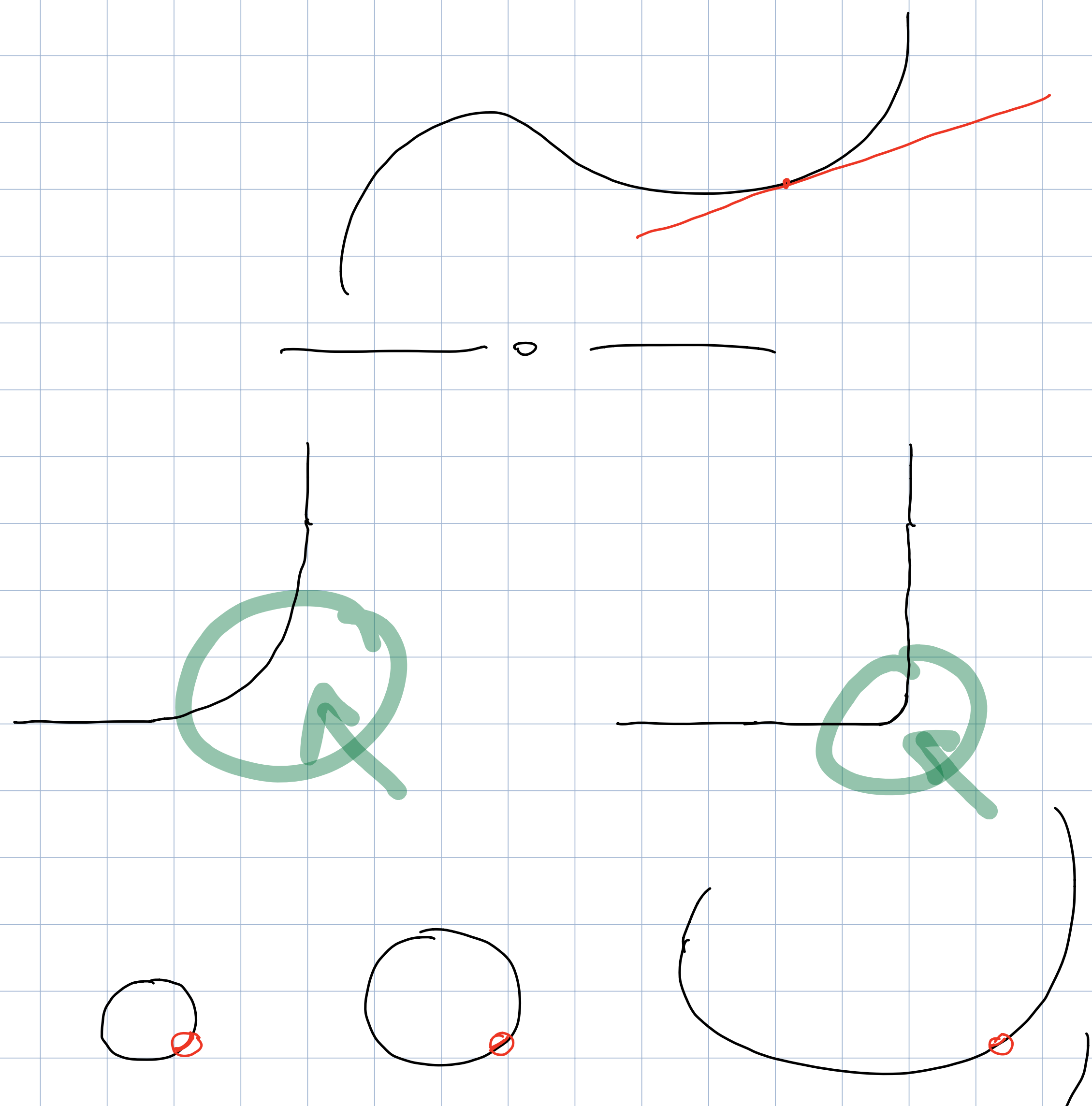
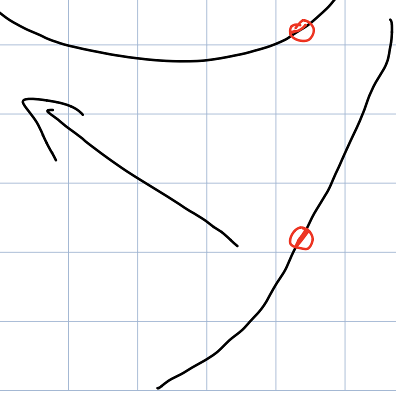
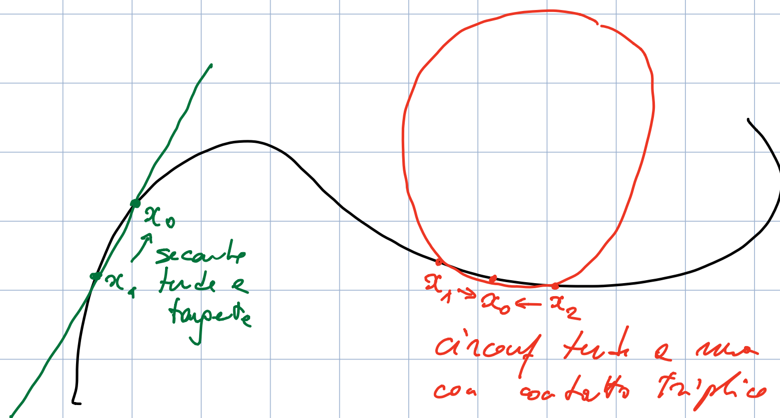
Geometria 6/5/20

