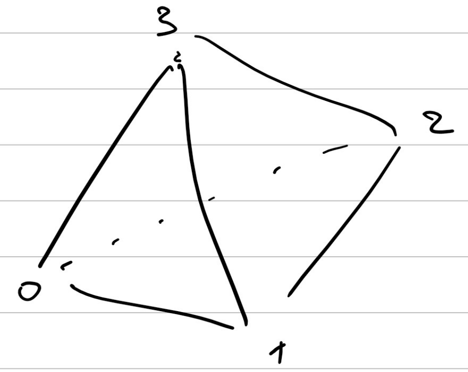
LDT - IUSS - 27/3/20

Omologia:

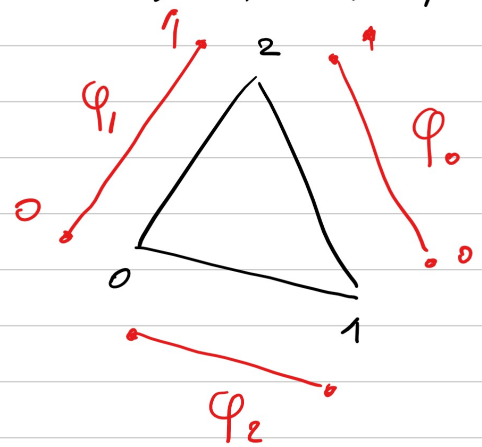


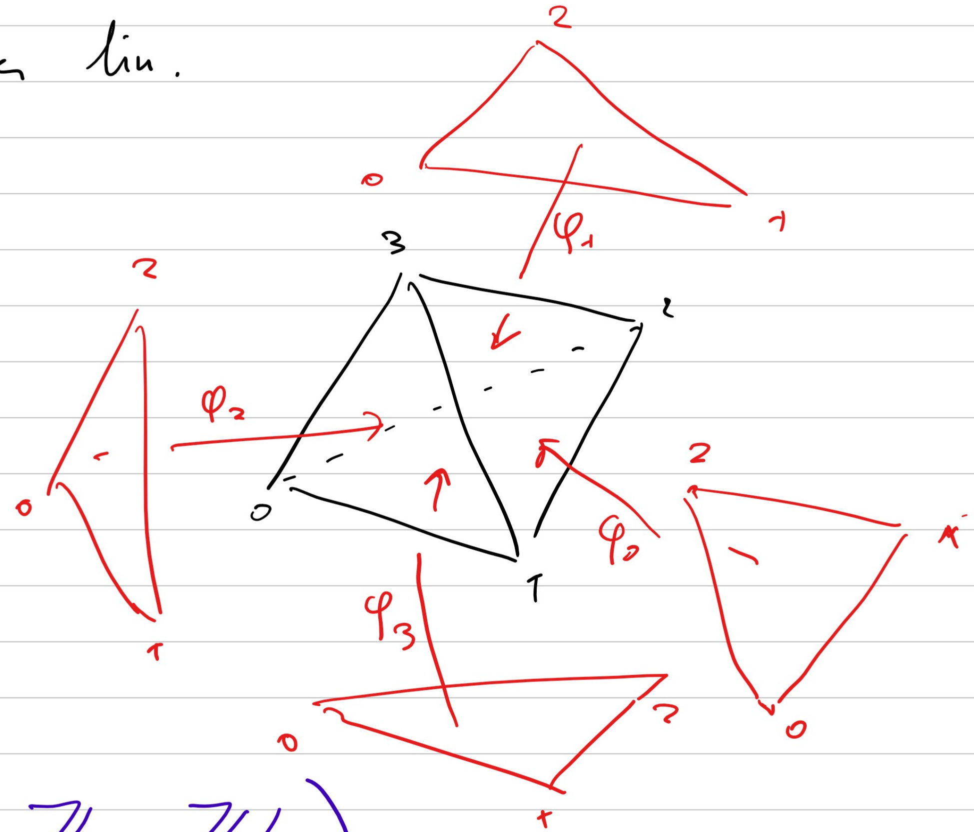
$\Delta_k = \mathrm{conv}(e_0, \dots, e_k) \subset \mathbb{R}^{k+1}$

$\varphi_i : \Delta_{k-1} \to \Delta_k$

$\varphi_i(e_j) = \begin{cases} e_j & j < i \\ e_{j+1} & j \geq i \end{cases}$ esteso per lin.

$\mathrm{Im}(\varphi_i) : \mathrm{Conv}(e_0, \dots, \hat{e}_i, \dots, e_k)$





X sp. top.

