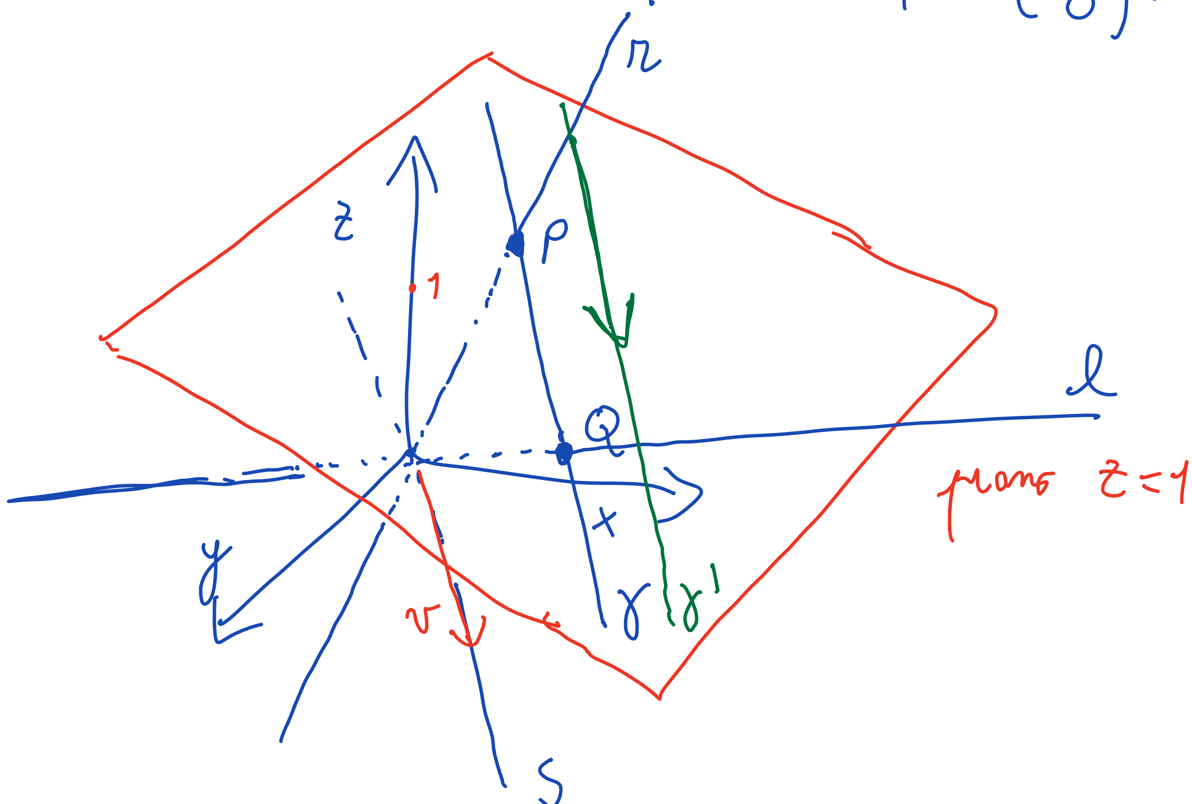
Nella scorsa lezione abbiamo descritto

$$\mathbb{P}(\mathbb{R}^2) = \mathbb{P}^1(\mathbb{R})$$

Studiamo ora $\mathbb{P}(\mathbb{R}^3) = \mathbb{P}^2(\mathbb{R})$

$\mathbb{P}^2(\mathbb{R})$ è in bigezione con l'insieme delle rette in $\mathbb{R}^3$ passanti per $\begin{pmatrix}0\\0\\0\end{pmatrix}$.



ad ogni retta r passante per $\begin{pmatrix}0\\0\\0\end{pmatrix}$ e che non giace sul piano $z=0$ posso associare un unico punto P sul piano $z=1$

Dunque dentro $\mathbb{P}^2(\mathbb{R})$ posso intanto "vedere" il piano $z=1$. Si tratta di un piano affine la cui giacitura è un sottospazio di $\mathbb{R}^3$ di dimensione 2, insomma è $\mathbb{R}^2 \times \{0\}$.

Sia s una retta che giace nel piano $z=0$.
Tale retta corrisponde ad un punto in $\mathbb{P}^2(\mathbb{R})$. Che significato possiamo dare a tale punto?

Prendo un'altra retta, l, che interseca il piano $z=1$ nel punto Q.
Sia γ la retta affine che passa per

P e Q e giace dunque sul piano $z=1$.

Noto che le rette r ed l generano un piano W in $\mathbb{R}^3$.

(↑ sottospazio)

In tale sottospazio W c'è anche il vettore $v=$ " $\overrightarrow{PQ}$ ".

Se ho scelto r ed l ed s in modo che $v \in S$, allora anche la retta s giace in W ed è parallela a $\jmath$.

