Geometria 13/5/2020

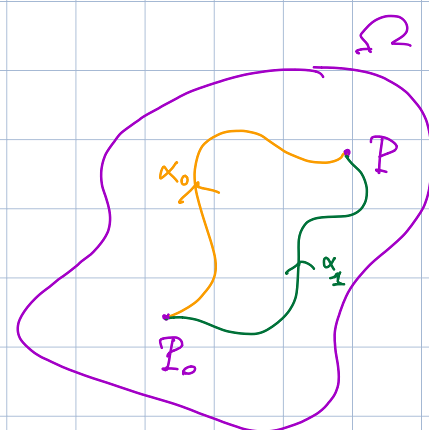
$\Omega \subset \mathbb{R}^2$ aperto; $\omega = f\,dx + g\,dy$ su Ω

ω esatta
$(\omega = dU)$
$\left(f = \frac{\partial U}{\partial x}, \; g = \frac{\partial U}{\partial y} \right)$?

Sì $\Longleftrightarrow$ $\boxed{\int_\alpha \omega \text{ dipende solo dagli estremi di } \alpha \; \forall \alpha : [a,b] \to \Omega}$

$\Longrightarrow \quad \int_\alpha \omega = U(\alpha(b)) - U(\alpha(a))$

$\Longleftarrow \quad U(P) = \int_\alpha \omega$ dove



Devo vedere che $\omega = dU$ cioè
$\boxed{f = \frac{\partial U}{\partial x}}$, $g = \frac{\partial U}{\partial y}$.

Devo vedere che

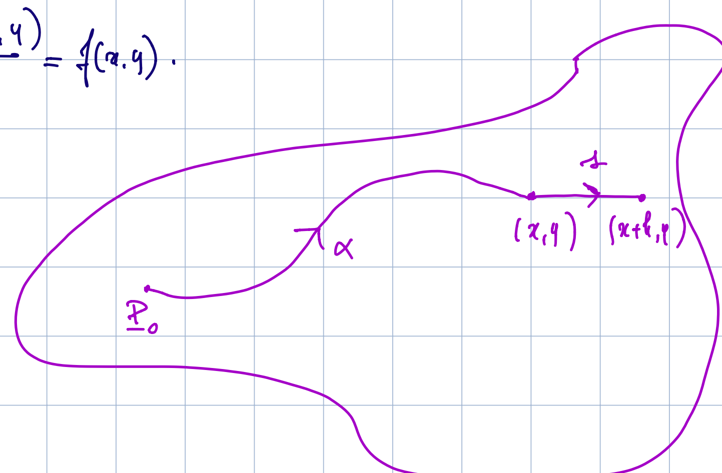
$$\lim_{h \to 0} \frac{U(x+h,y) - U(x,y)}{h} = f(x,y).$$

$$\| $$

$$\lim_{h \to 0} \frac{1}{h} \left(\int_{\alpha \cup s} \omega - \int_\alpha \omega \right)$$

$$\|$$

$$\lim_{h \to 0} \frac{1}{h} \left(\cancel{\int_\alpha \omega} + \int_s \omega - \cancel{\int_\alpha \omega} \right)$$



$$= \lim_{h\to 0} \frac{1}{h} \int_x^{x+h} \Big(f(t,y)\cdot 1 + g(t,y)\cdot 0 \Big)\, dt$$

$s(t) = (t,y)$
$x \le t \le x+h$
[modificare per $h<0$]

$$= \lim_{h\to 0} \frac{1}{h} \int_x^{x+h} f(t,y)\, dt = f(x,y)$$

teo fond. calcolo integrale.

Visto: ω esatta $(\omega = dU)$ $\Longleftrightarrow$ $\int_\alpha \omega$ dipende solo da estremi $\forall \alpha$

Anche: ω esatta $\Longleftrightarrow$ $\int_\beta \omega = 0$ $\forall \beta$ chiusa.

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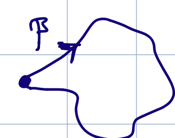
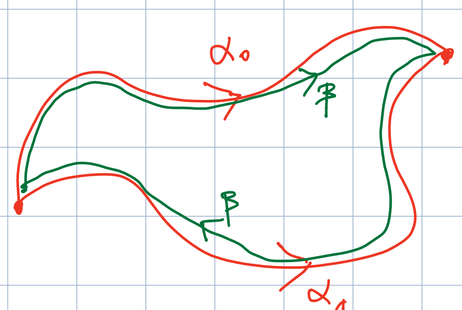
ω esatta $\Rightarrow$ $\int_\beta \omega = 0$ $\forall \beta$ chiusa : visto: $U(\beta(b)) - U(\beta(a)) = 0$



$\int_\beta \omega = 0$ $\forall \beta$ chiusa $\Rightarrow$ $\int_\alpha \omega$ dipende solo da estremi $\forall \alpha$

$\Downarrow$ infatti

date α_0, α_1 con stessi estremi ho:

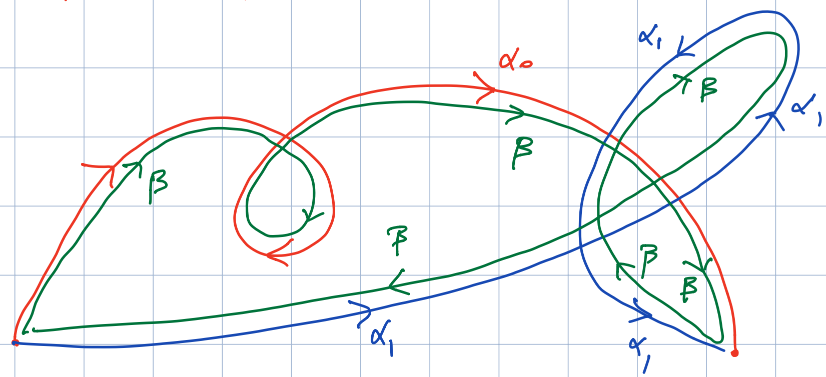


$\beta = \alpha_0 \cup (-\alpha_1)$ chiusa

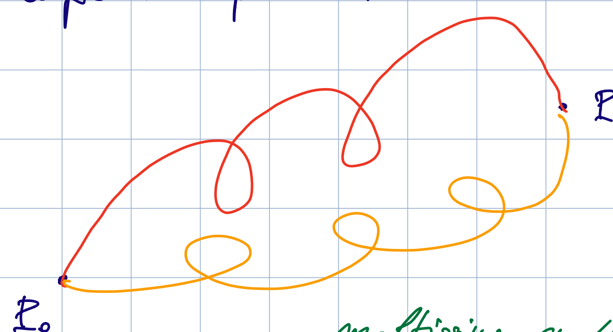
$$\int_\beta \omega = 0 \qquad \int_\beta \omega = \int_{\alpha_0} \omega + \int_{-\alpha_1} \omega = \int_{\alpha_0} \omega - \int_{\alpha_1} \omega$$

$$\Longrightarrow \int_{\alpha_0} \omega = \int_{\alpha_1} \omega \ . \quad \square$$