$\alpha : [a,b] \to \mathbb{R}^n \qquad (n = 2)$

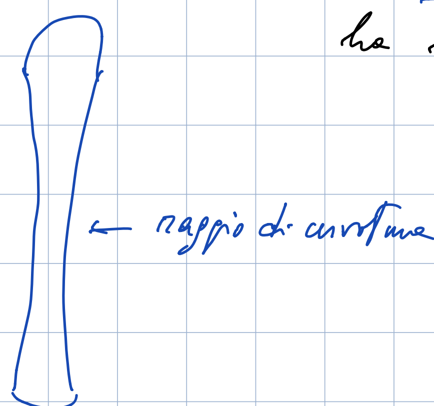




misura naturale per curvatura di circonferenza $\kappa = \frac{1}{R}$.



se la circonf con contatto triplice ha raggio R pongo $\kappa = \frac{1}{R}$.



circonf. osculatrice

Tangente : $\alpha(t_0) + \mathrm{Span}(\alpha'(t_0))$.

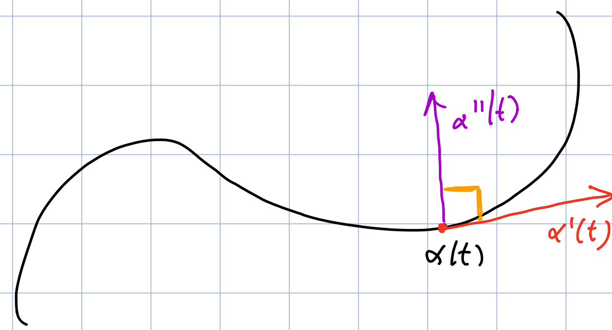
Lem: se $v : [a,b] \to \mathbb{R}^n$ e $\|v(t)\| \equiv 1$ allora $v'(t) \perp v(t)\ \forall t$.

Dim.: $\|v(t)\|^2 = v_1(t)^2 + \dots + v_n(t)^2 \equiv 1$

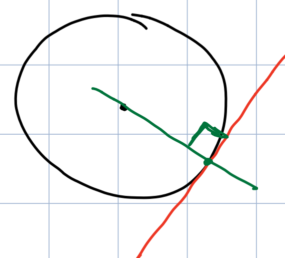
$\Longrightarrow 2v_1(t)\cdot v_1'(t) + \dots + 2v_n(t)\cdot v_n'(t) \equiv 0$

$\Longrightarrow \langle v(t) \mid v'(t) \rangle \equiv 0$ $\boxtimes$

Cor: se $\alpha : [a,b] \to \mathbb{R}^n$ è in p.d'a. $(\|\alpha'(t)\| \equiv 1)$
allora $\alpha''(t) \perp \alpha'(t)\ \forall t$.



Oss: se circonf C ha contatto triplice con α in $\alpha(t_0)$ allora a l'ha anche doppio $\Rightarrow$ ha stessa tangente in $\alpha(t_0)$ $\Rightarrow$ se è in p.d'a. ha centro su $\alpha(t_0) + \mathrm{Span}(\alpha''(t_0))$



Prop: se α è in p.d'a. la circonf. osculatrice in $\alpha(t_0)$ ha centro in $\alpha(t_0) + \dfrac{\alpha''(t_0)}{\|\alpha''(t_0)\|^2}$.