Come sappiamo ogni retta di $\mathbb{R}^3$ corrisponde ad un punto in $\mathbb{P}^2(\mathbb{R})$

$r \longrightarrow P$

$l \longrightarrow Q$

$s \longrightarrow S$

S non va pensato sul piano $z=1$
va pensato piuttosto come "punto all'infinito" che si aggiunge alla retta γ passante per P e Q. Infatti $\mathbb{P}(W)$ è un $\mathbb{P}^1$, contiene γ e il punto aggiuntivo S.

Notiamo ora che se consideriamo sul piano $z=1$ una retta γ' parallela a γ, il piano W' di $\mathbb{R}^3$ che (sottospazio) passa per l'origine e contiene γ' contiene anche il vettore v e dunque la retta s.

Allora anche γ' ha come punto all'infinito proprio S.

Dunque $\mathbb{P}^2(\mathbb{R})$ si può pensare come il piano $z=1$ a cui vengono aggiunti dei punti all'infinito (esattamente un $\mathbb{P}^1(\mathbb{R})$ ← cioè $\mathbb{P}(\mathbb{R}^2 \times \{0\})$) in modo tale che due rette parallele in $z=1$ abbiano associato lo stesso punto all'infinito

(nell'esempio sopra a γ e γ' viene associato S).

Par 12.2.4

SPAZIO PROIETTIVO COME COMPLETAMENTO ALL'INFINITO DELLO SPAZIO AFFINE

Proposizione. 12.2.3 Consideriamo in $\mathbb{R}^{n+1}$ il sottospazio affine $x_{n+1}=1$ che è $\mathbb{R}^n \times \{1\}$. Si ottiene in tal modo una identificazione di $\mathbb{R}^n$ con

un sottoinsieme di $\mathbb{P}^n(\mathbb{R}) = \mathbb{P}(\mathbb{R}^{n+1})$
Il complementare di $\mathbb{R}^n$ in $\mathbb{P}^n(\mathbb{R})$ è in modo naturale identificato a $\mathbb{P}^{n-1}(\mathbb{R})$

<u>Dim</u> Come sopra nel caso $\mathbb{P}^2(\mathbb{R})$, ogni retta passante per O interseca $\mathbb{R}^n \times \{1\}$ in al più un punto, inoltre il complementare cercato è in bigezione con le rette in $\mathbb{R}^n \times \{0\}$, ed è identificato con $\mathbb{P}^{n-1}(\mathbb{R})$.

$\square$

<u>Prop 12.2.4</u> $\mathbb{P}^n(\mathbb{R})$ si ottiene aggiungendo a $\mathbb{R}^n$ i punti di $\mathbb{P}^{n-1}(\mathbb{R})$ ciascuno dei quali rappresenta il comune punto all'infinito di un fascio di rette parallele in $\mathbb{R}^n$.

$\mathbb{R}^n = \mathbb{R}^n \times \{1\}$

Dim Analoga al caso di $\mathbb{P}^2(\mathbb{R})$.

Un altro modo di vedere $\mathbb{P}^2(\mathbb{R})$ è leggerlo come S^2 ← la sfera unitaria in $\mathbb{R}^3$

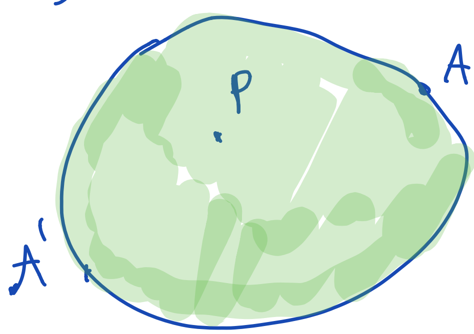
dove i punti antipodali sono identificati.

Un altro modo ancora è leggerlo come la semisfera di S^2 data da $S^2 \cap \{z \geq 0\}$ dove nella circonferenza S^1 data da $S^2 \cap \{z = 0\}$ identifico i punti antipodali

A e A' sono identificati

Posso con una trasformazione continua "schiacciarla" ad un disco

$\mathbb{P}^2(\mathbb{R})$ è un disco D



in cui i punti al bordo sono identificati con gli antipodali. (A è identificato ad A').

FINALE: per divertimento, date un'occhiata al gioco "SHAPLEY", che si gioca su una scacchiera proiettiva (vedi paragrafo 6 del bellissimo articolo di David Gale qui sotto).

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Topological games at Princeton, a mathematical memoir ☆

David Gale

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ABSTRACT

The games of the title are "Nash" (or Hex), "Milnor" (or *Y*), "Shapley" (or Projective Plane) and "Gale" (or Bridg-It) all of which were discovered (or re-discovered) in Princeton in 1948–1949. After giving the basic topological connections, I will discuss more recent ramifications related to computational complexity theory. A recurrent theme will be non-constructive proofs, or how we can know something can be done without having the slightest idea of how to do it.