Oss: vero anche se:



✱ : impossibile da verificare in pratica:



moltissime curve...

Q: come vedere in pratica se ω è esatta ($\omega = dU$).

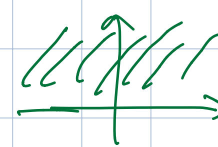
$$\boxed{\text{SEMPRE} \quad \omega = f\,dx + g\,dy \quad \text{su } \Omega}$$

Visto: ω esatta $\Longrightarrow$ ω chiusa

$(\omega = dU)$ $\qquad \dfrac{\partial g}{\partial x} = \dfrac{\partial f}{\partial y}$

<u>Esempio</u>: $\log(1+x/y)\,dx + \sin(xy^2)\,dy$

su $\Omega = \{(x,y) : y>0\}$



esatta? chiusa?

$$\frac{\partial g}{\partial x} = \boxed{y^2 \cdot \cos(xy^2)}$$

$$\frac{\partial f}{\partial y} = \frac{1}{1+x/y} \cdot \left(-\frac{x}{y^2}\right) = \boxed{-\frac{x}{y(x+y)}}$$

<u>No</u> (chiusa) $\Rightarrow$ <u>No</u> (esatta)

Q: quali forme chiuse posso concludere che sono anche esatte?

A: (1) non tutte

(2) dipende da Ω

<u>Es</u>: $\Omega = \mathbb{R}^2 \setminus \{(0,0)\}$ $\quad \omega(x,y) = \dfrac{-y\,dx + x\,dy}{x^2+y^2}$

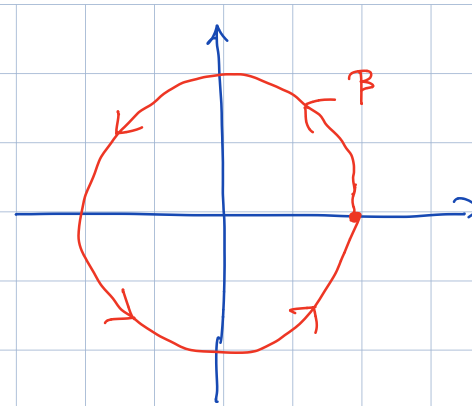
<u>chiusa</u>: $f(x,y) = -\dfrac{y}{x^2+y^2}$ $\quad g(x,y) = \dfrac{x}{x^2+y^2}$

$$\frac{\partial g}{\partial x} = \frac{1\cdot(x^2+y^2) - x\cdot(2x)}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$\frac{\partial f}{\partial y} = -\frac{1\cdot(x^2+y^2) - y\cdot(2y)}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2} \qquad \underline{\text{OK}}$$

<u>non esatta</u>: $\gamma(t) = (\cos(t), \sin(t))$ $\quad t \in [0, 2\pi]$

$$\int_\beta \frac{-y\,dx + x\,dy}{x^2+y^2}$$



$$= \int_0^{2\pi} \frac{-\sin(t)\cdot(-\sin(t)) + \cos(t)\cdot\cos(t)}{\cos^2(t)+\sin^2(t)}\,dt$$

$$= \int_0^{2\pi} dt = 2\pi$$ OK (per esatte $\int_{\text{chiuse}} = 0$)

$$\omega(x,y) = \frac{-y\,dx + x\,dy}{x^2+y^2}$$ perché chiusa? perché $\int_\beta = 2\pi$?

Coordinate polari : $(x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}$

$(x,y) = \rho\cdot(\cos(\theta), \sin(\theta))$ (ρ, θ) coord. polari

ρ = modulo $= \sqrt{x^2+y^2}$ θ = argomento.

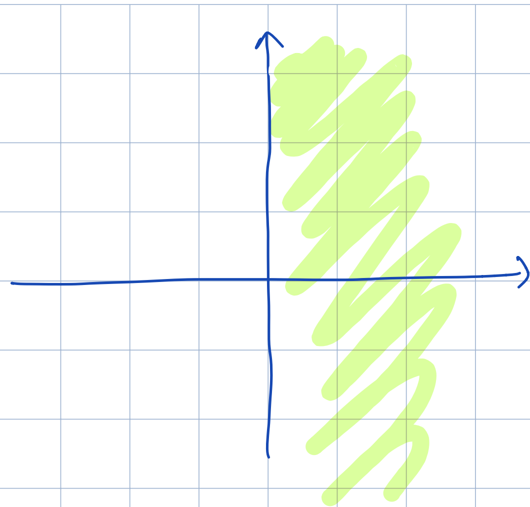
Però θ non è ben def : posso sommare $2k\pi$

θ è "l'angolo" con $\cos = \dfrac{x}{\rho}$ $\sin = \dfrac{y}{\rho}$

in particolare

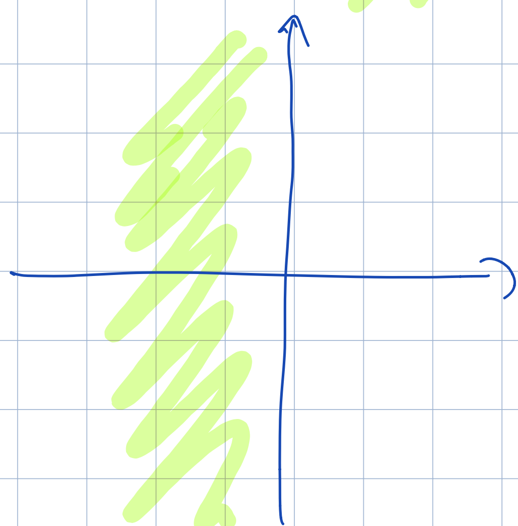
$$\tg = \frac{y}{x} \qquad \cotg = \frac{x}{y}$$

