow\rightarrow} (-1)$

(ognuno orientazione a caso $\leftarrow\uparrow_{+1} \rightsquigarrow -\downarrow\rightarrow_{+1}$)

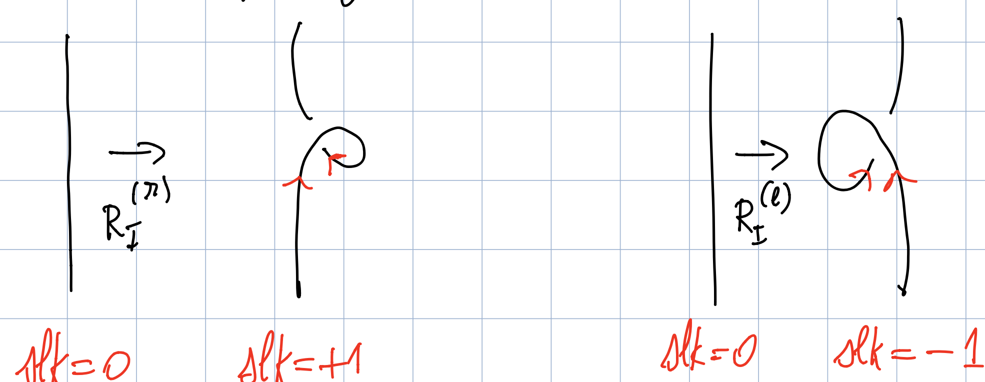
Prop: l'intero che corrisponde al framing diag è $slk(D)$

Dim:



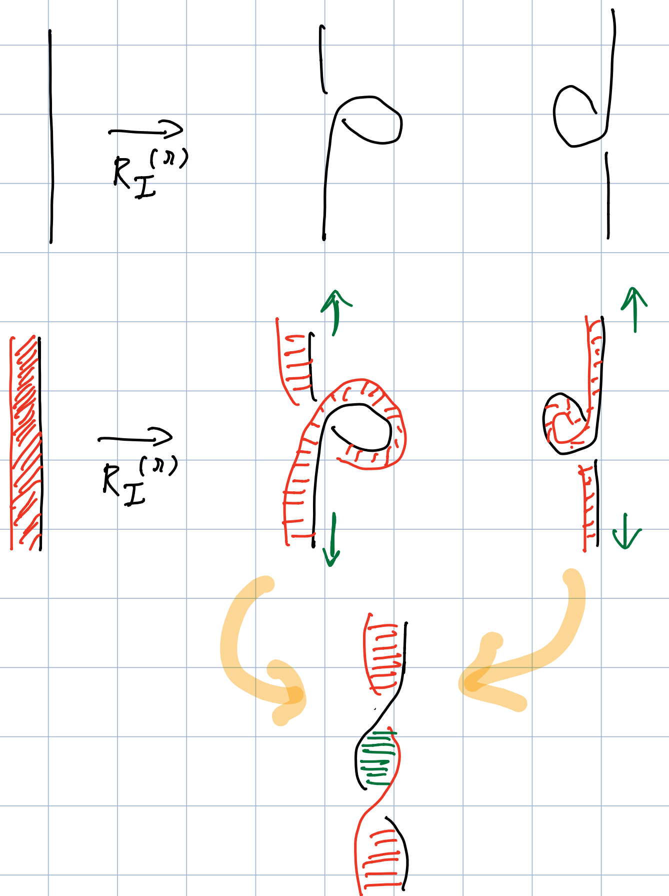
Oss: $\lambda \leftrightarrow p$
se $[\lambda] = [\lambda_0 + p\mu]$
$\Rightarrow p = lk(K, \lambda)$ ∎

Cor: due diagrammi legati da mosse R_I definiscono nodi con framing diversi:



Oss: R_{II} e R_{III} e isotopia attraverso ∞ preservano framing

Cor: $R_I^{(r)}$ è generata da R_{II}, R_{III}, isotopia attraverso ∞ + $R_I^{(\ell)^{-1}}$ ma non senza $R_I^{(\ell)^{-1}}$