1. The game of Nash/Hex

It was in the winter of 1948. John Nash was sitting across from me at breakfast in the dining hall of Princeton's Graduate College. He was a second year graduate student. I was in my third year. We were both part of a group working in the then new subject of game theory.

"I have an example of a game that I can prove is a win for the first player, but I have no idea of what a winning strategy would be," he said and then proceeded to describe the game. It is played on a checkerboard and Red[1] and Black take turns placing their pieces on the board. The red player is trying to make a red path between the top and bottom edges of the board while the black player is trying to make a black path between the two vertical sides. A path is a sequence of squares in which consecutive squares are neighbors either horizontally, vertically, or (key condition) diagonally but only in the "positive" direction, that is, two squares are adjacent if the upper right corner of one square is the lower left corner of the other. Fig. 1 shows an example.

Why does this game have the asserted properties? The reason is that the game cannot end in a draw (as is possible with most games from tic–tac-toe to chess). Intuitively one sees, as both Nash and I did, that if the board is full and there is no connected red path from top to bottom, then it must be the case that this is prevented by a black "fence" connecting the two sides. Somewhat more picturesque is the geographic argument where the United States replaces the checkerboard. The assertion is that either it is possible to sail across the country from the Atlantic to the Pacific or one can walk from Canada to Mexico without getting wet. Making these arguments rigorous is of course where the topology comes in. Also it is perhaps even more "obvious" that there can be only one winner, since a north–south path and an east–west path would have to cross. The reader can check that the one-sided diagonal condition is necessary because if you do not allow diagonals then there could be no winner and if you allow both diagonals there could be two winners.

The fact that the first player should be the winner follows from a familiar "strategy stealing" argument which applies to a wide class of games such as tic–tac-toe and Go Moku (five in a row). If the second player had a winning strategy, the first player could make an arbitrary first move and then adopt this second player's strategy. This would work unless the strategy required playing on the square occupied by the first move, in which case the first player would make another arbitrary move, and so on. Hence the second player cannot have a winning strategy.

☆ Presented at the Stony Brook Gale Feast, July 12, 2007. Katharine Gale supplied David Gale's draft of this manuscript. Joel Sobel lightly edited the draft and added the final section, which was inspired by Gale's concluding comments at Stony Brook. Sobel thanks Jack Edmonds, Katharine Gale, Sandra Gilbert, and Bernhard von Stengel for their comments.

[1] The on-line version of the article contains colored figures. Red appears more lightly shaded than Blue in the black-and-white images.

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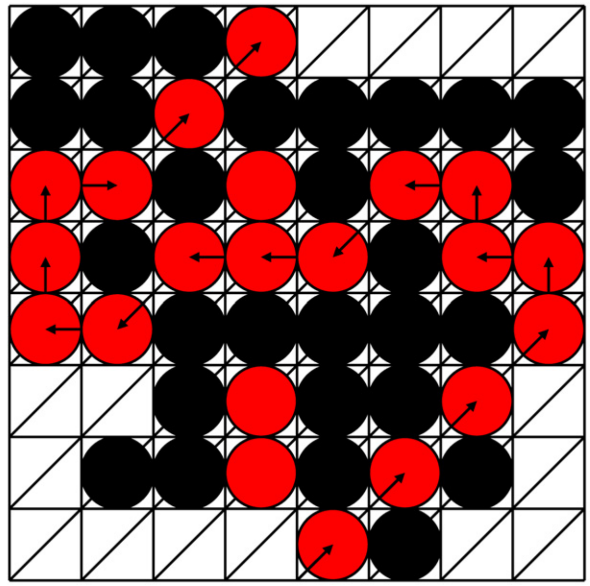

Fig. 1. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

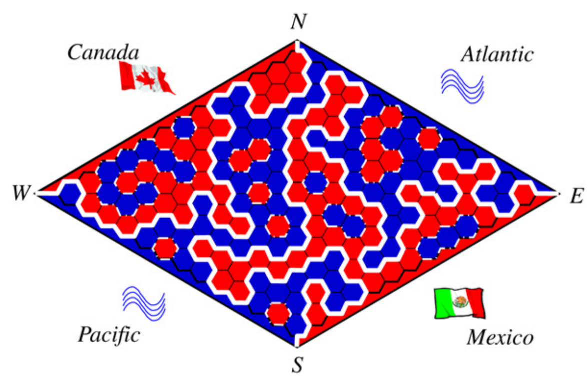

Fig. 2. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

I was naturally enthralled by Nash's charming argument, but it also occurred to me that his game might actually be fun to play. It did not take much geometric imagination to realize one could get around the asymmetric diagonal constraint by replacing the squares by hexagons, and I spent the rest of that day constructing a 14×14 Nash board. I donated it to the Fine Hall common room and it was an immediate hit and in fact the board was still being played on several years later. The game was sometimes called John rather than Nash because in those days the most familiar use of hexagons was for tiling bathroom floors. There is much more to the history of Nash, or Hex as it is now called, and I will return to it.