Dunque posso definire θ a pezzi :



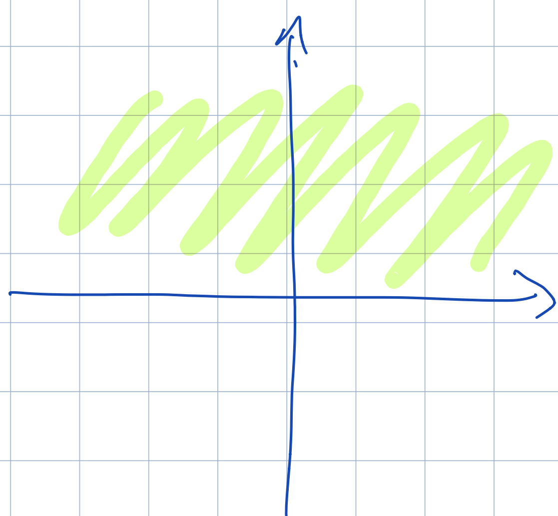
$$\boxed{\theta = \operatorname{arctg}\left(\frac{y}{x}\right) + 2k\pi}$$

su $\{x > 0\}$



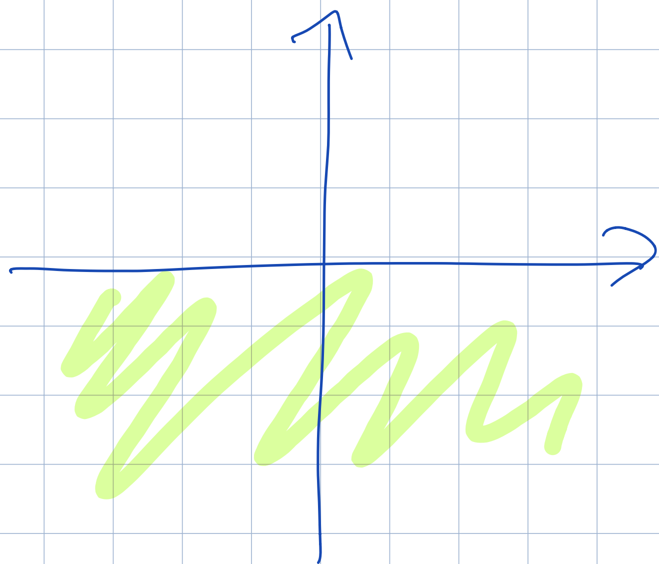
$$\boxed{\theta = \operatorname{arctg}\left(\frac{y}{x}\right) + \pi + 2k\pi}$$

su $\{x < 0\}$



$$\boxed{\theta = \operatorname{arccotg}\left(x/y\right) + 2k\pi}$$

su $\{y > 0\}$



$$\boxed{\theta = \mathrm{arccotg}\left(x/y\right) + \pi + 2k\pi}$$
in $\{y<0\}$

<u>Oss</u>: le diverse def. differiscono per costante
$\Rightarrow$ θ non ben def. ma $d\theta$ è ben definito

Verifico che $\dfrac{-y\,dx + x\,dy}{x^2+y^2}$ è proprio $d\theta$:

$\theta = \mathrm{arctg}\left(\dfrac{y}{x}\right)$

$\theta = \mathrm{arccotg}\left(\dfrac{x}{y}\right)$

$$\frac{d}{dt}\,\mathrm{arctg}(t) = \frac{1}{1+t^2}$$

$$\frac{d}{dt}\,\mathrm{arccotg}(t) = -\frac{1}{1+t^2}$$

$$d\theta = \frac{1}{1+\left(\frac{y}{x}\right)^2}\cdot\left(-\frac{y}{x^2}\right)dx + \frac{1}{1+\left(\frac{y}{x}\right)^2}\cdot\frac{1}{x}\,dy$$

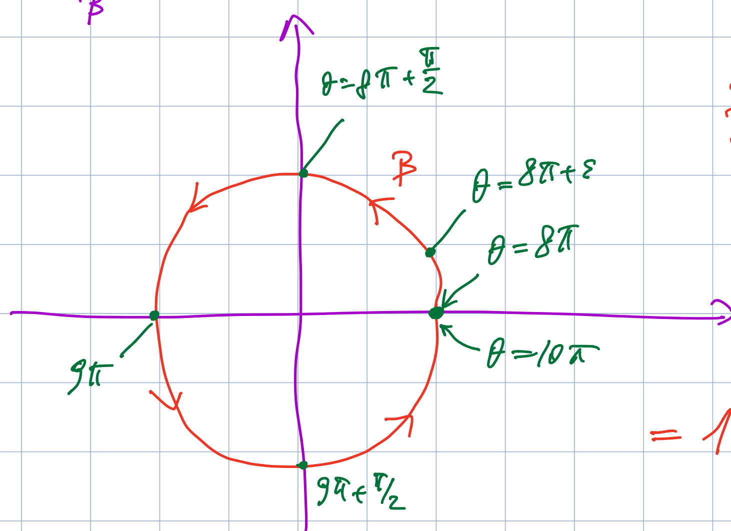
$$d\theta = -\frac{1}{1+\left(\frac{x}{y}\right)^2}\cdot\left(\frac{1}{y}\right)dx + \left(-\frac{1}{1+\left(x/y\right)^2}\right)\cdot\left(-\frac{x}{y^2}\right)dy$$

$$\frac{-y\,dx + x\,dy}{x^2+y^2}$$

Visto: $\dfrac{-y\,dx + x\,dy}{x^2+y^2} = d\theta$.

chiusa: visto se $dU = f\,dx + g\,dy$ allora $\dfrac{\partial g}{\partial x} = \dfrac{\partial f}{\partial y}$
$\Rightarrow dU$ chiusa
$\Rightarrow d\theta$ chiusa

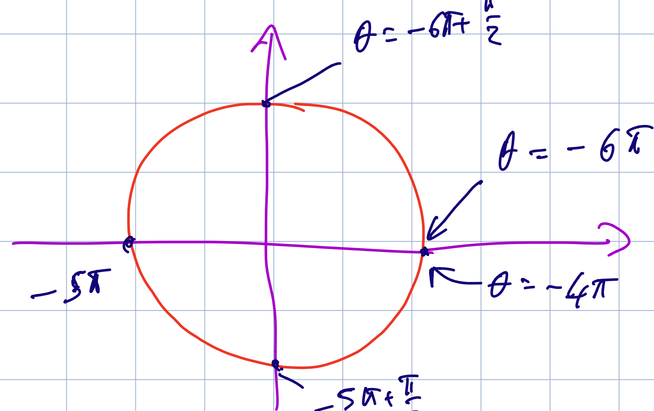
non esatta: $\displaystyle\int_\beta = 2\pi$.