R anello commut. con 1 $\quad (R = \mathbb{Z} \text{ o } \mathbb{Z}/2)$

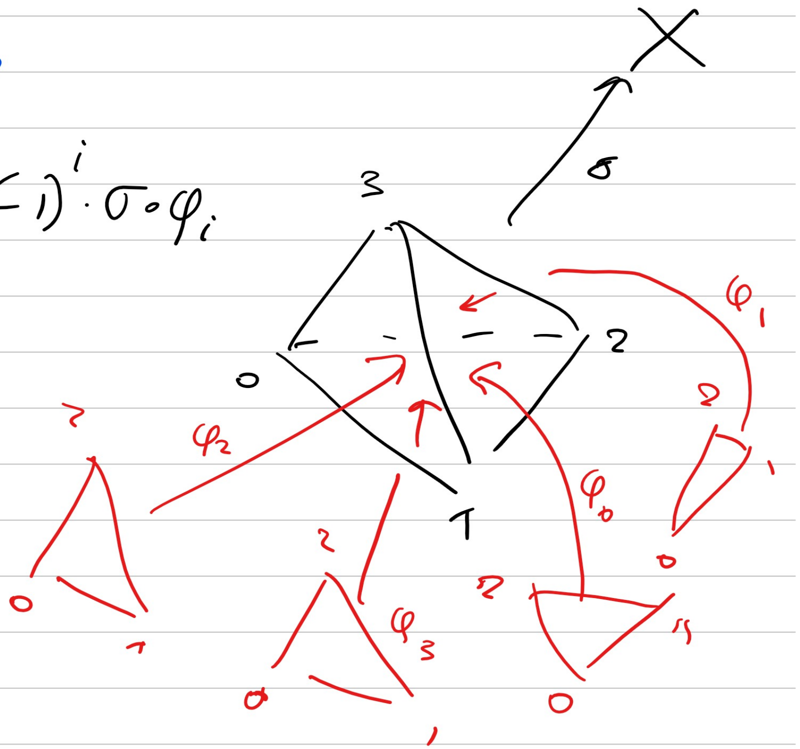
k-catene singolari

$C_k(X,R)$ = R-modulo libero generato dalle $\sigma : \Delta_k \to X$

$= \left\{ \sum_{i=1}^{m} r_i \cdot \sigma_i \ : \ m \in \mathbb{N} \ r_i \in R \ \sigma_i : \Delta_k \to X \text{ cont} \right\}$

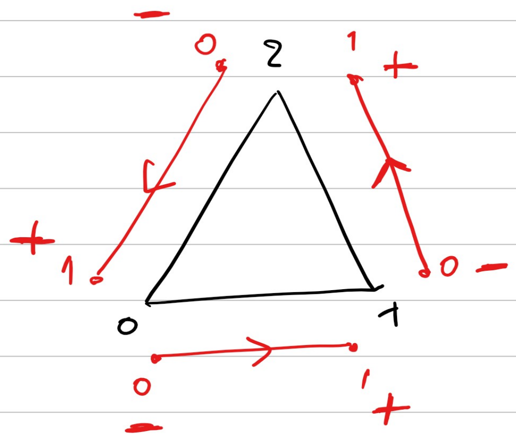
Oss: $R = \mathbb{Z}$ $\quad$ R-mod. = gruppo abeliano

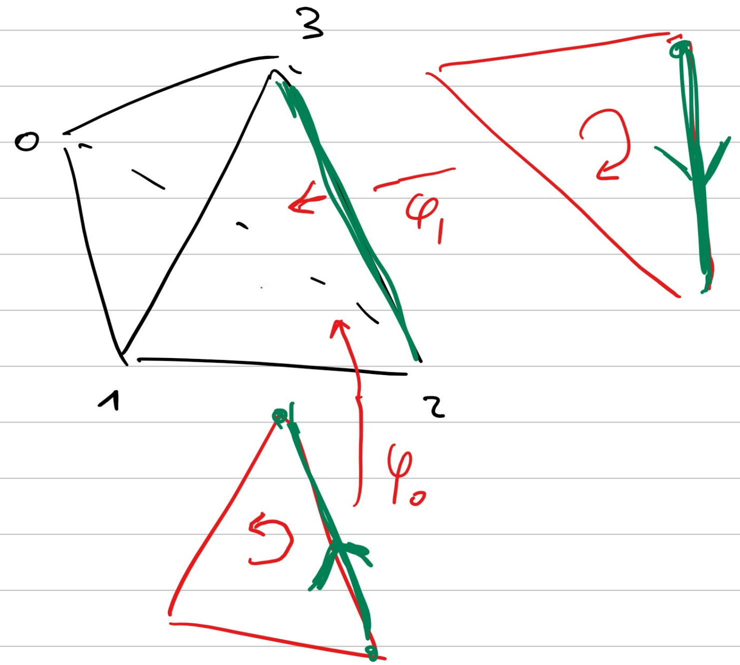
$$\partial_k : C_k(X;R) \to C_{k-1}(X;R) \qquad \partial_k \sigma = \sum_{i=0}^{k} (-1)^i \cdot \sigma \circ \varphi_i$$



Prop: $\boxed{\partial_k \circ \partial_{k+1} = 0}$

Dim:





$B_k(X;R) = \mathrm{Im}(\partial_{k+1})$ "bordi"

$Z_k(X;R) = \mathrm{Ker}(\partial_k)$ "cicli"

Def: $H_k(X;R) = Z_k(X;R) / B_k(X;R)$

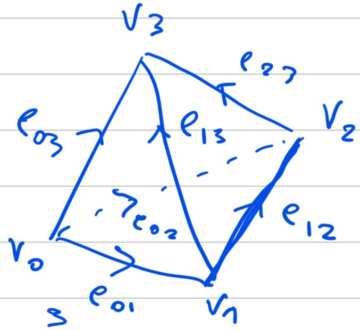
k-gruppo di omologia di X a coeff. in R

Fatto: se X è triangolato o realizzato come successivo attaccamento di celle allora posso calcolare $H_k(X;R)$ usando:

$\tilde{C}_k$ = R-modulo lib. generato dalle k-celle o simplessi

$\tilde{\partial}_k$ = modo naturale di esprimere il bordo di una cella o simplesso come comb. lin. di quelli più piccoli di 1 dim.