What is the precise result, which we will call the No-Draw Theorem, and how is it proved? During the Princeton period no one worried about this question, but as it later turned out this "obvious" result is at least as deep as the Brouwer fixed-point theorem that, as we will see, is an easy consequence.

For aid in understanding, it is helpful to generalize. In particular, the fact that the tiles are hexagons is irrelevant. We consider tilings of a square by tiles of arbitrary shape. Thus we are given a "map" on the square, in the sense of Euler and the Four Color Theorem, with vertices, edges, and faces (or tiles). The one crucial condition is that all vertices lie on exactly 3 edges. That is, in the edge graph all vertices have degree three.

No-Draw Theorem 1. *If all of the tiles of the board are either red or blue, then there is either a blue path that meets the blue boundaries or a red path that meets the red boundaries, but not both.*

Fig. 2 shows a standard 13×13 Hex board where all of the tiles have been "painted" red or blue. The four boundary areas have also been colored appropriately and may be thought of as four additional tiles. The four corner vertices correspond to the points of the compass.

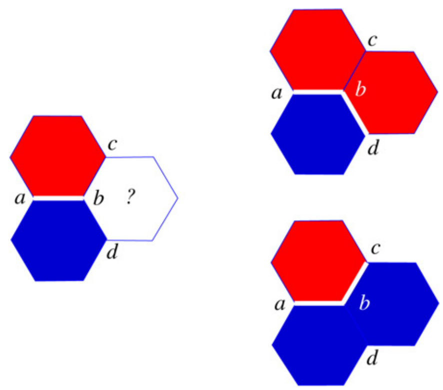

Fig. 3. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

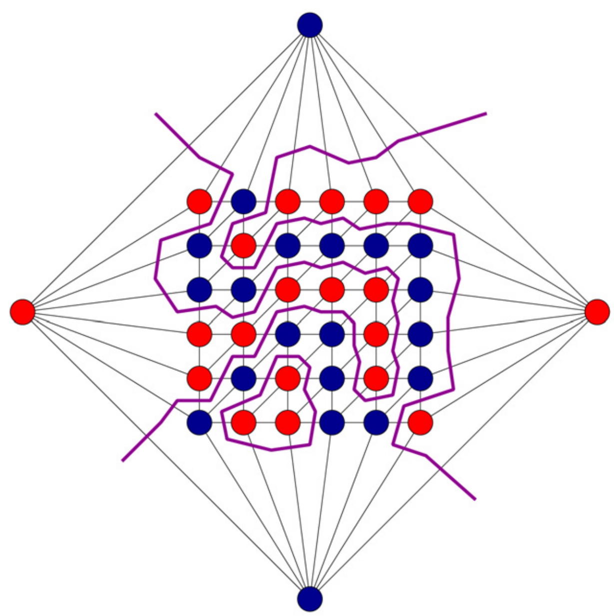

Fig. 4. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

Let us continue with the geographic metaphor and think of the blue tiles as water and the red tiles as land, so we have an aerial view of a land with lakes, rivers, bays, islands and peninsulas. The idea is now to look at the "shore lines," that is, the edges that separate a red and blue tile. These have been painted white in Fig. 2. The set of all white edges form a graph G.

Key Lemma. *The four corner vertices of G have degree* 1. *All other vertices of G have degree* 2.

Fig. 3 establishes the lemma. The separating white edge $[a, b]$ has a red tile above and a blue tile below. If the vertex a is one of the corner vertices, then it will have degree 1. The edge $[b, c]$ or $[b, d]$ will be separating (white) according as tile "?" is red or blue, and in either case vertex b will have degree 2.

We now use the elementary fact that a graph with all vertices of degree 2 or 1 consists of disjoint paths and cycles (both cases occur in the figure). Therefore the four corner vertices must be paired off as endpoints of paths. Further, a path cannot connect opposite corners, since, for example, the path starting from the South has blue tiles on its right, while the edge ending at the North corner has the blue tile on its left. The game has been won by red if the paired end points are N with E and S with W and by blue in the alternate case.

Please note that the argument above gives a simple algorithm for finding a winning path. Start, say, with the separating edge at S and follow it to either E or W.

For some purposes the original Nash checkerboard representation or rather its dual, where players move by coloring vertices rather than faces, is convenient. Fig. 4 gives an example along with a coloring of the vertices and separating edges.