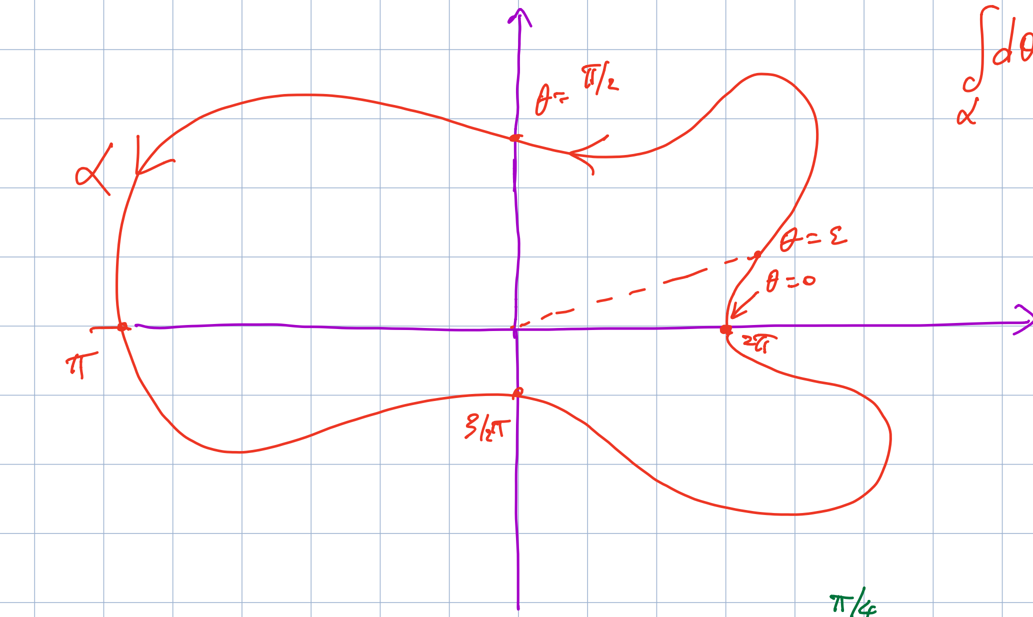
$\displaystyle\int_\beta d\theta = \theta(2\pi) - \theta(0)$

se ho scelto θ continuo su $[0, 2\pi]$

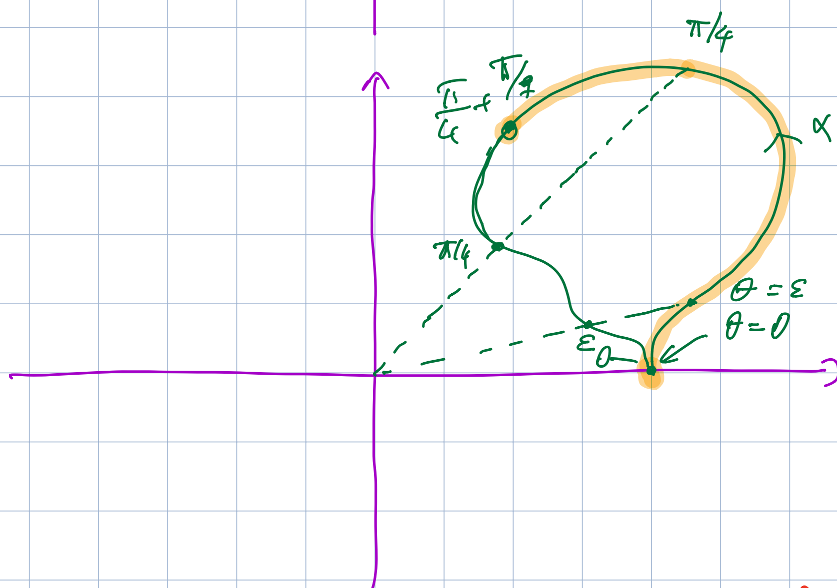
$= 10\pi - 8\pi = 2\pi$



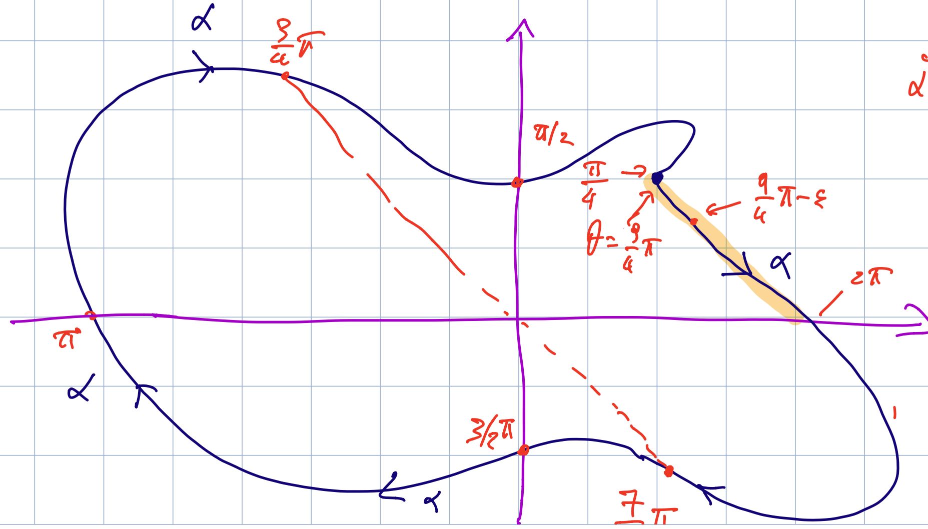
$\displaystyle\int = -4\pi - (-6\pi)$
$= 2\pi$