Cor: raggio $= \left\| \alpha(t_0) - \left(\alpha(t_0) + \dfrac{\alpha''(t_0)}{\|\alpha''(t_0)\|^2} \right) \right\|$

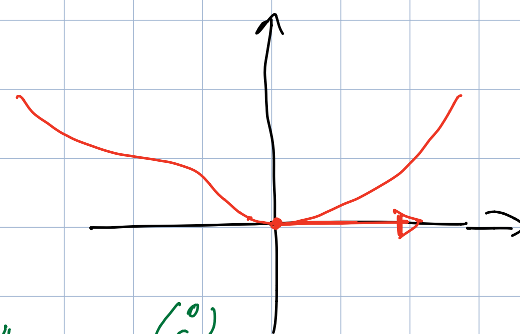
$$= \left\| \frac{\alpha''(t_0)}{\|\alpha''(t_0)\|^2} \right\| = \frac{1}{\|\alpha''(t_0)\|}$$

$$\Rightarrow \kappa(t_0) = \|\alpha''(t_0)\|.$$

Oss: $\alpha''(t_0) = 0 \Rightarrow$ la circonf osculatrice degenera in retta (tangente ha contatto triplice).

Dim: suppongo $t_0 = 0$ $\quad \alpha(t_0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $\quad \alpha'(t_0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$$\Rightarrow \boxed{\alpha''(0) = \begin{pmatrix} 0 \\ c \end{pmatrix}}$$

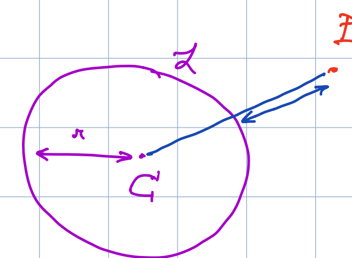


Oss: centro $\begin{pmatrix} 0 \\ r \end{pmatrix}$ $\quad r \in \mathbb{R}$

Tesi $\quad r = \frac{1}{c}$ $\quad$ poiché $\quad \frac{\alpha''(0)}{\|\alpha''(0)\|^2} = \frac{\begin{pmatrix} 0 \\ c \end{pmatrix}}{c^2} = \begin{pmatrix} 0 \\ 1/c \end{pmatrix}$.

$$f(t) = d(\alpha(t), \mathcal{L}) = d\left(\alpha(t), \begin{pmatrix} 0 \\ r \end{pmatrix}\right) - |r|$$

$$= \sqrt{X(t)^2 + (Y(t) - r)^2} - |r| .$$



$d(P, \mathcal{L}) = d(P, C) - r$

contatto triplice:

$$\left.\begin{array}{ll} f(0) = 0 & \text{(passa)} \\ f'(0) = 0 & \text{(stessa tangente)} \end{array}\right\} \text{garantite}$$

$$f''(0) = 0$$

$$f'(t) = \frac{X(t) \cdot X'(t) + (Y(t) - r) \cdot Y'(t)}{\sqrt{}}$$

$$f''(t) = -\frac{(X \cdot X' + (Y - r) Y')^2}{(\sqrt{})^3} + \frac{X'^2 + X \cdot X'' + Y'^2 + (Y - r) Y''}{\sqrt{}}$$

