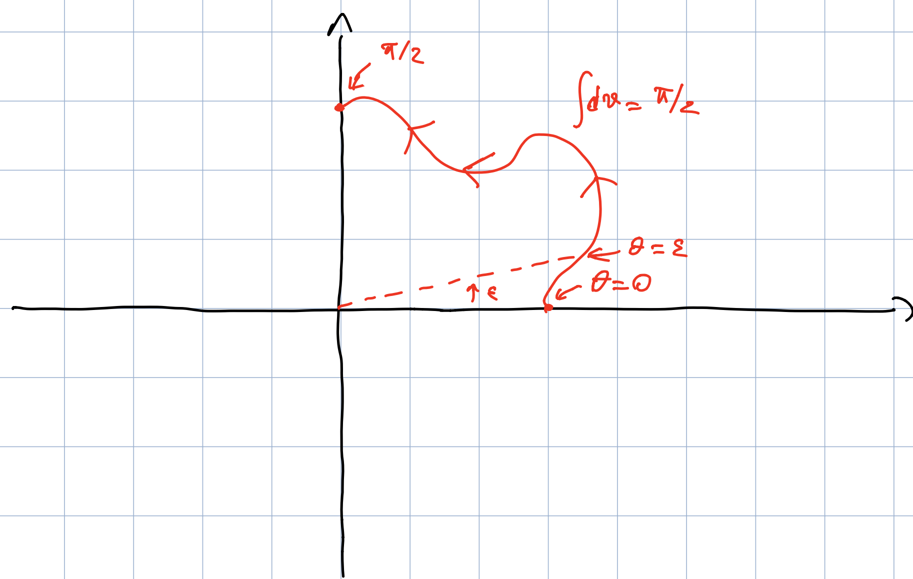
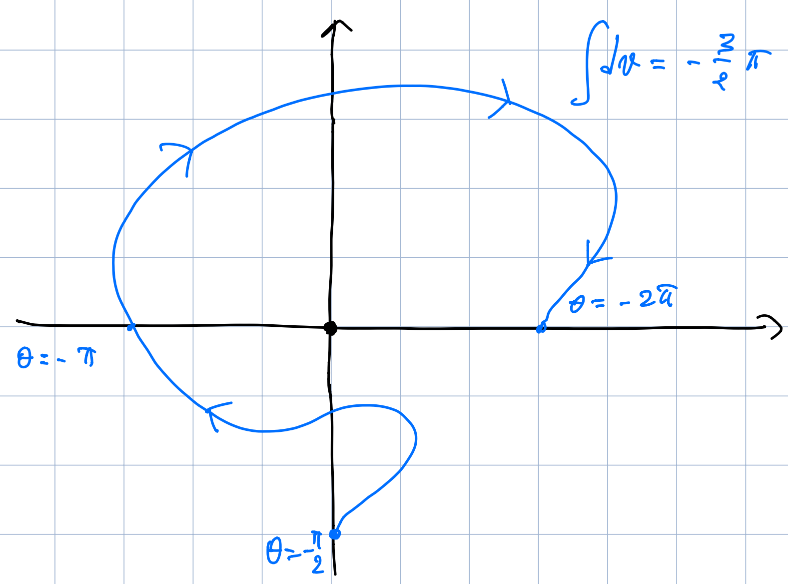
Geometria 20/5/20

in $\mathbb{R}^2 \setminus \left\{\begin{pmatrix}0\\0\end{pmatrix}\right\}$ $\qquad d\vartheta = \dfrac{-y\,dx + x\,dy}{x^2+y^2}$ $\qquad$ "$\int_\alpha d\vartheta = \theta(\alpha(b)) - \theta(\alpha(a))$"





<u>Thm</u> (Gauss-Green): $\Omega \subset \mathbb{R}^2$ aperto limitato con $\partial\Omega = \bigcup$ curve ; ω 1-forma su aperto che contiene $\Omega \cup \partial\Omega = \overline{\Omega}$ (chiusura di Ω)

$$\Longrightarrow \int_{\Omega} d\omega = \int_{\partial\Omega} \omega .$$

$\omega = f\,dx + g\,dy$

$d\omega = \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dx \wedge dy$

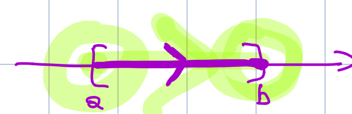
TFCI: $\int_a^b F'(t)\,dt = F(b) - F(a)$