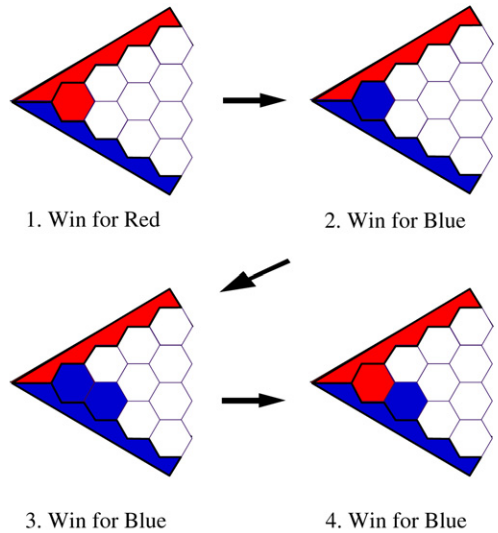

Fig. 5. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

To see that the No-Draw Theorem really is topological we show that it implies the celebrated

Brouwer Fixed-Point Theorem. *A continuous mapping f of the unit square S into itself has a fixed point.*

To prove the theorem it suffices to prove that

(1) for any $\varepsilon > 0$ there is a point x of S such that $|f(x) - x| \leqslant \varepsilon$ where $|x| = \max[x_1, x_2]$.

Proof. By uniform continuity, we can make an n by n dual Hex board H on the square so that

(2) if x and y are adjacent nodes then $|f(x) - f(y)| \leqslant \varepsilon$.

Now suppose (1) is false so that every point is moved by more than ε. We now label the vertices as follows. If a node v is moved north (south) by at least ε, then it is labeled B^+ (B^-) and painted blue. Next, vertices that are moved east (west) by at least ε are labeled R^+ (R^-) and painted red. Now note that from (2) no vertex in B^- is adjacent to a vertex in B^+. Further, no vertex of B^- is on the south boundary of the board and no vertex of B^+ is on the north boundary. It follows that there can be no blue path between north and south. The same argument shows that there is no red path between east and west. But this contradicts the No-Draw Theorem since every vertex has been painted. Hence (1) has been proved.[2] □

The No-Draw Theorem generalizes to n dimensions and an n-player game. Given a triangulation of the n-cube, the players take turns, labeling vertices with their own "color." An argument generalizing that for the two-player case shows that if the board is filled, then at least one of the players has connected his/her assigned pair of opposite faces. The n-dimensional Brouwer theorem is again an easy consequence. However, unlike the two-dimensional case, it is possible for several or all of the players to have winning paths on a filled board, since the strategy stealing result that the first player has a winning strategy only works for two players.

I want to mention two more developments in Hex theory. Let us assume from now on that Player 1 is Red.

2. Losing moves

In 1969, Anatole Beck (see Beck et al., 2000), showed that Hex becomes a win for B if R makes the "mistake" of playing his first move in one of the acute corners. The proof, once again, is completely non-constructive. Assume that the position (1) in Fig. 5 is a win for R (with B to move), then from symmetry it follows that (2) is a win for B (with R to move), which implies (3) a win for B since, as in strategy stealing, having an extra blue tile cannot be a disadvantage for B, which implies (4) is a win for B. The last claim follows because any winning path for (3) must also be a winning path for (4), but position (4) can be obtained by B.

[2] Gale (1979) first presented this argument (and contains the generalization to n dimensions).

3. Misère Hex

Because of the No-Draw Theorem, there must always be a winner for the so-called Misère (or give away) variation of Hex in which the player who has a connecting path is the loser. For this case it is not clear who the winner should be. The strategy stealing argument is clearly not applicable. Lagarias and Sleator (1999) solved the problem with the following:

Lemma 1. *In Misère Hex there exists a strategy for the losing player so that the game will not be decided until the board is completely full.*

If Misère Hex is a win for R (B), then B (R) must make the last move, so R (B) wins if the board has an even (odd) number of tiles.

The proof of Lemma 1 is non-constructive and quite subtle.

All of these non-constructive results are an echo of Nash's original insight that the fact that knowing R had a winning strategy was no help in finding the strategy and thus "solving" the game. Perhaps the crowning result in this direction was a discovery by Reisch (1981) that the problem of determining which player has a winning strategy in Hex from a given position of the board is PSPACE-complete, which means it is a "hard" problem in the sense of complexity theory.

4. Hex history

Because of the success of Nash/Hex in the Fine Hall common room, Nash and I thought there might be a chance of cashing in on it commercially. At one point, I made a round of some of the game companies in New York using the homemade board for demonstrations. In fact, some of the people I talked to were intrigued by the game, but the consensus was that it would not sell. People will not go for "games of no chance" they claimed. They prefer throwing dice or spinning spinners. I even had some correspondence with Parker Brothers, the biggest of the board game companies. No go.

At about the same time we learned that Hex/Nash had been discovered/invented independently six years earlier by the famous Danish engineer-poet Piet Hein, a quite remarkable inventor of games, puzzles, and short verses that appeared regularly in one of the Danish newspapers.[3]

A few years later Hein on a visit to America did manage to sell his game, which he called Polygon, to Parker Brothers and it was available for several years in an 11×11 version. I recall getting a phone call from John Nash in around 1957. By this time he was at MIT and I was at Brown. John, who did not know about Piet Hein, had just seen the commercial board, and I had to assure him that it was Hein, not I, who had sold the game to Parker Brothers.