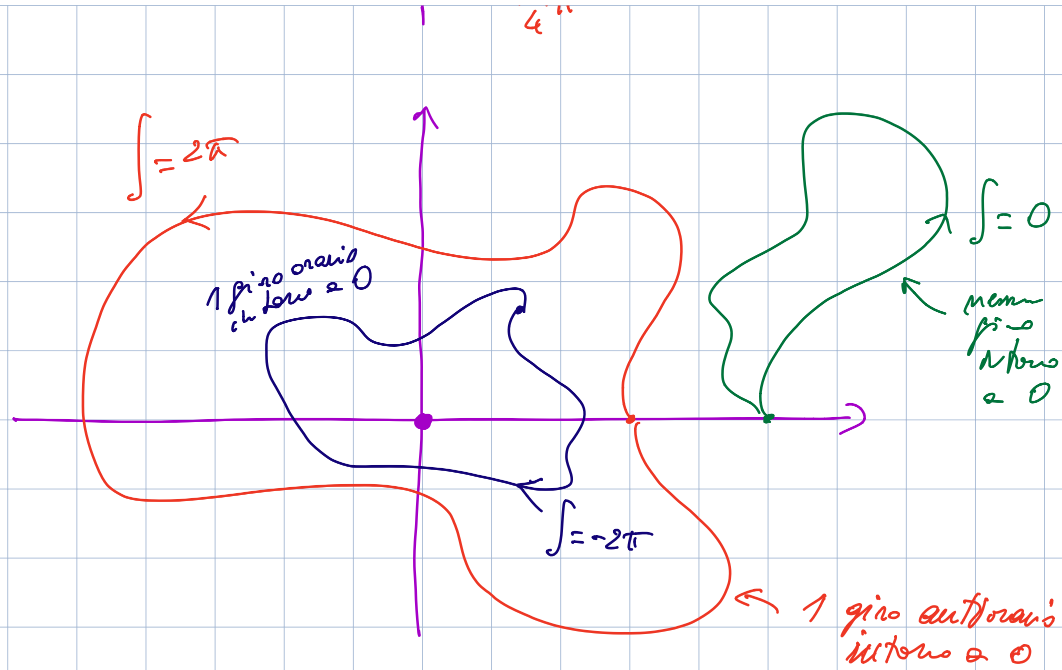
$$\int_\alpha d\theta = 2\pi$$



$$\int_\alpha d\theta = 0$$

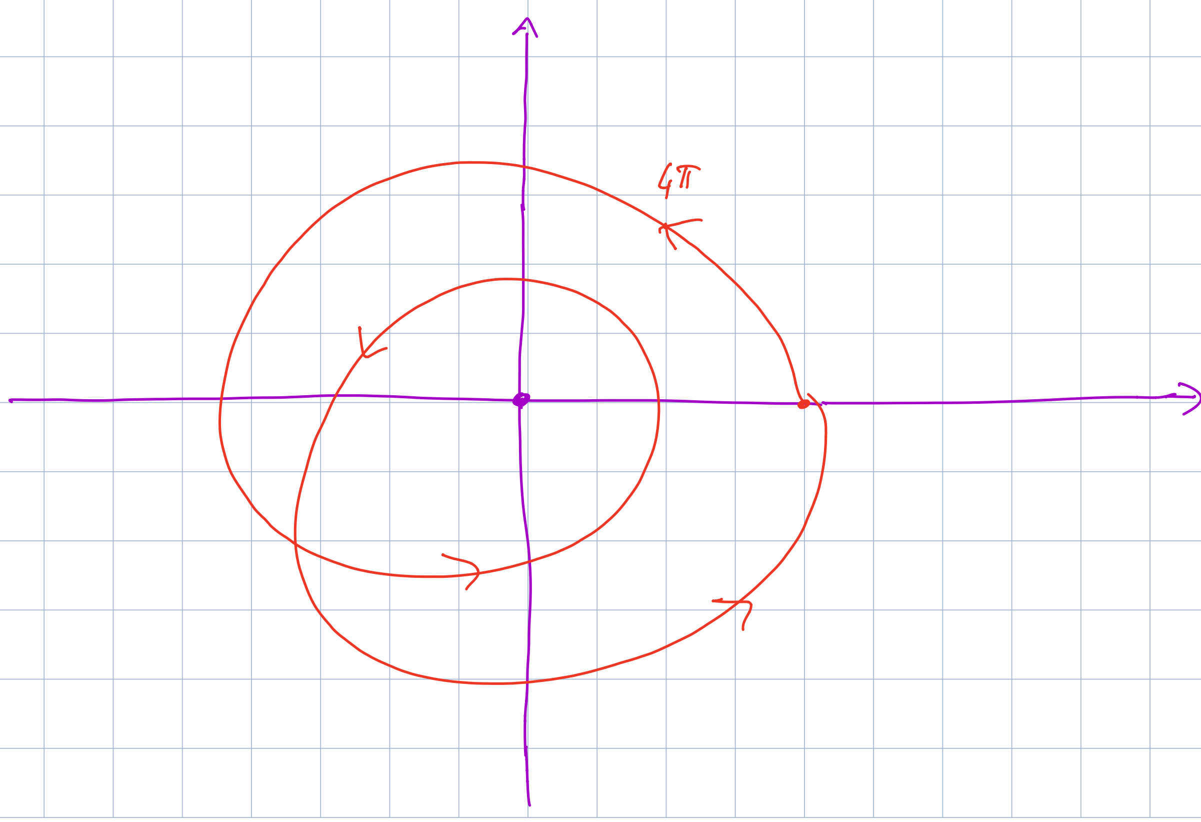


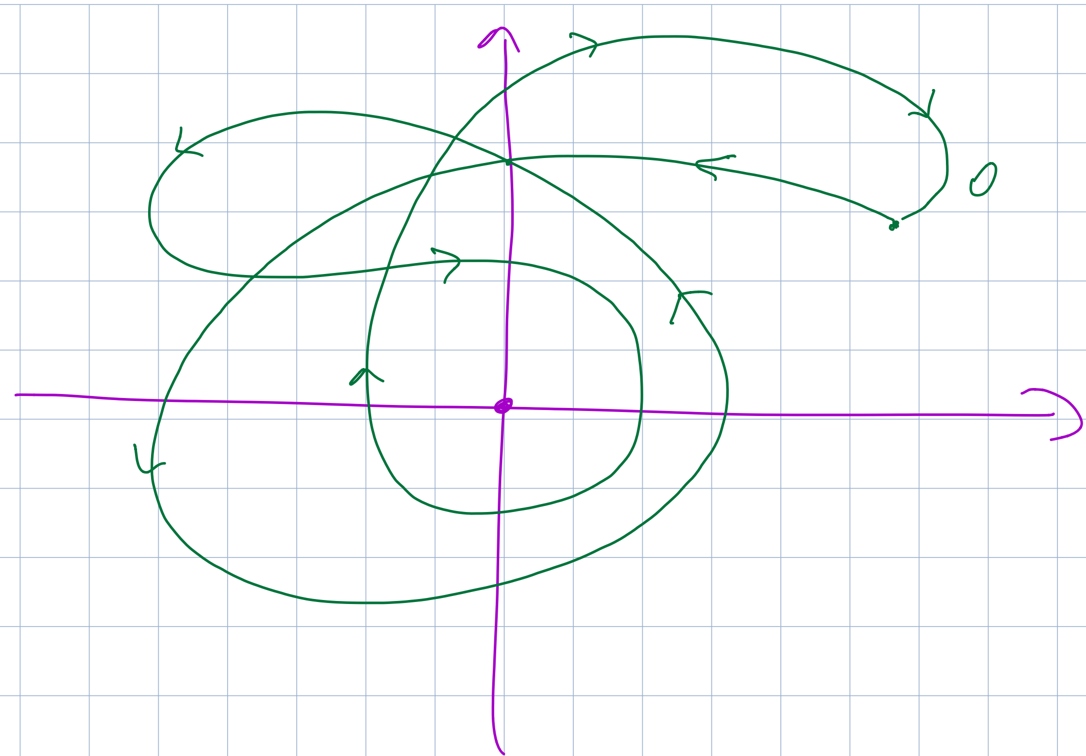
$$\int_\alpha d\vartheta = \frac{\pi}{4} - \frac{9}{4}\pi$$
$$= -2\pi$$

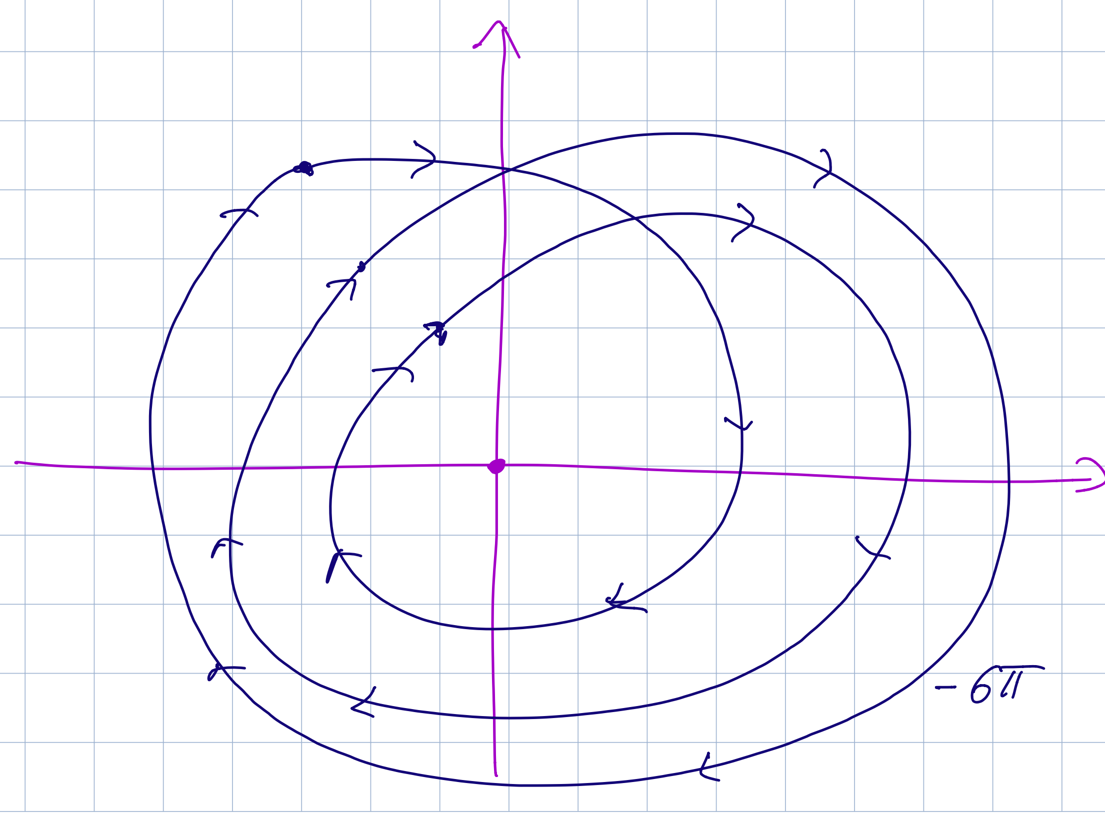


In generale:

$$\int_\alpha \frac{-y\,dx + x\,dy}{x^2+y^2} = 2\pi \cdot \left(\begin{array}{l}\text{numero algebrico}\\ \text{di giri che } \alpha\\ \text{fa intorno a } 0\end{array}\right)$$




0


-6π

ω esatta $(\omega = dU)$ $\Longrightarrow$ ω chiusa $(\partial g/\partial x = \partial f/\partial y)$