nulli in $t = 0$
ma rapporto tende a 0

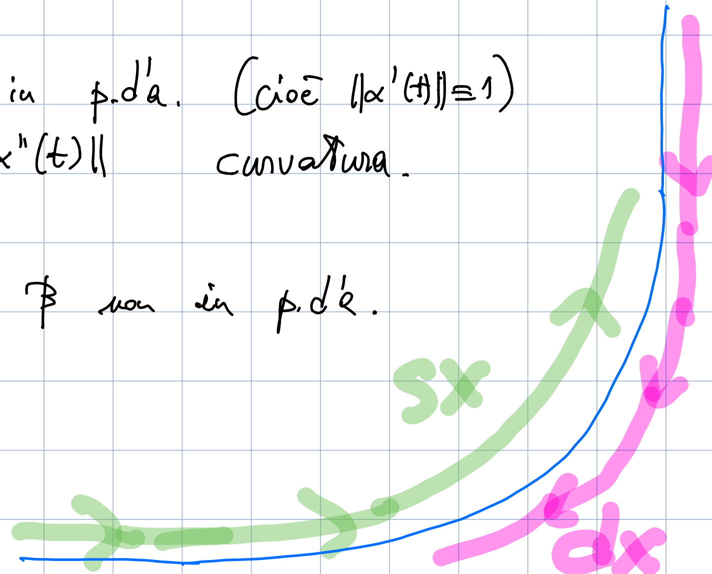
deve annullarsi

$$1 + 0 + 0 + (0 - r) \cdot c = 0 . \quad \blacksquare$$

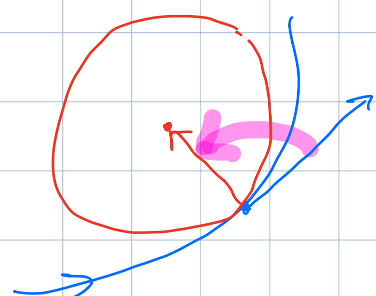
Recap: se α è in p.d'a. (cioè $\|\alpha'(t)\| \equiv 1$)

$\kappa(t) = \|\alpha''(t)\|$ curvatura.

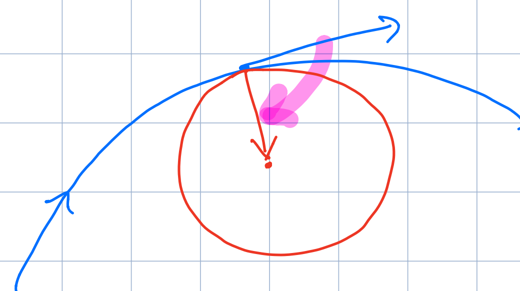
Q: come calcolarla per β non in p.d'a.



Def: diciamo che α orientata ha curvatura positiva in $\alpha(t_0)$ se l'angolo tra $\alpha'(t_0)$ e la congiungente di $\alpha(t_0)$ con il centro della circonf. osculatrice è il verso antiorario



$\kappa > 0$



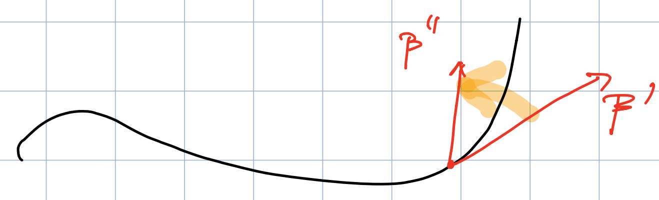
$\kappa < 0$

Prop: se β è curva orientata qualsiasi la sua curvatura nel punto $\beta(t)$ è

$$\frac{\det(\beta'(t), \beta''(t))}{\|\beta'(t)\|^3}.$$

Idea: $\alpha(s) = \beta(\sigma(s))$ $\quad \|\alpha'(s)\| \equiv 1 \quad \sigma'(s) = \dfrac{1}{\|\beta'(\dots)\|}$

si calcola $K = \|\alpha''(0)\|$ derivando ...



$$\gamma(t) = \beta(k \cdot t) \qquad k \in \mathbb{R}$$

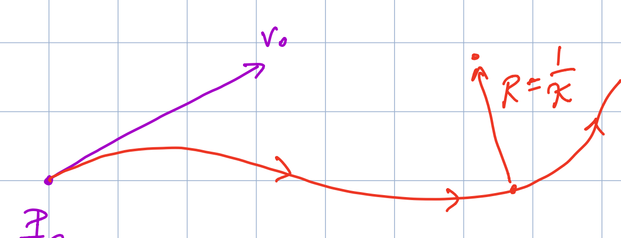
$$\frac{\det(\gamma'(t), \gamma''(t))}{\|\gamma'(t)\|^3} = \frac{\det(k \cdot \beta'(kt), k^2 \cdot \beta''(kt))}{\|k \cdot \beta'(kt)\|^3}$$