The legendary science writer Gardner (1957) devoted one column to Hex where he says, "Hex may well become one of the most widely played and thoughtfully analyzed new mathematical games of the century." Despite this prediction, Hex has not been commercially successful, but it is still being played all over the world. The Russian mathematician Vadim Anshelevich succeeded in writing a very strong computer program for his virtual player "Hexy" using a technique that has come to be called H-search, and he/she/it will happily take you on.[4] Hexy won the gold medal in the 5th Computer Olympiad in London in 2000. More recently, a program named Six using H-search methods won the tournaments in 2003, 2004 and in Turin in 2006. There are also sites for on-line tournaments that are constantly in progress.[5]

Returning to Princeton 1948, the subject from here on will be three other games that were "spin offs" of Hex.

5. The game of "Milnor" or "Triangle" or "*Y*"

John Milnor, still an undergraduate in 1949, was among the people who contributed to the topological game portfolio. He called the game Triangle, but around Fine Hall it was referred to as Milnor. It seems Claude Shannon independently invented the game. The rules are, if anything, even simpler than those of Nash. The hexagonal tiles this time are used to pave an equilateral triangle in the natural way. Players alternately paint tiles with their respective colors and the winner is the one whose pieces contain a winning set, that is, a connected set that meets all three boundary edges of the triangle. Such a set is called a Y, since it is topologically equivalent to a slightly thickened letter Y, where the players "color" the vertices rather than the cells.

Actually the game is more interesting if the board is a hexagon rather than a triangle, with the boundary divided into three parts. A completely painted Y board is shown in Fig. 6.

As with Nash, Milnor can be played on any tiling of a disk where the boundary is partitioned into three arcs, as long as it satisfies the crucial condition that at most three tiles meet at any vertex. Alternatively, one can play the dual game using any triangulation of a disc where one colors the vertices rather than the tiles. This is how it is done in the currently available commercial version of the game, called the game of Y.[6]

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[3] For more on Hein see: http://www.piethein.com/usr/piethein/HomepagUK.nsf or http://en.wikipedia.org/wiki/Piet_Hein_(Denmark).

[4] http://home.earthlink.net/~vanshel/.

[5] http://www.littlegolem.net/jsp/games/gamedetail.jsp?gtid=hex.

[6] http://www.gamepuzzles.com/gameofy.htm.

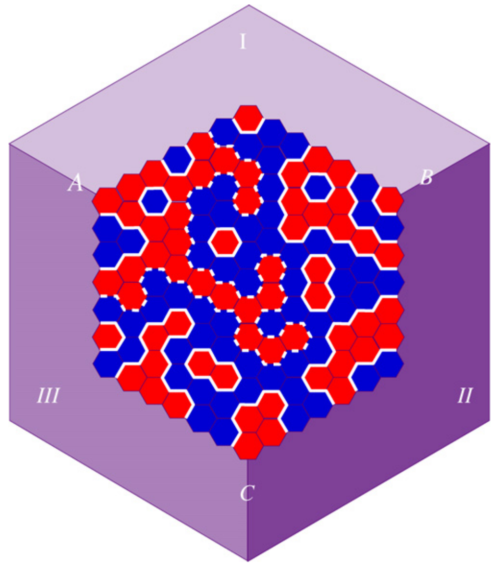

Fig. 6. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

No-Draw Theorem 2. *If all of the tiles have been painted red or blue, then there is a red or blue Y.*

Here is Milnor's lovely proof. The tiles A, B, C that are adjacent to two of the colored boundary arcs will be called corners. The geographic analogy is particularly illuminating for this game. The tiles painted red are land and the blue tiles are water. A completely painted board is again made up of islands, lakes, peninsulas, bays and the like. The boundaries between land and sea, the shorelines, are again colored white and consist either of cycles (surrounding islands or lakes), or paths connecting a pair of points on the boundary. In each case the shoreline curve divides the board into two (connected) regions. The proof (due to Milnor) that there exists a winning set is by induction on the number of shoreline curves. If the set is empty, then the board is all one color and there is nothing to prove. If there is just one white curve it must divide the board into a red and a blue region one of which contains at least two of the three corners and is therefore winning. For the general case, pick any white curve X and, say, the region S on the blue side of X contains two of the corners, say, B and C. Then it follows that no tile on the red side S' can be connected to a winning set since S' does not meet the arc BC of the boundary. Thus, if we "flood" S', changing all the tiles to blue water it cannot affect whether there is a winning set. But we have now reduced the number of white curves (by at least one) so by the induction hypothesis region S contains a Y.

The proof is again constructive. Instead of flooding successive sets, we can simply erase all the tiles on the side that does not contain two corners. Eventually we will be left with the unique winning Y. Fig. 7 illustrates how the board of the game in Fig. 6 changes after removing tiles.