$\partial\,[a,b] = \{a,b\}$

$F'(t)\,dt = d\,F(t)$

$\partial\,\overrightarrow{[a,b]} = \{ b^{(+)}, a^{(-)} \}$

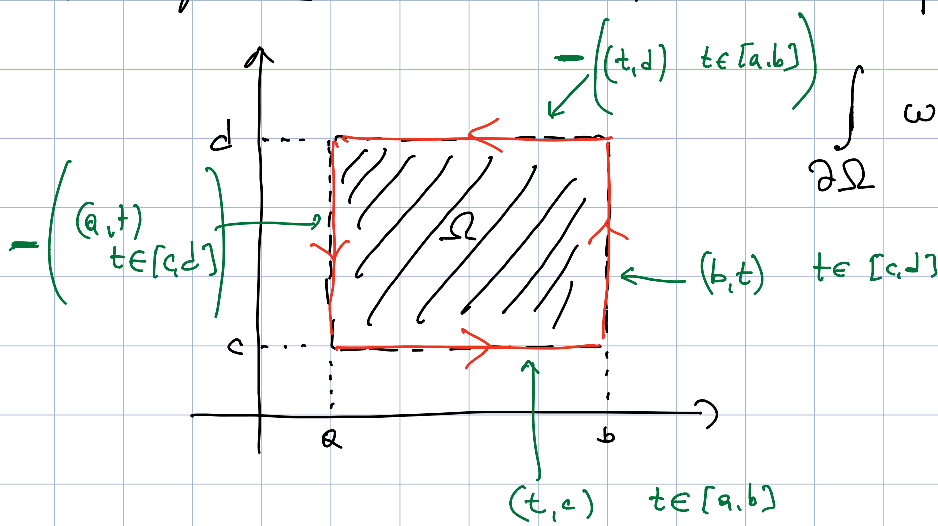
$$\int_{\overrightarrow{[a,b]}} dF = F(b) - F(a) = F(b^{(+)}) + F(a^{(-)}) = \int_{\partial \overrightarrow{[a,b]}} F$$



Dim per $\Omega = [a,b] \times [c,d]$.

$\omega = f\,dx + g\,dy$

$$\int_{\partial\Omega} \omega \overset{?}{=} \int_{\Omega} d\omega$$



$$\int_{\partial\Omega}\omega = \int_a^b \big(f(t,c)\cdot 1 + g(t,c)\cdot 0\big)\,dt + \int_c^d \big(f(b,t)\cdot 0 + g(b,t)\cdot 1\big)\,dt$$

$$- \int_a^b \big(f(t,d)\cdot 1 + g(t,d)\cdot 0\big)\,dt - \int_c^d \big(f(a,t)\cdot 0 + g(a,t)\cdot 1\big)\,dt$$

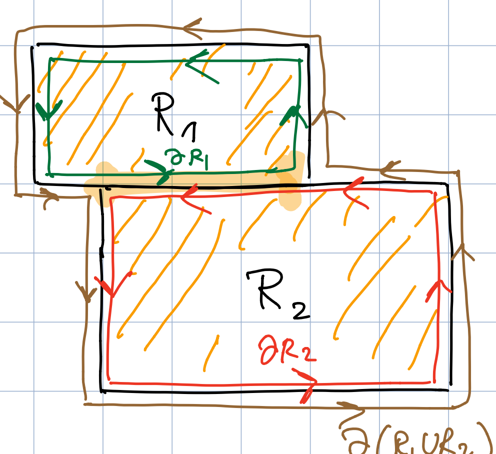
$$= \underbrace{\boxed{\int_a^b f(t,c)\,dt}}_{I} + \underbrace{\boxed{\int_c^d g(b,t)\,dt}}_{II} \underbrace{\boxed{- \int_a^b f(t,d)\,dt}}_{III} \underbrace{\boxed{- \int_c^d g(a,t)\,dt}}_{IV}$$