<u>Teo</u>: due diagrammi rappresentano lo stesso nodo con framing

$\Longleftrightarrow$ si ottengono uno dall'altro tramite

- R_{II}, R_{III} + isotopia su S^2
- R_{II}, R_{III} + mosse 

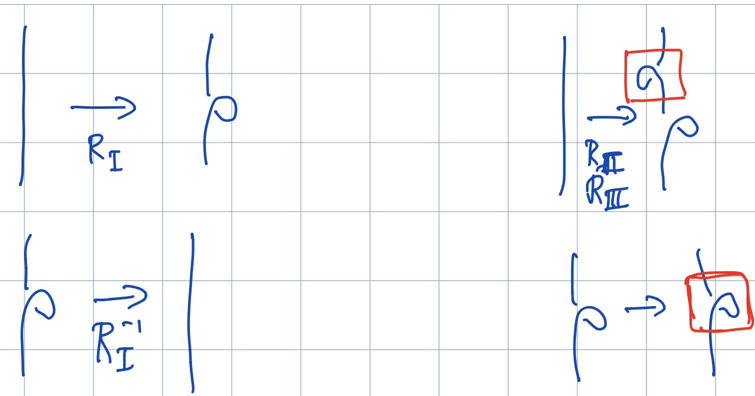
Dimo: equivalenza dei due set segue da quanto visto all'inizio.

Faccio dimo con secondo.

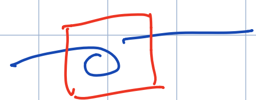
D_1, D_2 rappresentano stesso nodo

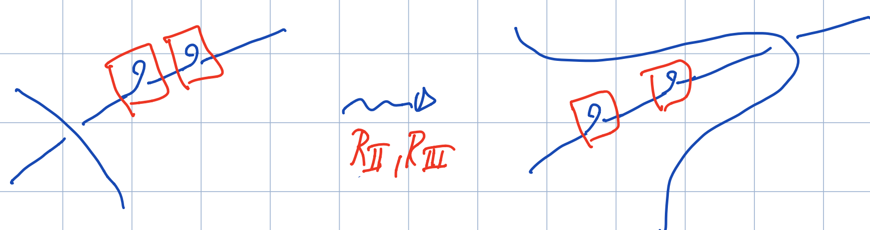
$\Rightarrow$ legati da R_I, R_{II}, R_{III}.

Modifico successione così:



eseguo solo R_{II} e R_{III} ma laccio compare



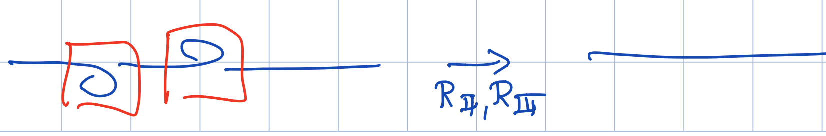


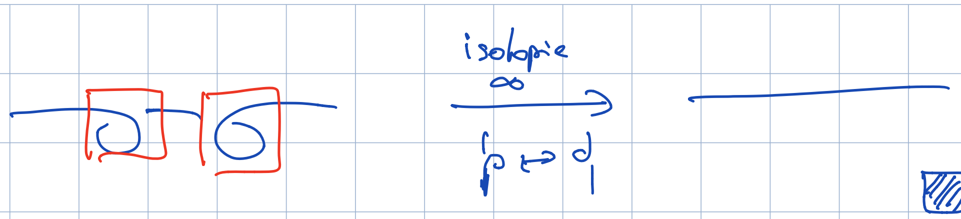
Alla fine ho $D_1 \xrightarrow[R_{II}, R_{III}]{} \overline{D}_2 = D_2$ + alcuni riccioli

$\Rightarrow$ $\overline{D}_2$ e D_2 danno lo stesso framing:

$\Rightarrow$ numero algebrico di riccioli $= 0$

li faccio scivolare vicini $\Rightarrow$ ne ho due discordi adiacenti





$\mathcal{F} = \{\text{link con framing}\}/_{\sim}$

Kauffman bracket $\qquad [\cdot] : \mathcal{F} \longrightarrow \mathbb{Z}[A^{\pm 1}]$

- $\left[\times \right] = A \cdot \left[\asymp \right] + A^{-1} \left[ )(\right]$

  sopra verso sinistra $\qquad$ sopra verso destra

- $[D \sqcup \bigcirc] = (-A^2 - A^{-2}) \cdot [D] \qquad D \neq \emptyset$

- $[\emptyset] = 1$

oppure: chiamo stato s la scelta di segni $+$ o $-$ per ciascun incrocio; $p(s) = \#$ "$+$" $\quad n(s) = \#$ "$-$"