As with Hex, the No-Draw Theorem implies that the Misère version must also have a winner and, in fact, Lemma 1 applies verbatim to this and all the other games to be discussed here. This is the result that says in the Misère game the losing player can prolong the agony so that the winning set is not achieved until the board has been completely painted. All of these games have the following simple structure: there is a given set B (the board) and a family of winning sets W with the property: that, given any subset S of B either S or its complement S', but not both, contain a member of W.

6. The game of Shapley/projective plane

Perhaps the most sophisticated of the topological games is due to Lloyd Shapley and involves tilings of the real projective plane. The projective plane, as we know, cannot be embedded in 3-space, much less on a two-dimensional playing board, so one needs to impose some other conditions in order to represent the game. Recall that the projective plane is topologically a disk with antipodal points identified. In the board game, the players move by coloring vertices and the corresponding rule is that when a player colors a boundary vertex the antipodal vertex is automatically given the same color. A winning set for this game is a path connecting any point with its antipode. It is clear that there can be at most one since any two such paths would have to cross.

Unfortunately, however, the No-Draw Theorem does not hold in general for games defined in this way, as shown by the game on the left side of Fig. 8, where there is no path of either color connecting antipodes. To get around this, we extend the definition of a path by allowing jumps from a boundary tile to its antipode. In the left picture Fig. 8 there is a red path from any red boundary point to the corresponding antipode with two jumps. The right picture in Fig. 8 illustrates that there is still a problem with this definition: there is a red path with one jump and a blue path with no jumps so the uniqueness property is lost. The right definition of a winning set is a path with an even number of jumps connecting a pair

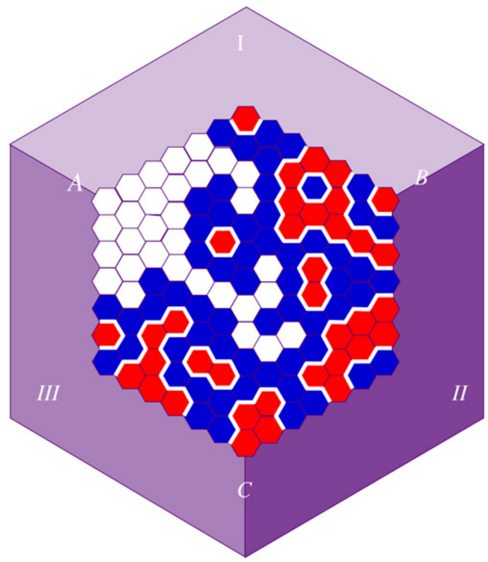

Fig. 7. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

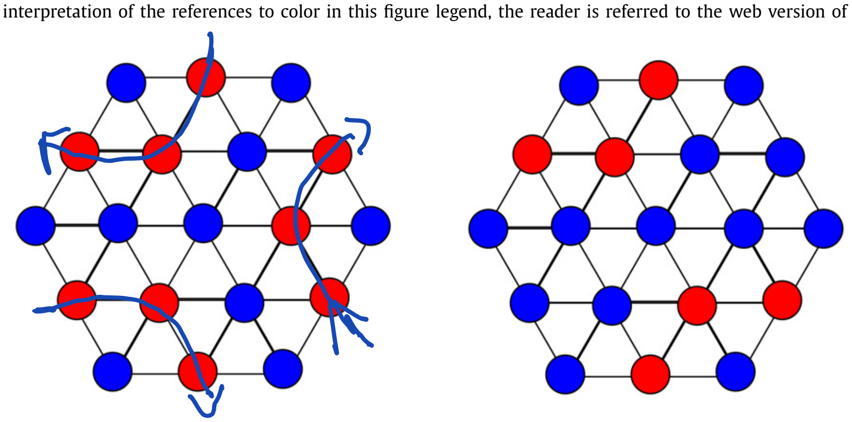

Fig. 8. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

of antipodes. Such sets correspond exactly to the topological concept of non-bounding cycles and can be shown to satisfy a No-Draw Theorem.

7. The game of Gale/Bridg-It

Another Hex inspired connection game I thought up was what was later known as Bridg-It. It was played on a square board rather than a rhombus, as shown in Fig. 9. Players connect (with pen or pencil) dots of their own color and try to connect north and south (resp., east and west) boundaries. This was an attempt by me to make a rectangular analogue of hexagonal Nash. I do not know that anyone around Princeton actually played the game. In the pre-computer age it would have been quite a task to mass-produce pads with the desired pattern. Once again, the No-Draw Theorem is intuitively clear though I have never seen a formal proof of it. Likewise the strategy stealing argument applies so the game is a first player win.

Like Nash/Hex, Gardner (1987, which reprints a 1961 article) publicized Bridg-It and named it Gale. From there on, however, the histories of Bridg-It and Hex diverge. In 1966 Oliver Gross of the Rand Corporation solved Bridg-It (see Gardner, 1995, pages 212–213). Fig. 10 shows Gross's solution, a pairing strategy. Red makes the indicated first move. Thereafter, when Blue's move covers one end of a dotted line, Red moves to cover the other end.