$$\int_\Omega d\omega = \int_a^b \int_c^d \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dx\,dy = \int_c^d \left(\boxed{\int_a^b \frac{\partial g}{\partial x}\,dx}\right) dy - \int_a^b \left(\boxed{\int_c^d \frac{\partial f}{\partial y}\,dy}\right) dx$$

$$= \int_c^d g(x,y)\Big|_{x=a}^{x=b} dy - \int_a^b f(x,y)\Big|_{y=c}^{y=d} dx$$

$$= \underbrace{\boxed{\int_c^d g(b,y)\,dy}}_{II} \underbrace{\boxed{- \int_c^d g(a,y)\,dy}}_{IV} \underbrace{\boxed{- \int_a^b f(x,d)\,dx}}_{III} + \underbrace{\boxed{\int_a^b f(x,c)\,dx}}_{I} \qquad \blacksquare$$

E se Ω non è un rettangolo? Se è $R_1 \cup R_2$:

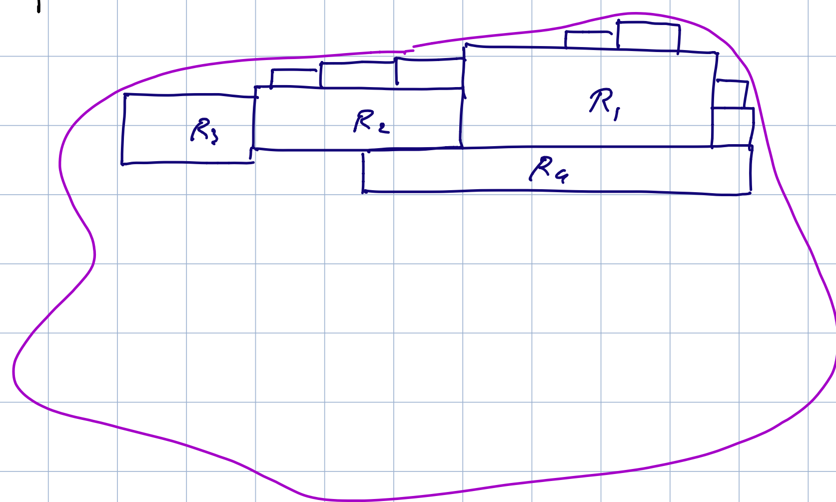


$$\underbrace{\int_{R_1 \cup R_2} d\omega}_{\text{ovvio}} \overset{?}{=} \underbrace{\int_{\partial(R_1 \cup R_2)} \omega}_{\parallel *}$$

$$\int_{R_1} d\omega + \int_{R_2} d\omega \underset{\text{visto}}{=} \int_{\partial R_1} \omega + \int_{\partial R_2} \omega$$

* perché gli integrali sui tratti condivisi si cancellano OK

Ω qualsiasi:



approssimo Ω con $\bigcup_{i=1}^{N} R_i$ che si toccano solo lungo ∂.

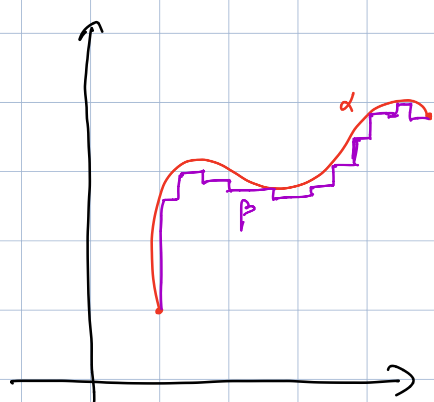
* estendendo dal caso $N=2$

$$\int_{\cup R_i} d\omega \overset{*}{=} \int_{\partial(\cup R_i)} \omega$$

OK — migliorando approx — non ovvio *

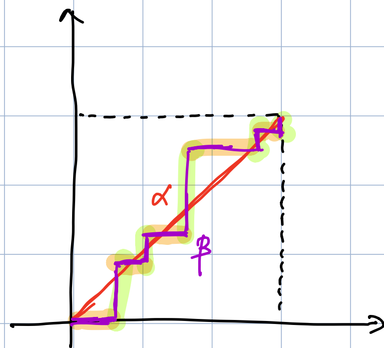
$$\int_{\Omega} d\omega \qquad \int_{\partial\Omega} \omega$$