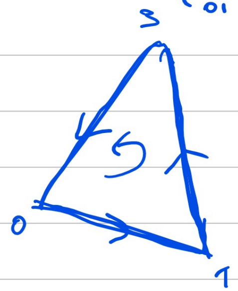
Es: S^2



T_{ijk} = faccia di vertici $i<j<k$

$\partial e_{ij} = v_j - v_i$

$\partial T_{ijk} = e_{ij} + e_{jk} - e_{ik}$



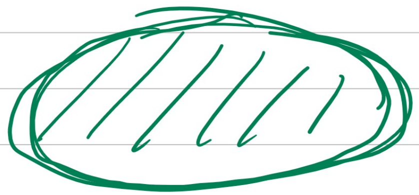
$Z_0 = \langle v_0, v_1, v_2, v_3 \rangle$ $\quad B_0 = \langle v_j - v_i \mid \dots \rangle$ $\quad \frac{Z_0}{B_0} = H_0 = \mathbb{Z}$

Ese: $Z_1 = B_1 = \langle \partial T_{ijk} \rangle$ $\quad H_1 = 0$

$Z_2 = \langle T_{123} - T_{023} + T_{013} - T_{012} \rangle$ $\quad B_2 = 0$ $\quad H_2 = \mathbb{Z}$

Ese: $S^n = \{x \in \mathbb{R}^{n+1} : \|x\| = 1\}$

$S^n = \{pto\} \cup_{cost} D^n \qquad cost : \partial D^n \to \{pto\}$

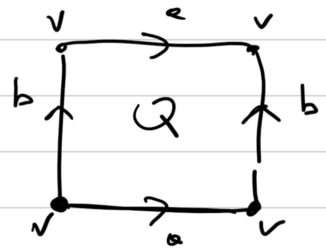


D^2

$C_0 = C_n = \mathbb{Z} \qquad C_i = 0 \quad 0 < i < n$

$H_0 = \mathbb{Z} \qquad H_n = \mathbb{Z} \qquad H_i = 0 \quad \forall i \neq 0, n$

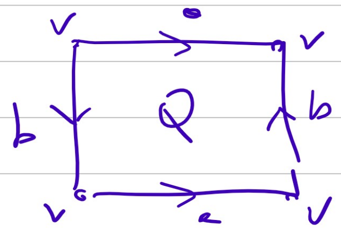
Ese: $T =$



$\partial v = 0$
$\partial a = \partial b = 0$
$\partial Q = 0$

$H_0 = \mathbb{Z}$
$H_1 = \mathbb{Z} \oplus \mathbb{Z}$
$H_2 = \mathbb{Z}$

Ese: $K =$



$\partial v = 0$
$\partial a = \partial b = 0$
$\partial Q = 2b$

$H_0 = \mathbb{Z}$
$H_1 = \mathbb{Z} \oplus \mathbb{Z}/2$
$H_2 = 0$

Fatto: $\pi_1 \twoheadrightarrow H_1$ $\quad Ker = \langle [a,b] : a,b \in \pi_1 \rangle$

$\Rightarrow H_1 = Ab(\pi_1)$

Successione esatta di omomorfismi di R-moduli

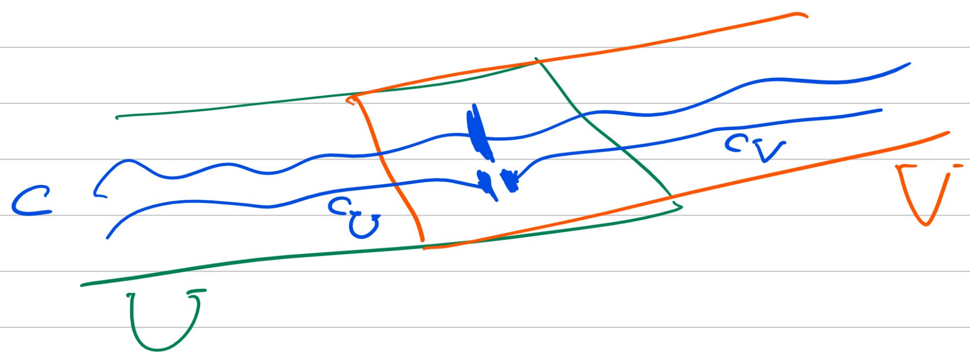
$$\cdots \longrightarrow G_{k+1} \xrightarrow{\psi_{k+1}} G_k \xrightarrow{\psi_k} G_{k-1} \longrightarrow \cdots$$

se $Ker(\psi_k) = Im(\psi_{k+1})$.