$$= \frac{\det(\beta', \beta'')}{\|\beta'\|^3}$$

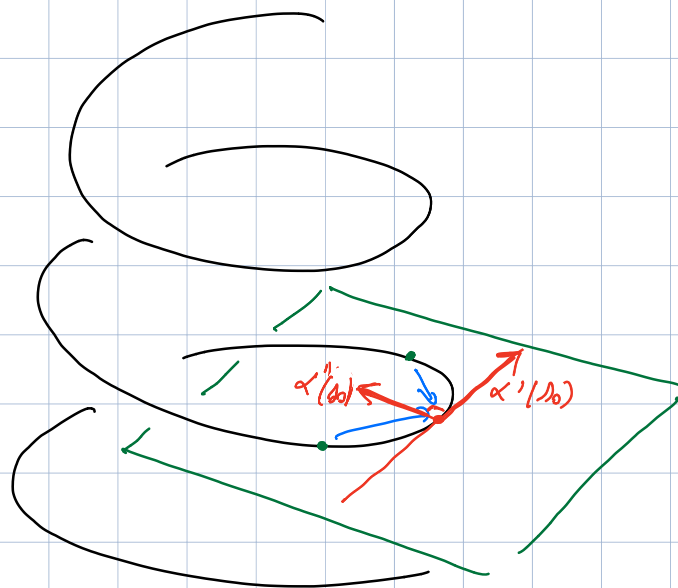
Prop: dati $P_0 \in \mathbb{R}^2$, $v_0 \in \mathbb{R}^2$, $\|v_0\|=1$, $\kappa : [0,L] \to \mathbb{R}$

$\exists$ unica $\alpha : [0,L] \to \mathbb{R}^2$ curva in p.d'a.

t.c. $\alpha(0) = P_0$, $\alpha'(0) = v_0$ con curvatura $\kappa(t)$ in $\alpha(t)$ $\forall t$.



$\alpha : [a,b] \to \mathbb{R}^3$



- $\alpha'(t_0)$ dà retta tg. (contatto doppio)

- esiste un unico piano osculatore con contatto triplice (di solito)

e su tale piano giace la circonferenza osculatrice con contatto triplice:

Per α in p.d'a.

piano: $\alpha(s_0) + \mathrm{Span}(\alpha'(s_0), \alpha''(s_0))$

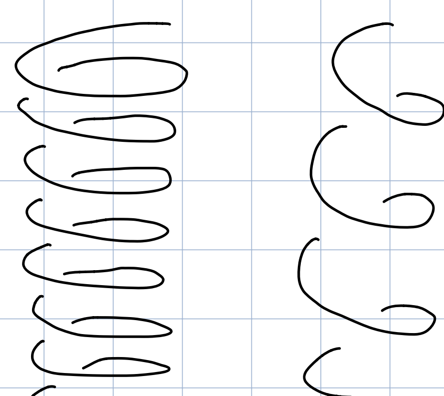
circonf: quella per $\alpha(s_0)$ con centro $\alpha(s_0) + \dfrac{\alpha''(s_0)}{\|\alpha''(s_0)\|^2}$

$\Rightarrow \kappa(s_0) = \|\alpha''(s_0)\|$ curvatura

Q: come si calcola K per β non in p.d'a?

<u>Oss</u>: nello spazio non ha senso segno di K

curve con stessa curvatura ma diversa non-planarità



Q: come misurare la non planarità?

Def: chiamo riferimento di Frénet per α (in p. d'a.) nel punto $\alpha(s)$ la terna $\big(t(s), n(s), b(s)\big)$ ortonormale positiva

tangente — normale — binormale

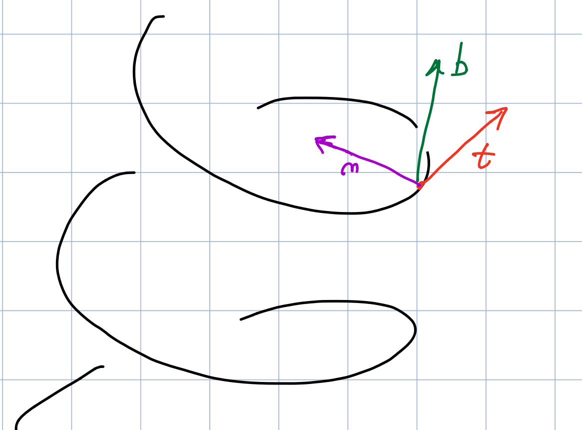
$$t(s) = \alpha'(s) \qquad \big(\|t(s)\| \equiv 1 \big)$$

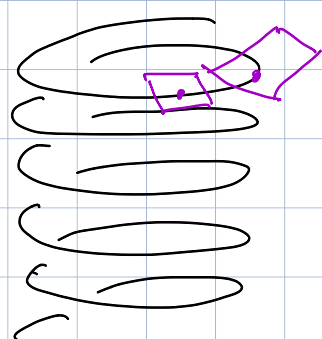
$$n(s) = \frac{\alpha''(s)}{\|\alpha''(s)\|} \qquad n(s) \perp t(s)$$