$|s| = \#$ circonferenze del diagramma semplificato

$\overset{+}{\times} \rightsquigarrow)($

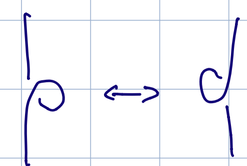
$\overset{-}{\times} \rightsquigarrow \asymp$

$$[D] = \sum_s A^{p(s)-n(s)} \cdot (-A^2 - A^{-2})^{|s|}$$

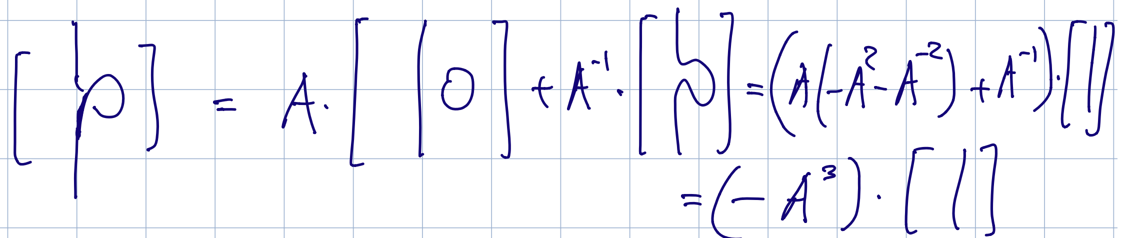
E' ben def. sui diagrammi.

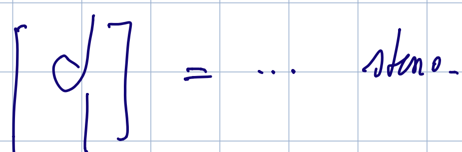
Teo: è invariante rispetto alle mosse con framing $\Rightarrow$ è ben def. su $\mathcal{F}$.

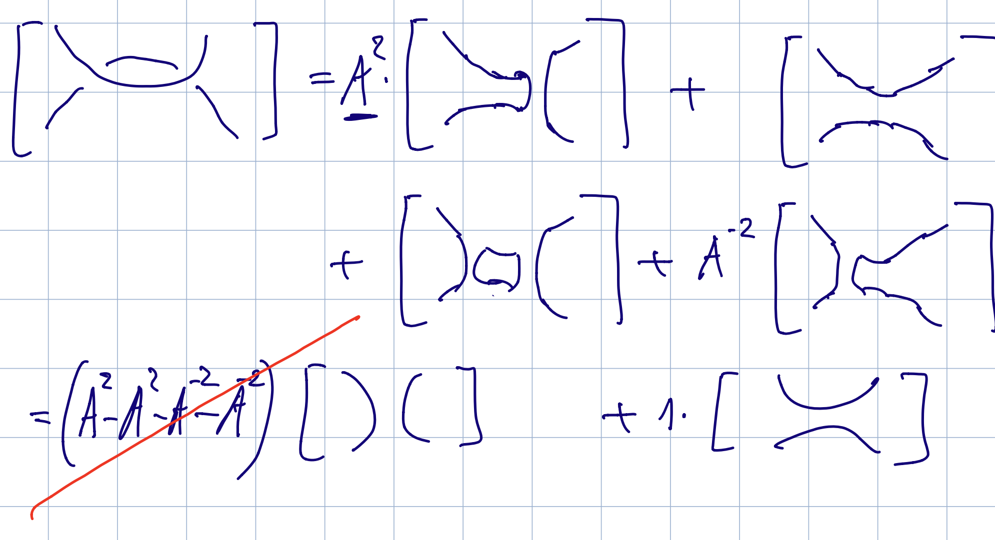
Dimo:

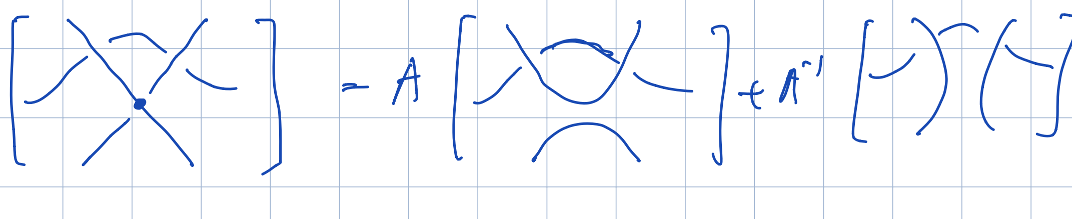
- 

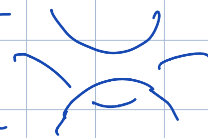
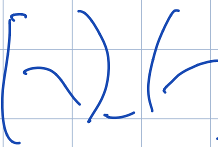
provo invece che hanno stesso effetto su [.]

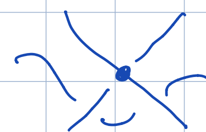




- 

- 

$= A \cdot \Big[$  $\Big] + A^{-1} \Big[$  $\Big]$

$= \Big[$  $\Big]$. $\blacksquare$