Teo (Mayer-Vietoris): $X = U \cup V$, $U, V, U \cap V$ ragionevoli

$$\Rightarrow \cdots \to H_{n+1}(X) \to H_n(U \cap V) \to H_n(U) \oplus H_n(V) \to H_n(X) \to \cdots$$

inclusione



$c = c_U - c_V$

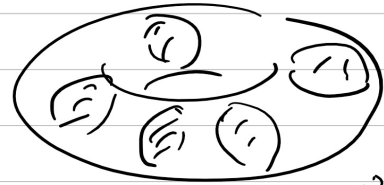
$\partial c = 0 \quad \partial c_U - \partial c_V = 0$

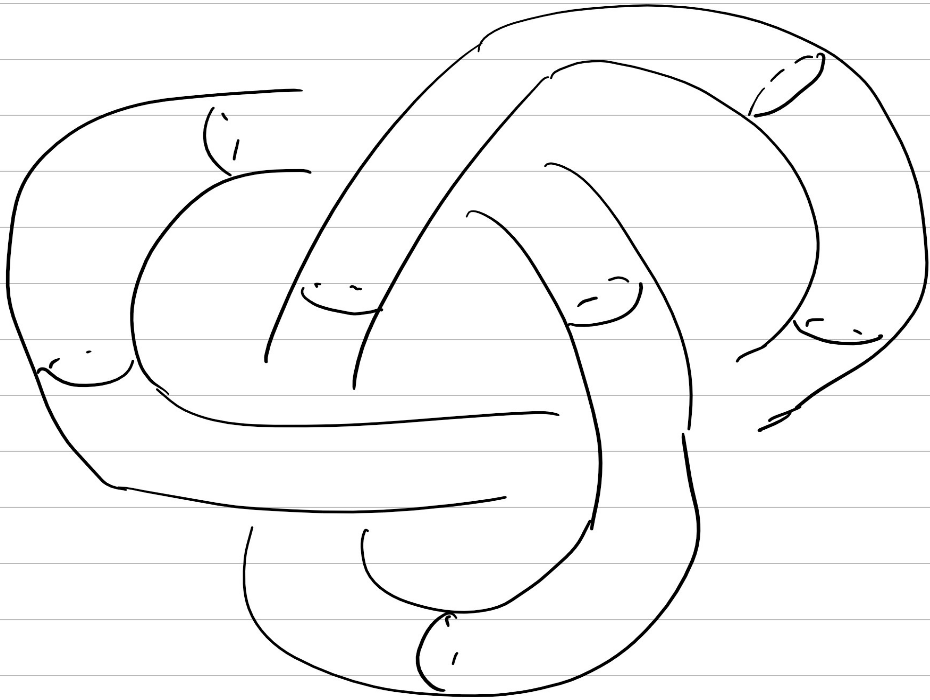
$[c] \mapsto [\partial c_U] = [\partial c_V]$

$\partial c = 0$

Considero in $S^3 = \mathbb{R}^3 \cup \{\infty\}$

$U \cong D^2 \times S^1$ sottoinsieme diffeomorfo a toro solido





Pongo $T = \partial U$

$E = \overline{S^3 \setminus U} = S^3 \setminus \mathring{U}$

$= (S^3 \setminus U) \cup T$

$$\underbrace{H_2(S^3)}_{=0} \xrightarrow{0} H_1(T) \xrightarrow{\psi} H_1(U) \oplus H_1(E) \xrightarrow{0} \underbrace{H_1(S^3)}_{=0}$$

$\ker(\psi) = \mathrm{Im}(0) = 0 \Longrightarrow \psi$ iniettiva

$\mathrm{Im}(\psi) = \ker(0) \Longrightarrow \psi$ suriettiva

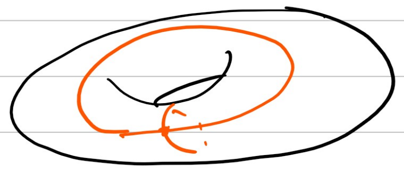
ψ isomorfismo.

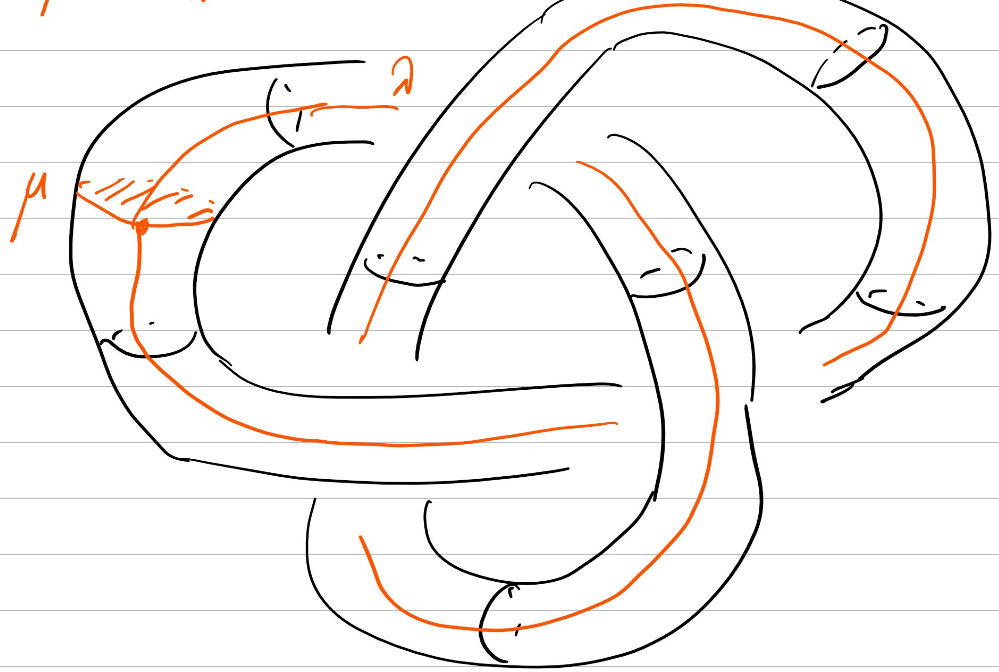
$R=\mathbb{Z}$

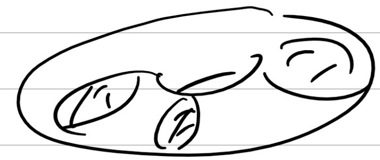
$$H_1(T) \longrightarrow H_1(U)\oplus H_1(E) \quad \text{isomorf.}$$

$$H_1(T) = \mathbb{Z}_\mu \oplus \mathbb{Z}_\lambda \qquad H_1(U) = \mathbb{Z}_\lambda$$