*



$\int_\beta \omega \longrightarrow \int_\alpha \omega$ **VERO**

$\int_\beta F \longrightarrow \int_\alpha F$ **FALSO**



$$L(\alpha) = \int_\alpha 1 = \sqrt{2}$$

$$L(\beta) = \int_\beta 1 = 2$$

$\omega = f \cdot dx + g \cdot dy$ su Ω

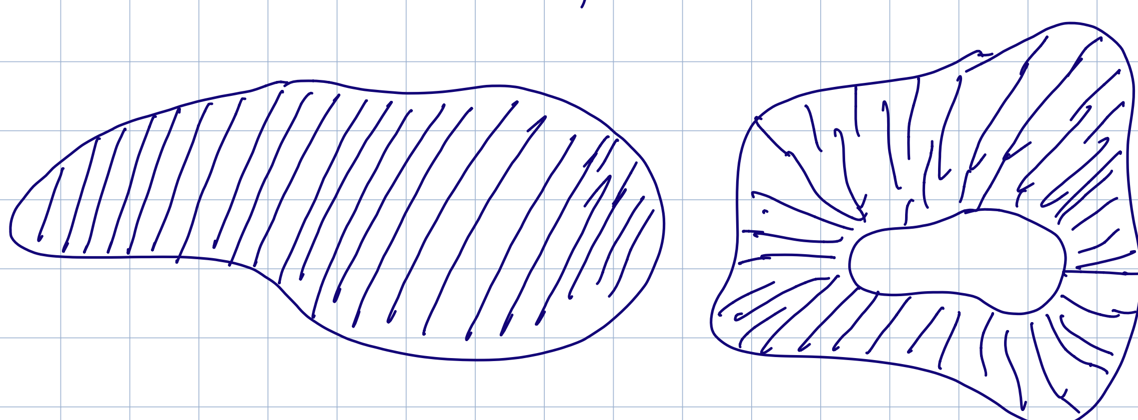
esatta se $\omega = dF = \frac{\partial F}{\partial x} dx + \frac{\partial F}{\partial y} dy$

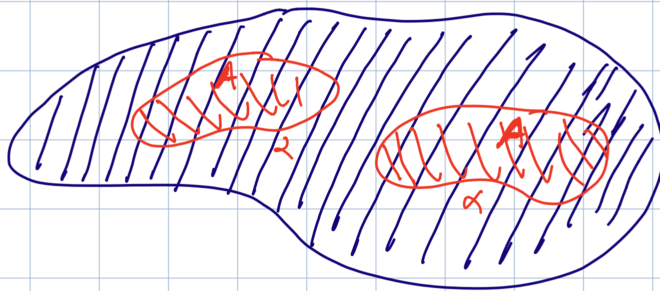
chiusa se $\frac{\partial g}{\partial x} = \frac{\partial f}{\partial y}$

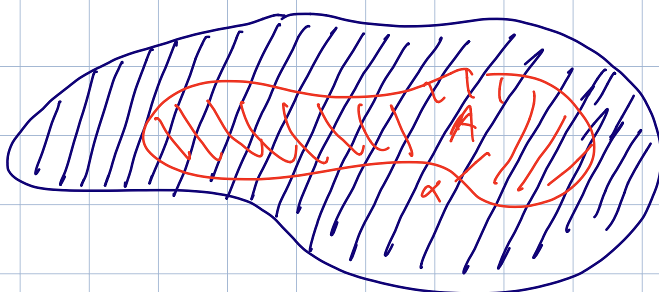
esatta $\Rightarrow$ chiusa. Viceversa <u>non sempre</u> ($d\theta$).