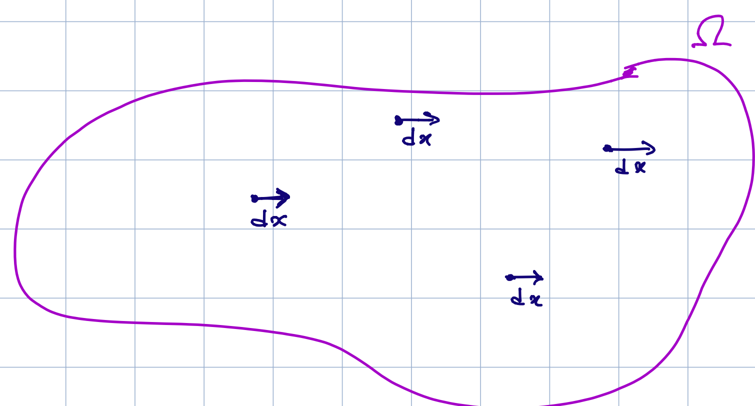
$\Longleftarrow$ non sempre

$U \rightsquigarrow$ differenziale $\boxed{dU = \frac{\partial U}{\partial x}\cdot dx + \frac{\partial U}{\partial y}\cdot dy}$.

data $\omega = f\cdot dx + g\cdot dy$ proviamo a definire $d\omega$:

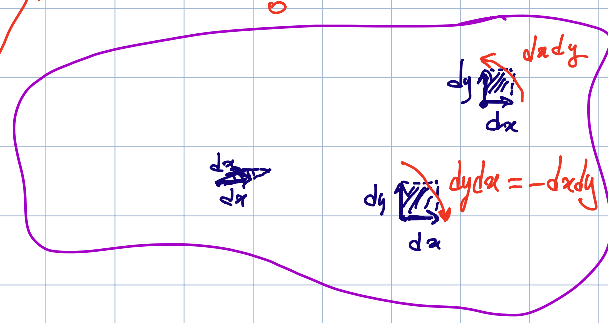
differenziale $d\omega = d(f\cdot dx + g\cdot dy)$

$$= \underbrace{df}_{\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy}\cdot dx + f\cdot \underbrace{ddx}_{=0} + \underbrace{dg}_{\frac{\partial g}{\partial x}dx + \frac{\partial g}{\partial y}dy}\cdot dy + g\cdot \underbrace{ddy}_{=0}$$



$$= \frac{\partial f}{\partial x} \underbrace{dx\,dx}_{=0} + \frac{\partial f}{\partial y} \underbrace{dy\,dx} + \frac{\partial g}{\partial x} \cdot \underbrace{dx\,dy} + \frac{\partial g}{\partial y} \cdot \underbrace{dy\,dy}_{=0}$$

elemento d'area



$$= \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dx\,dy$$

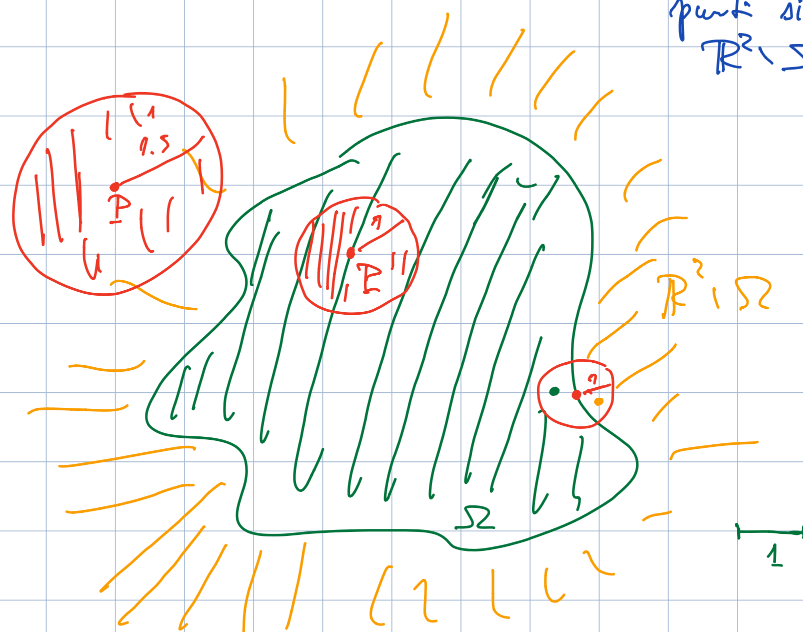
Def: $$\underbrace{d}_{\text{differenziale}}\big(\underbrace{f\,dx + g\,dy}_{\text{1-forma}}\big) = \underbrace{\underbrace{\boxed{\left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right)}}_{\text{funz. scalare}} \underbrace{dx\,dy}_{\text{el. d'area}}}_{\text{si integra su } \Omega \subset \mathbb{R}^2}$$

Ricordo: $\omega = f\,dx + g\,dy$ è chiusa se $\boxed{\frac{\partial g}{\partial x} = \frac{\partial f}{\partial y}}$

Dunque: ω chiusa $\iff d\omega = 0$.

Def: $$d(f\,dx + g\,dy) = \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dx\,dy$$

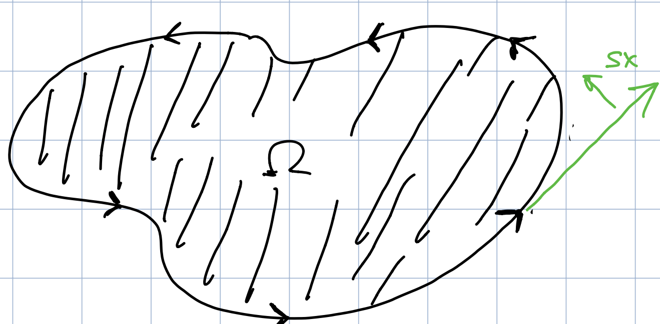
Bordo di $\Omega \subset \mathbb{R}^2$: $\partial\Omega = \{P :\ \forall r>0$ esistono punti sia di Ω sia di $\mathbb{R}^2 \setminus \Omega$ a distanza da P meno di $r\}$