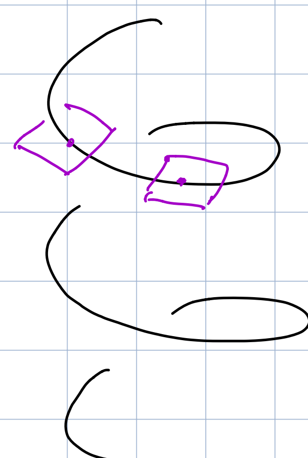
$$b(s) = t(s) \wedge n(s).$$

Oss: piano osculatore è $\mathrm{Span}(t, n) = b^{\perp}$

$\Rightarrow$ misura della non planarità è b'







Prop: data $\alpha : [a,b] \to \mathbb{R}^3$ in p. d'a. esistono $\varkappa, \tau : [a,b] \to \mathbb{R}$ curvatura, torsione t.c.

$$\begin{cases} t' = \varkappa \cdot n \\ n' = -\varkappa \cdot t + \tau \cdot b \\ b' = -\tau \cdot n \end{cases}.$$

Dim: $t = \alpha'$ $\quad n = \frac{\alpha''}{\|\alpha''\|} \Rightarrow \alpha'' = (\alpha')' = t'$

$\alpha'' = \|\alpha''\| \cdot n \Rightarrow \varkappa = \|\alpha''\|$

$b = t \wedge n$; $\quad \|b\| \equiv 1 \Rightarrow b' \perp b$

$b = t \wedge n \Rightarrow b' = t' \wedge n + t \wedge n'$

$= \varkappa \underbrace{n \wedge n}_{0} + t \wedge n' = t \wedge n'$

$\Rightarrow b' \perp t$

$\Rightarrow b'$ multiplo di n;

pongo τ t.c. $\quad b' = -\tau \cdot n$. $\quad$ Ora:

(t, n, b)

$b = t \wedge n$
$n = b \wedge t$
$t = n \wedge b$

$n' = (b \wedge t)' = b' \wedge t + b \wedge t'$

$= -\tau \cdot n \wedge t + b \wedge \varkappa \cdot n$

$= \varkappa \cdot \underbrace{b \wedge n}_{-t} - \tau \cdot \underbrace{n \wedge t}_{-b}$

$= -\varkappa \cdot t + \tau \cdot b$. $\quad \blacksquare$

Oss:

$$\begin{cases} t' = \varkappa \cdot n \\ n' = -\varkappa \cdot t + \tau \cdot b \\ b' = -\tau \cdot n \end{cases} \Rightarrow (t,n,b)' = \underbrace{(t,n,b)}_{\text{matr. ortog}} \cdot \underbrace{\begin{pmatrix} 0 & -\varkappa & 0 \\ \varkappa & 0 & -\tau \\ 0 & \tau & 0 \end{pmatrix}}_{\text{antisimm.}}$$

Prop: $M : [a,b] \to \mathcal{M}_{n\times n}(\mathbb{R})$ $\quad M(t)$ ortog. $\forall t$

$\Longrightarrow M'(t) = M(t)\cdot A(t)$ $\quad A(t)$ antisimm.

Dim: ${}^tM\cdot M \equiv I_n$

$$\underbrace{{}^tM'\cdot M}_{{}^tA} + \underbrace{{}^tM\cdot M'}_{A} = 0 \quad ; \quad M' = I_n\cdot M' = M\cdot \underbrace{{}^tM\cdot M'}_{A}$$

$\Longrightarrow A$ antisimmetrica ∎

Prop: se β è curva qualsiasi la sua curvatura e torsione nel punto $\beta(s)$ sono:

$$\kappa(s) = \frac{\|\beta'(s)\wedge\beta''(s)\|}{\|\beta'(s)\|^3} \qquad \tau(s) = \frac{\langle \beta'(s)\wedge\beta''(s) \,|\, \beta'''(s)\rangle}{\|\beta'(s)\wedge\beta''(s)\|^2}$$

Oss $\gamma(s) = \beta(k\cdot s)$

$$\frac{\langle \gamma'\wedge\gamma'' \,|\, \gamma'''\rangle}{\|\gamma'\wedge\gamma''\|^2} = \frac{\langle k\beta'\wedge k^2\beta'' \,|\, k^3\beta'''\rangle}{\|k\beta'\wedge k^2\beta''\|^2}$$