G-G: Se $\bar{A} \subset \Omega$ ho $\int_A d\omega = \int_{\partial A} \omega$.

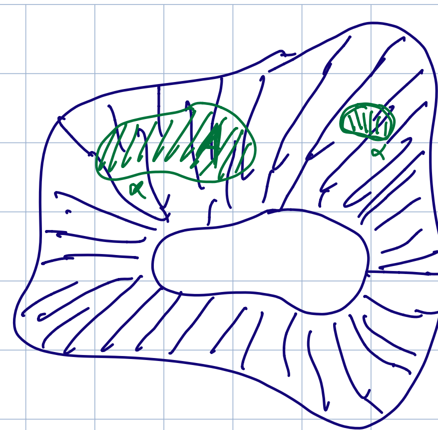
<u>Def</u>: dico Ω semplicemente connesso ("senza buchi") se per ogni curva (semplice) chiusa α in Ω, se $A \subset \mathbb{R}^2$ è l'insieme t.c. $\partial A = \alpha$, ho che $A \subset \Omega$.

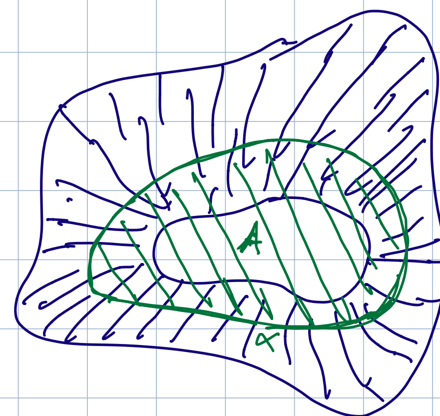






semplicemente connesso

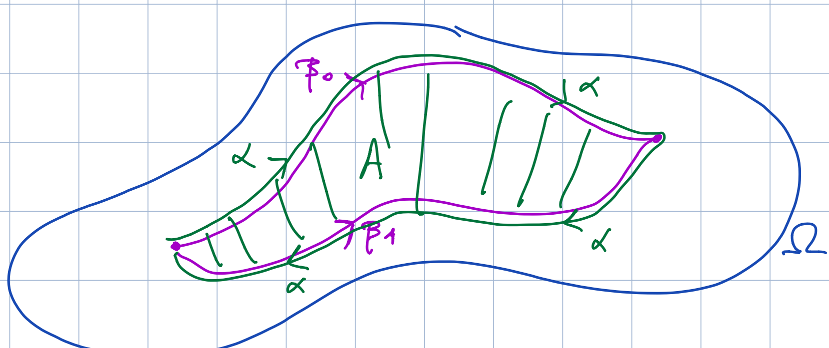




Non semplicemente connesso

Prop: se Ω è semplicemente connesso e ω è 1-forma su Ω chiusa allora è anche esatta.

"Dim": Sappiamo: ω esatta $\Longleftrightarrow$ $\int_\beta \omega$ dipende solo da estremi di β $\forall \beta$



$\alpha = \beta_0 \cup \beta_1^{(-)}$

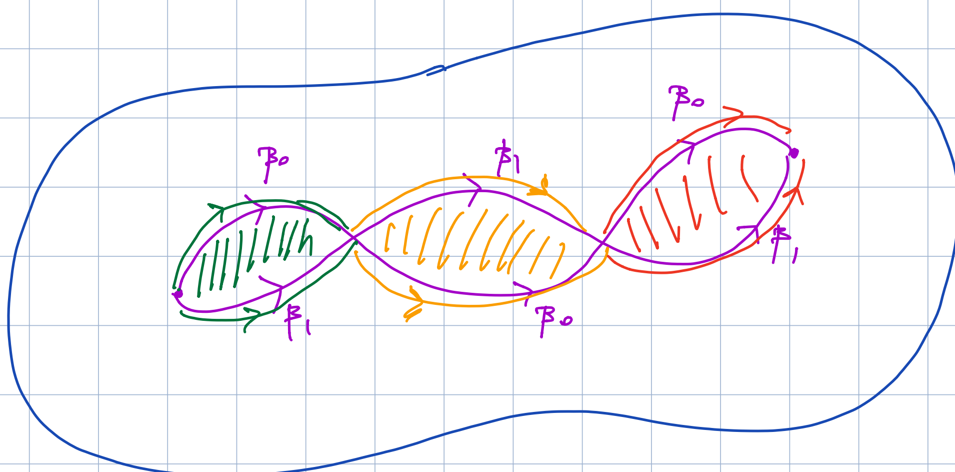
$\Rightarrow$ ha senso $\int_A d\omega = \int_A 0 = 0$