> Fatto: l'omologia non cambia deformando uno spazio in modo continuo



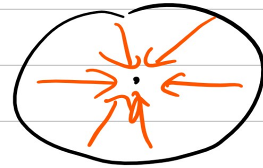




$U = D^2 \times S^1$

$D^2 \times S^1$

$\mu = \partial D^2 \times \{*\}$


$\times S^1$

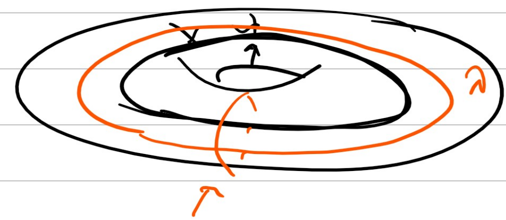
Teorema

$$\Longrightarrow \boxed{H_1(E) = \mathbb{Z}_\mu}$$

$$H_1(U) = H_1(S^1) = \mathbb{Z}$$

Oss: μ scelto come $\partial D^2 \times \{*\}$ meridiano

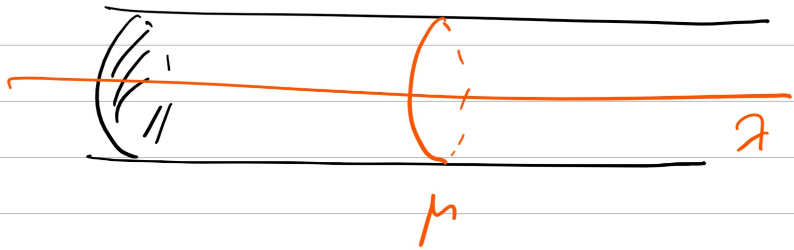
λ scelto come qualsiasi laccio con $\mu \cap \lambda = \{pto\}$
longitudine

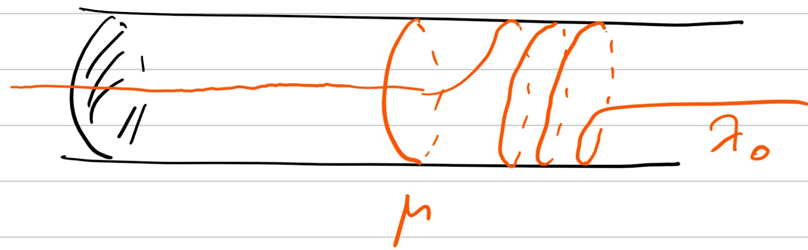


Dunque $H_1(T) \longrightarrow H_1(E)$ mappa indotta da inclusione

$$\lambda \longmapsto k\cdot\mu$$

$$\Rightarrow \lambda_0 = \lambda - k\cdot\mu \in H_1(T)$$





λ_0 = unica longitudine di U che è nulla in $H_1(E)$.

—— o ——

Versione relativa dell'omologia: (X,Y) $\quad Y\subset X$ $\quad R$

$$C_k(X,Y;R) = \frac{C_k(X;R)}{C_k(Y;R)} \; ; \quad \partial_k \qquad H_k(X,Y;R).$$

Oppure: $\tilde{Z}_k(X,Y;R) = \{c\in C_k(X) : \partial c \in C_{k+1}(Y)\}$

$\tilde{B}_k(X,Y,R) = \{\partial c + d : c\in C_{k+1}(X),\ d\in C_k(Y)\}$

$$H_k(X,Y) = \frac{\tilde{Z}_k}{\tilde{B}_k}.$$

Fatto: esiste una successione esatta

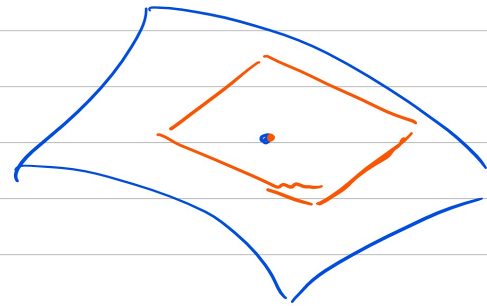
$$\dots \to H_k(X) \xrightarrow[\text{incl}]{} H_k(X,Y) \xrightarrow[\partial]{} H_{k-1}(Y) \xrightarrow[\text{incl}]{} H_{k-1}(X) \to \dots$$

Orientazione di varietà

$M^{(n)}$ n-varietà (anche con ∂M)

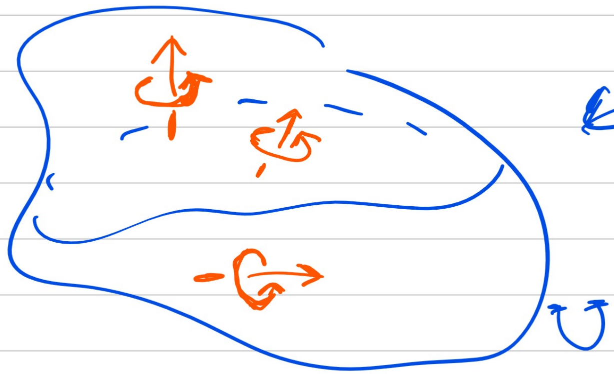
(1) Fatto: $\forall x \in M$ ha senso $T_xM \cong \mathbb{R}^n$ sp. tang.