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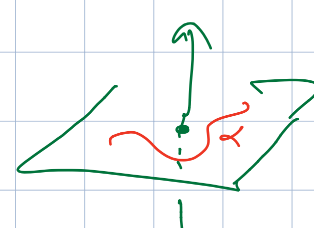
Oss: $\alpha : [a,b] \to \mathbb{R}^2$ qualsiasi

$\beta = \begin{pmatrix}\alpha\\ 0\end{pmatrix}$



$$\frac{\det(\alpha',\alpha'')}{\|\alpha'\|^3} \qquad \frac{\|\beta'\wedge\beta''\|}{\|\beta'\|^3}$$

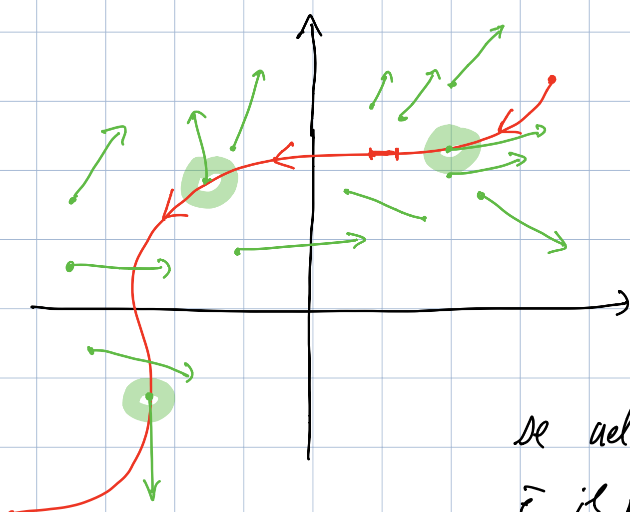
$\alpha = \begin{pmatrix} X \\ Y \end{pmatrix} \qquad \beta = \begin{pmatrix} X \\ Y \\ 0 \end{pmatrix}$

$$\|\beta' \wedge \beta''\| = \left\|\begin{pmatrix} X' \\ Y' \\ 0 \end{pmatrix} \wedge \begin{pmatrix} X'' \\ Y'' \\ 0 \end{pmatrix}\right\| = \left\|\begin{pmatrix} 0 \\ 0 \\ X'Y'' - Y'X'' \end{pmatrix}\right\|$$

$$= |X'Y'' - Y'X''| = \left|\det\left(\begin{pmatrix} X' \\ Y' \end{pmatrix} \begin{pmatrix} X'' \\ Y'' \end{pmatrix}\right)\right|$$

— o —

1-forme e loro integrazione.



se nel punto $\binom{x}{y}$ il vento è il vettore $\begin{pmatrix} f(x,y) \\ g(x,y) \end{pmatrix}$ il costo elementare di percorrenza nel punto $\alpha(t) = \begin{pmatrix} X(t) \\ Y(t) \end{pmatrix}$ è

$$\boxed{f(X(t),Y(t)) \cdot X'(t)\,dt + g(X(t),Y(t)) \cdot Y'(t)\,dt}$$

Dato $\Omega \subset \mathbb{R}^2$ aperto chiamo 1-forme su Ω un oggetto $\omega = f \cdot dx + g \cdot dy$

$$\omega(x,y) = f(x,y) \cdot dx + g(x,y) \cdot dy$$

con $f, g : \Omega \to \mathbb{R}$.

spostamenti elementari nelle due direzioni

Su Ω c'è campo vettoriale $\begin{pmatrix} f \\ g \end{pmatrix}$

Se $\alpha : [a,b] \to \mathbb{R}^2$ pongo

$$\int_\alpha \omega = \int_a^b \Big(f(\alpha(t)) \cdot X'(t) + g(\alpha(t)) \cdot Y'(t) \Big)\, dt .$$

ATTENZIONE

$F : \Omega \to \mathbb{R}$

$$\int_\alpha F = \int_a^b F(\alpha(t)) \cdot \|\alpha'(t)\|\, dt$$

integrale di funzione scalare

$f, g : \Omega \to \mathbb{R}$

$$\int_\alpha f\,dx + g\,dy = \int_a^b \Big(f(\alpha(t)) \cdot X'(t) + g(\alpha(t)) \cdot Y'(t) \Big)\, dt$$

integrale di 1-forma (NO $\|\alpha'(t)\|$)