$$\shortparallel$$

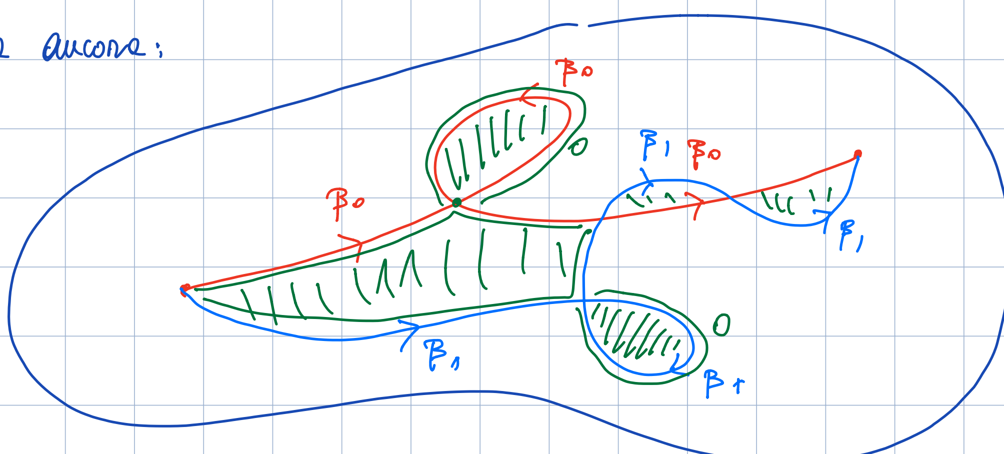
$$\int_{\partial A} \omega = \int_\alpha \omega = \int_{\beta_0 \cup \beta_1^{(-)}} \omega = \int_{\beta_0} \omega - \int_{\beta_1} \omega$$

$$\Rightarrow \int_{\beta_0} \omega = \int_{\beta_1} \omega \qquad \underline{OK}!$$

In realtà:



Ma ancora:



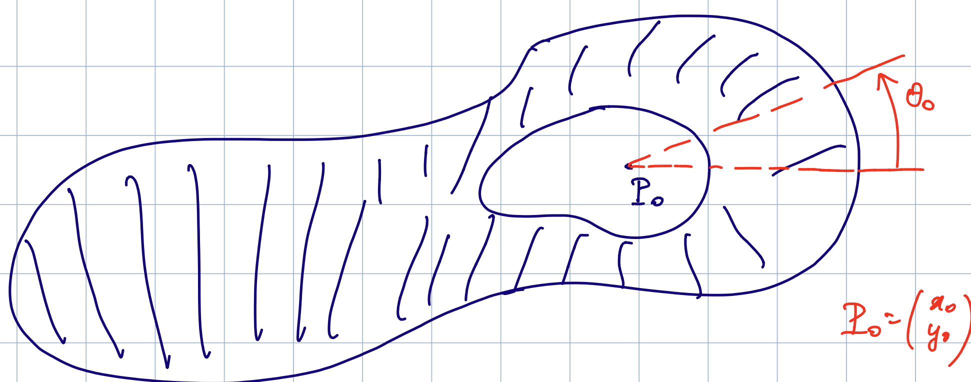
$\boxed{/\!/}$

Ω qualsiasi : esatte $\Longrightarrow$ chiuse

Ω sempl. conn: chiuse $\Longrightarrow$ esatte

Ω non sempl. conn : **alcune** forme chiuse su Ω non sono esatte

ma altre lo so



$$d\theta_0 = \frac{-(y-y_0)\,dx + (x-x_0)\,dy}{(x-x_0)^2 + (y-y_0)^2}$$

$\boxed{14.1.5}$ $\displaystyle\int_\alpha xy^2$ $\qquad \alpha : [0, \pi/2] \to \mathbb{R}^2 \qquad \alpha(t) = \begin{pmatrix} \cos(t) \\ \sin(t) \end{pmatrix}$