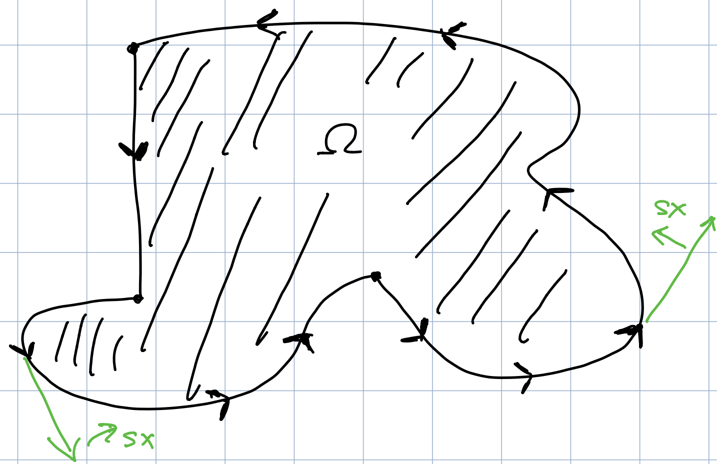


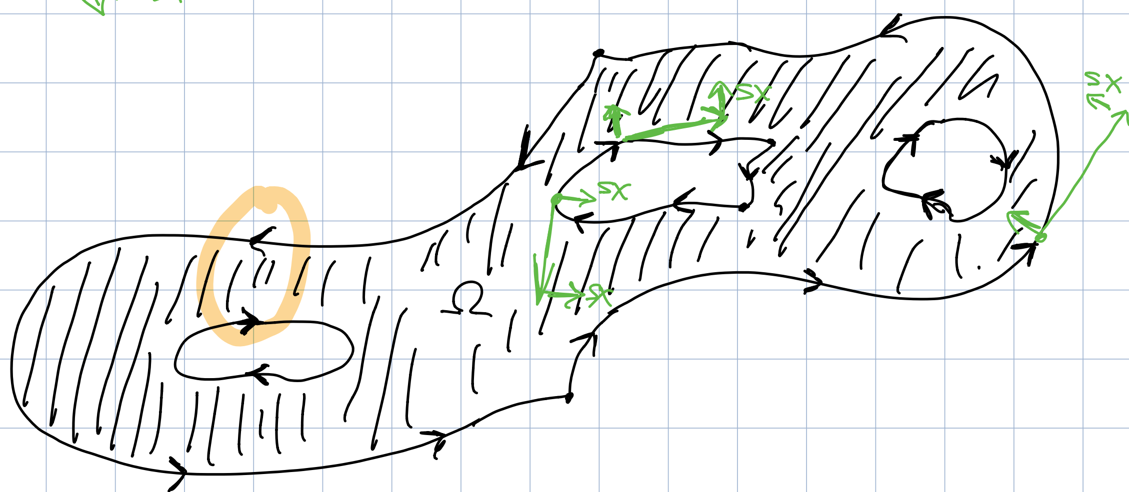
Oss: $\partial\Omega$ può essere bruttissimo.

Se $\Omega \subset \mathbb{R}^2$ e $\partial\Omega$ è una curva o una unione finita di curve regolari, allora $\partial\Omega$ ha naturale orientazione data da:

"$\partial\Omega$ deve lasciare Ω sulla sua sinistra".







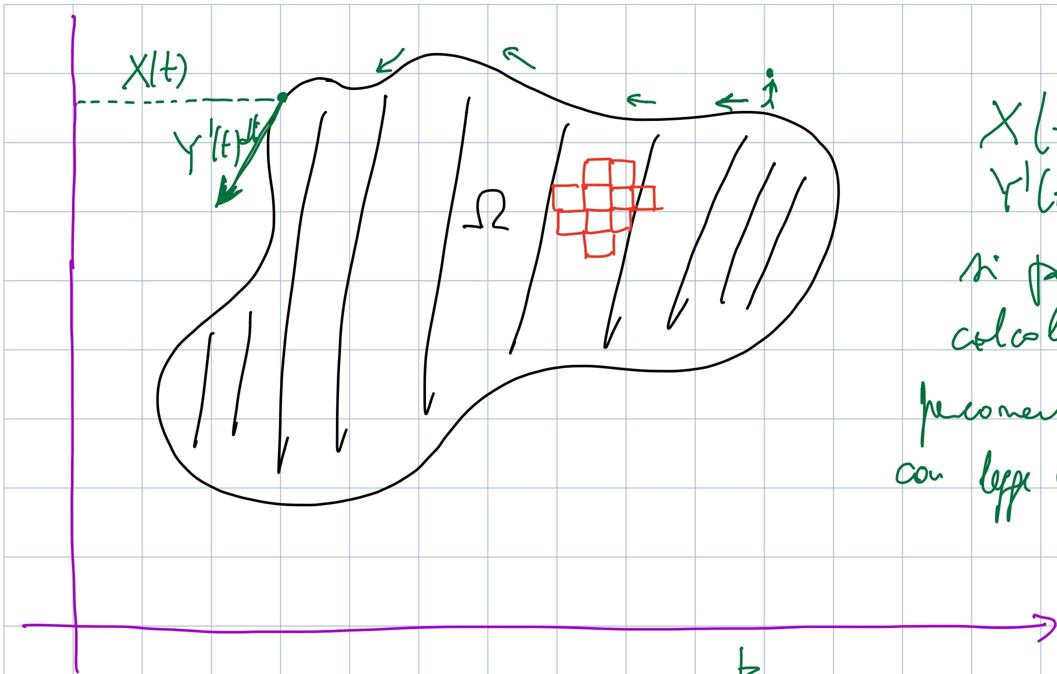
<u>Teo</u> (Gauss-Green): $\Omega \subset \mathbb{R}^2$ aperto limitato di $\mathbb{R}^2$
con $\partial\Omega$ = unione finita di curve

ω 1-forma definita su un aperto che contiene $\Omega \cup \partial\Omega$

$$\Rightarrow \int_\Omega d\omega = \int_{\partial\Omega} \omega .$$

<u>Cor</u>: $\omega = x\,dy \qquad d\omega = dx\,dy$

$$\boxed{\int_{\partial\Omega} x\,dy} = \int_\Omega dx\,dy = \text{Area}(\Omega).$$



$X(t)$
$Y'(t)$
si possono calcolare percorrendo $\partial\Omega$ con legge $\alpha(t) = \begin{pmatrix} X(t) \\ Y(t) \end{pmatrix}$

$$\Longrightarrow \int_a^b X(t)\cdot Y'(t)\,dt = \text{Area}(\Omega)$$