(ad es.: è sempre possibile realizzare M come sottovar. DIFF di $\mathbb{R}^N$ $N \gg 0$)



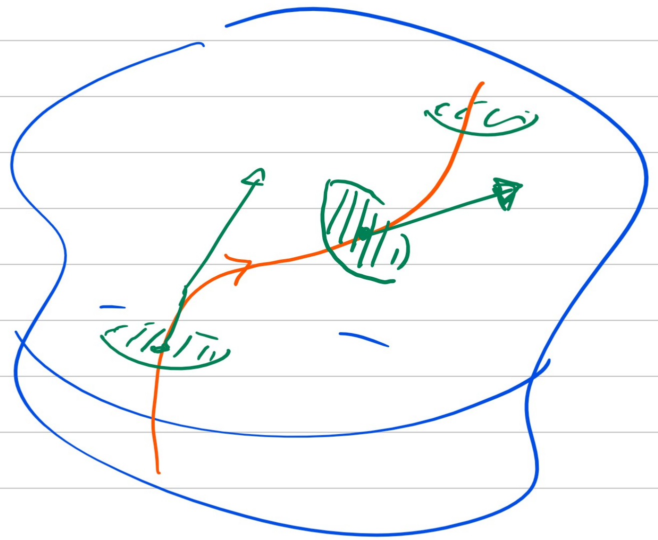
orientaz. di sp. vett. = scelta base / cambio base con matrice con $\det > 0$

Orientaz su M: orientaz per ogni T_xM "loc. costante"



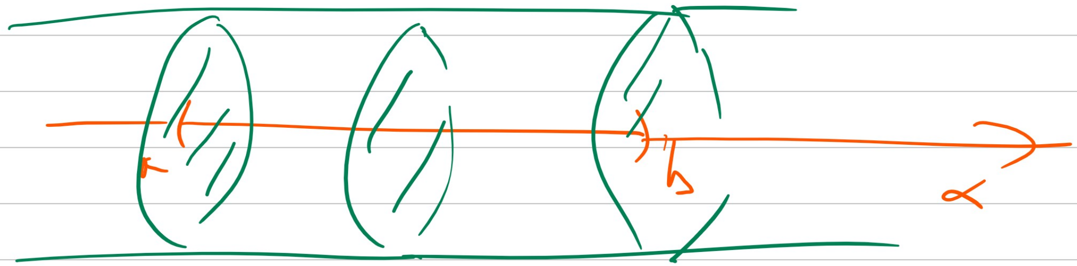
Oss: M connessa ha ≤ 2 orientazioni.

(2) M^{DIFF} $\alpha : S^1 \to M$ laccio semplice $\alpha'(t) \neq 0 \ \forall t$

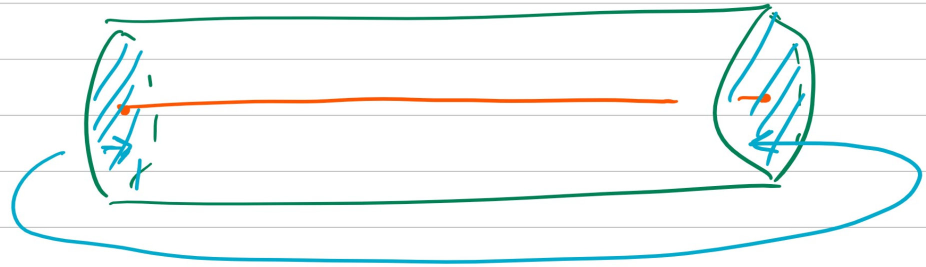


Si trova un intorno di $\mathrm{Im}(\alpha)$ che è localmente un tubo

$D^{m-1} \times (a,b)$



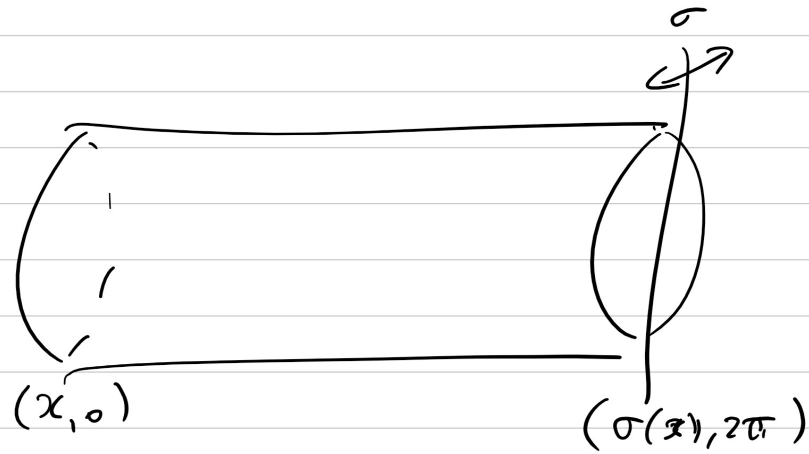
Globalmente:



$D^2 \times [0, 2\pi]$

Il risultato può essere solo:

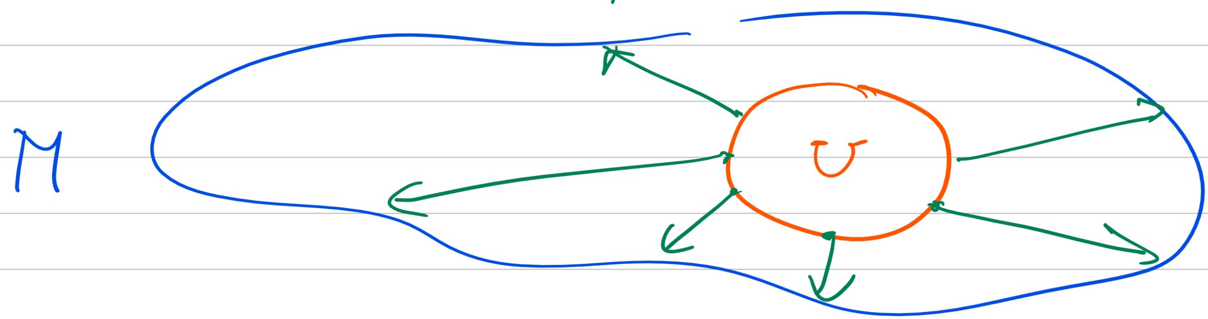
- $D^{n-1} \times S^1$ incollo con id
- incollo con una riflessione



M è <u>orientabile</u> se e solo se $\forall \alpha$ il suo intorno come sopra è $D^{n-1} \times S^1$.

(3) $x \in M \setminus \partial M$, $x \in U$ intorno $\cong B^n$, $\overline{U} \cong D^n$

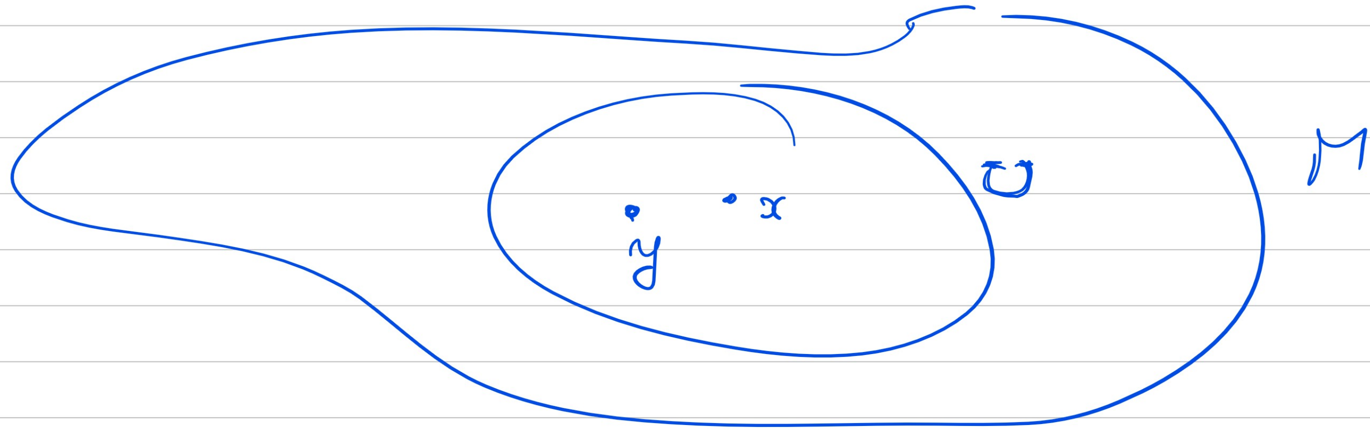
<u>Fatto</u>: $H_n\underbrace{(M, M \setminus U)}_{\text{nullo qui}} \cong H_n(\overline{U}, \partial U) = H_n(D^n, S^{n-1})$



$$\begin{array}{ll}
H_n(D^n) & 0 \\
\downarrow & \\
H_n(D^n, S^{n-1}) & \Rightarrow \mathbb{Z} \\
\downarrow & \\
H_{n-1}(S^{n-1}) & \mathbb{Z} \\
\downarrow & \\
H_{n-1}(D^n) & 0
\end{array}$$

Visto $H_n(M, M \setminus U) \cong \mathbb{Z}$

<u>Orientazione omologica</u>: $\forall x \in M \setminus \partial M$ scelta di un generatore per $H_n(M, M \setminus U)$ localmente costante

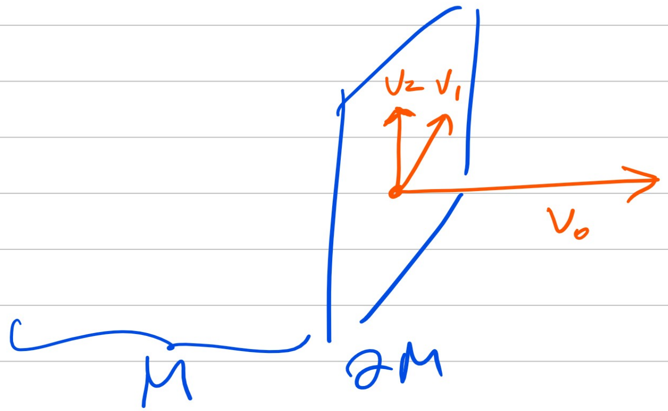


<u>Fatti</u>:

(1) M orientabile con $\partial M \Rightarrow \partial M$ orientabile:

"outer normal first"

diciamo v_1, v_2 pos se v_0, v_1, v_2 pos



(2) $M^{(n)}$ chiusa $\Rightarrow$ compatta senza bordo

$$\Rightarrow H_n(M) = \begin{cases} \mathbb{Z} & \text{se } M \text{ è orientabile} \\ 0 & \text{altrimenti} \end{cases}$$

(S^2, T)

$(K, \mathbb{P}^2)$

$B_n = 0$; $Z_n = ?$

Prendo triangolaz. di M con simplessi $S_1, \dots, S_p \cong \Delta_n$

$$c = \sum_{j=1}^{p} \alpha_j S_j \qquad \alpha_j \in \mathbb{Z}$$

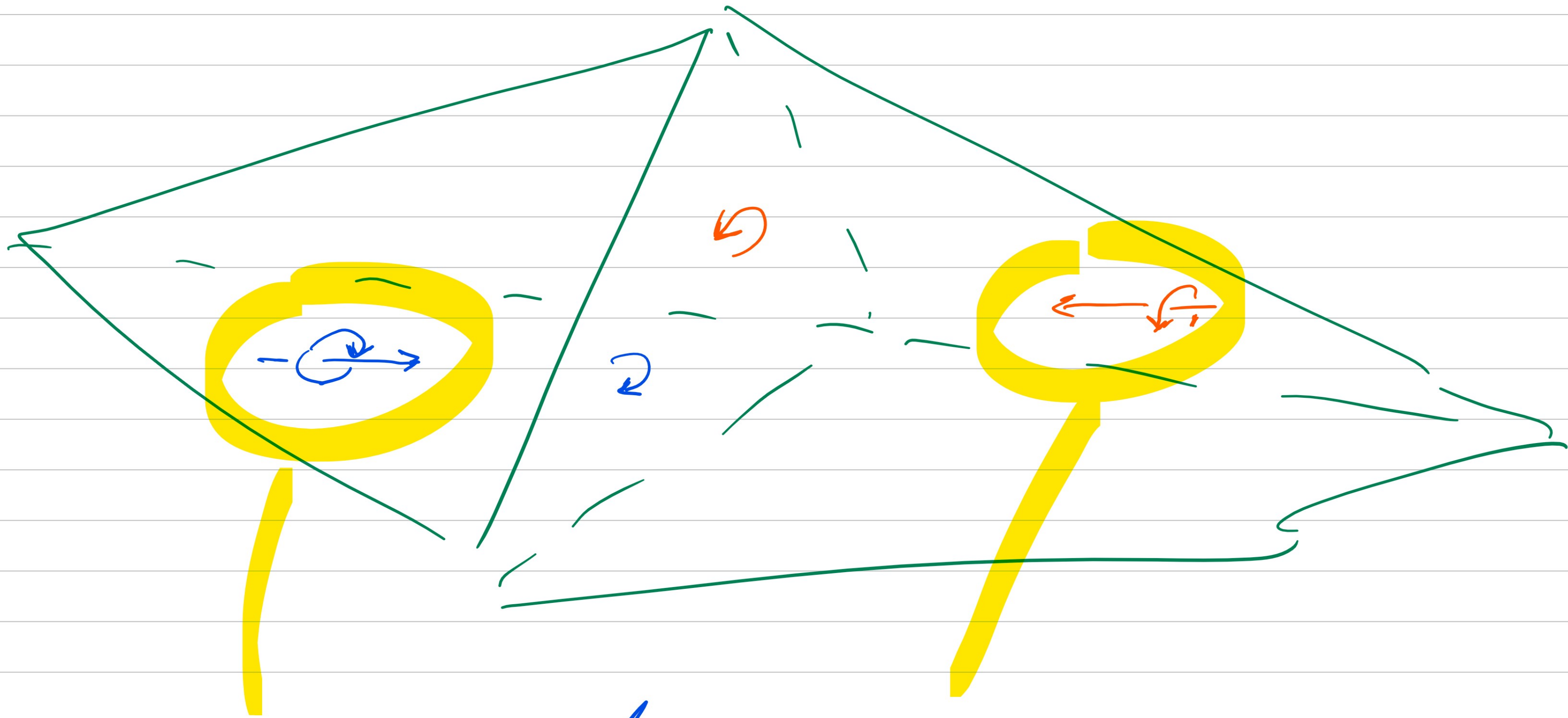
Per avere $\partial c = 0$

S_j orientazione prefissata a caso

devo avere coeff di T in $\partial c = 0$

$$\Rightarrow \alpha_{j_1} = \pm \alpha_{j_2} \, ; \quad \Rightarrow c = \alpha \cdot \sum_{j=1}^{p} (\pm S_j)$$

Riesco a realizzare c con $\partial c = 0 \Leftrightarrow$ posso riorientare i simplessi in modo che ogni triangolo riceva orientazioni opposte dai due incidenti



uguali

$\exists c$ t.c. $\partial c = 0 \iff$ posso orientare gli Σ_j in modo coerente

$\iff$ posso orientare M.