non ci sono dx/dy

$\Longrightarrow$ non è 1-forma, è funzione

$\longrightarrow$ ci vuole $\|\alpha'(t)\|$

$$\int_\alpha xy^2 = \int_0^{\pi/2} \cos(t)\cdot\sin^2(t)\cdot \underbrace{\sqrt{(-\sin(t))^2 + (\cos(t))^2}}_{\|\alpha'(t)\| \equiv 1}\cdot dt$$

$$= \int_0^{\pi/2} \cos(t)\cdot\sin^2(t)\,dt = \frac{1}{3}\sin^3(t)\Big|_0^{\pi/2} = \frac{1}{3}$$

$\boxed{14.1.6}$ $\displaystyle\int_\alpha \sqrt{12+x}$ $\alpha : [0,1] \to \mathbb{R}^2$ $\alpha(t) = \begin{pmatrix} 3t^2 \\ 1+t^3 \end{pmatrix}$

no $dx/dy \Longrightarrow$ serve $\|\alpha'(t)\|$

$$\int_0^1 \sqrt{12+3t^2} \cdot \sqrt{(6t)^2+(3t^2)}\, dt = \int_0^1 \sqrt{(12+3t^2)(36t^2+9t^4)}\, dt$$

$$= \int_0^1 \sqrt{3(4+t^2)\cdot 9t^2 \cdot (4+t^2)}\, dt = \int_0^1 3\sqrt{3}\, t(4+t^2)\, dt$$

$$= 3\sqrt{3}\left(2t^2 + \frac{1}{4}t^4\right)\Big|_0^1 = \frac{27}{4}\sqrt{3}$$

$\boxed{14.1.7}$ $\displaystyle\int_\alpha \sqrt{1+y^2}$ $\alpha : [0,1] \to \mathbb{R}^2$ $\alpha(t) = \begin{pmatrix} t \\ e^t \end{pmatrix}$

$$\int_0^1 \sqrt{1+e^{2t}} \cdot \sqrt{1+e^{2t}}\, dt = \int_0^1 (1+e^{2t})\, dt = \dots$$

$\boxed{14.1.9}$ $\displaystyle\int_\alpha xy$ $\alpha : [0, \frac{\pi}{2}] \to \mathbb{R}^2$ $\alpha(t) = \begin{pmatrix} \sin(t) \\ 2\cos(t) \end{pmatrix}$

$$\int_0^{\pi/2} 2\cdot \sin(t)\cos(t) \cdot \sqrt{\cos^2(t) + 4\sin^2(t)}\, dt$$

$$= \int_0^{\pi/2} \sin(2t) \cdot \sqrt{1+3\sin^2(t)}\, dt$$

$$= \int_0^{\pi/2} \sin(2t) \cdot \sqrt{1+\frac{3}{2}(1-\cos(2t))}\, dt$$

$\cos(2t) = \cos^2(t) - \sin^2(t)$

$= 1 - 2\sin^2(t)$

$\sin^2(t) = \frac{1}{2}(1-\cos(2t))$

$$= \int_0^{\pi/2} \sin(2t) \sqrt{\frac{5}{2} - \frac{3}{2}\cos(2t)}\, dt$$

$$= \frac{2}{3} \cdot \frac{2}{3} \cdot \frac{1}{2} \cdot \left(\frac{5}{2} - \frac{3}{2}\cos(2t)\right)^{\frac{3}{2}} \Bigg|_0^{\pi/2} = \frac{2}{9} \cdot (8-1) = \frac{14}{9}$$

$\boxed{14.1.1}$ $\quad C_1 = \left\{ \binom{x}{y} \in \mathbb{R}^2 : e^{3x-y} + \sin(x+2y) = 1 \right\}$

Provare che C_1 è una curva vicino a $\binom{0}{0}$ e trovare tangente in $\binom{0}{0}$.

$f(x,y) = e^{3x-y} + \sin(x+2y) - 1 \qquad C_1 = \{ f = 0 \}$

$f(0,0) = 0$ ✓

$$\frac{\partial f}{\partial x}(0,0) = 3 \cdot e^0 + \cos(0) = 3$$

$$\frac{\partial f}{\partial y}(0,0) = -e^0 + 2 \cdot \cos(0) = 1$$

$\text{grad} f(0,0) \neq 0 \Rightarrow$ ok, curva.

Tangente: $\quad 3x + y = 0$.

$\boxed{14.1.12}$ $\quad C_1 = \left\{ \binom{x}{y} \in \mathbb{R}^2 : 2xy^2 + 3x^2y = 1 \right\}$

Trovare i pti di C_1 vicino a cui C_1 è curva ed ha il vettore $\binom{-1}{4}$ come tangente.

$$\begin{cases} 2xy^2 + 3x^2y = 1 \\ \left\langle \begin{pmatrix} -1 \\ 4 \end{pmatrix} \middle| \begin{pmatrix} 2y^2+6xy \\ 4xy+3x^2 \end{pmatrix} \right\rangle = 0 \end{cases}$$

$$-2y^2 - 6xy + 16xy + 12x^2 = 0$$

$$y^2 - 5xy - 6x^2 = 0$$

$$(y-6x)(y+x) = 0$$

$$\begin{cases} y = -x \\ 2x^3 - 3x^3 = 1 \end{cases} \Rightarrow \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$\begin{cases} y = 6x \\ (72+18)x^3 = 1 \end{cases} \Rightarrow \begin{pmatrix} 1/\sqrt[3]{90} \\ 6/\sqrt[3]{90} \end{pmatrix}$$

$\boxed{14.2.4}$ calcolare curvatura con segno di α in t_0

(b) $\alpha(t) = \begin{pmatrix} 4t - t^5 \\ 4t^2 + t^4 \end{pmatrix}$ $t_0 = 2$.

$$\kappa(t) = \frac{\det(\alpha'(t)\ \alpha''(t))}{\|\alpha'(t)\|^3}$$

$$\alpha'(t) = \begin{pmatrix} 4 - 5t^4 \\ 8t + 4t^3 \end{pmatrix} \qquad \alpha''(t) = \begin{pmatrix} -20t^3 \\ 8 + 12t^2 \end{pmatrix}$$

$$\alpha'(2) = \begin{pmatrix} -76 \\ 48 \end{pmatrix} \qquad \alpha''(2) = \begin{pmatrix} -160 \\ 56 \end{pmatrix}$$

$$\kappa(2) = \frac{\det\begin{pmatrix} -76 & -160 \\ 48 & 56 \end{pmatrix}}{(76^2 + 48^2)^{3/2}} = \dots$$

(d) $\alpha(t) = \begin{pmatrix} \sin(3t) + t^2 \\ t - e^{-4t} \end{pmatrix} \qquad t_0 = 0$

$$\alpha'(t) = \begin{pmatrix} 3\cos(3t) + 2t \\ 1 + 4e^{-4t} \end{pmatrix} \qquad \alpha''(t) = \begin{pmatrix} -9\sin(3t) + 2 \\ -16e^{-4t} \end{pmatrix}$$

$$\alpha'(0) = \begin{pmatrix} 3 \\ 5 \end{pmatrix} \qquad \alpha''(0) = \begin{pmatrix} 2 \\ -16 \end{pmatrix} = \cancel{2} \cdot \begin{pmatrix} 1 \\ -8 \end{pmatrix}$$

NO

$$\kappa(0) = \frac{\det\begin{pmatrix} 3 & 2 \\ 5 & -16 \end{pmatrix}}{\left(\sqrt{34}\right)^3} = -\frac{58}{\left(\sqrt{34}\right)^3}$$