8. Shannon switching games

Bridg-It is a special case of a general class of games invented by Claude Shannon called Shannon switching games. Fig. 11 shows a board for a game that is "isomorphic" to Bridg-It, but with slightly different rules.

On his turn Player I, the *connector*, paints an edge, indicated by a solid red line, trying to make a path between vertices *A* and *B*. On her turn Player II, the *cutter*, removes an edge, as indicated by the dashed blue lines in the figure, trying to

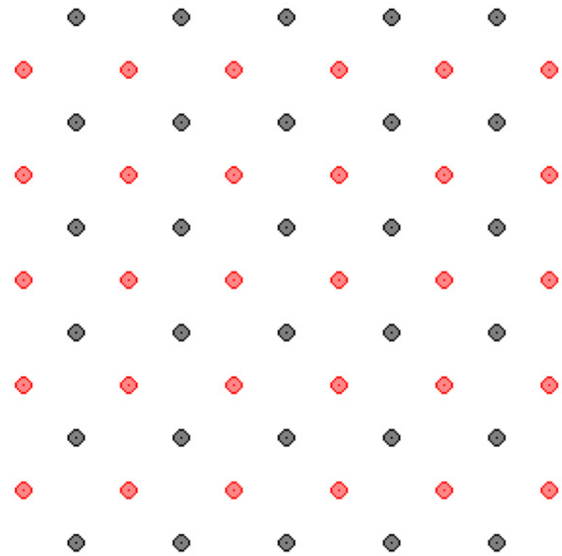

Fig. 9. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

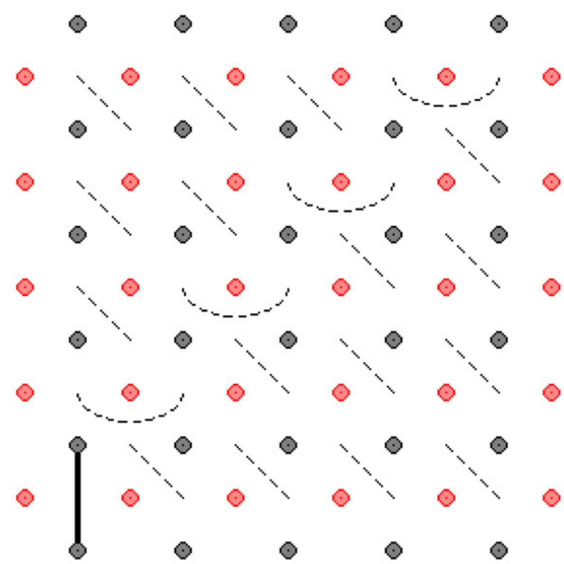

Fig. 10. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

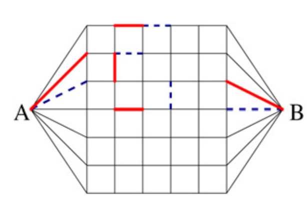

Fig. 11. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

prevent Player I from connecting A and B. Notice that the No-Draw Theorem holds from the definition of the game. The important point, however, is that the fact that the switching game can be played on a board consisting of any connected graph with a pair A and B of distinguished vertices. Fig. 12 gives four simple examples.

In a remarkable paper, Lehman (1964) (see also Edmonds (1965)) solved the general switching game by proving the following

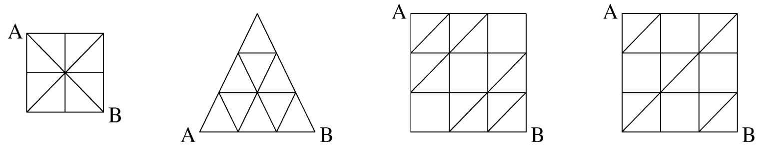

Fig. 12.

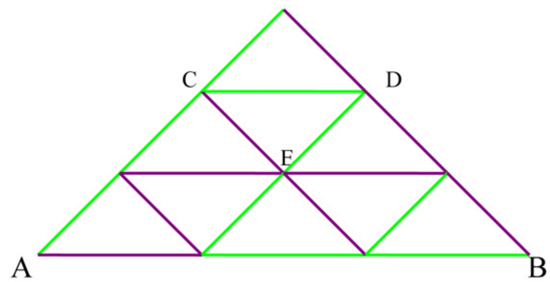

Fig. 13. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

Theorem. *A switching game graph G is a win for the connecting player playing second if and only if G has a subgraph containing A and B that admits two disjoint spanning trees.*

The condition is satisfied in all of the examples above. The proof that this condition is necessary for the connector to win is highly non-trivial. The proof of sufficiency, however, is easy to describe and is illustrated in Fig. 13.

Here is a winning strategy for the connector. The cutter plays first, removing some edge in one of the two trees, for example, the lighter green tree in Fig. 13 (edge CD). This disconnects the green tree into two connected subtrees. The connector then chooses and paints an edge in the darker purple tree (edge CE) that reconnects the two parts of the green tree. As the game progresses the painted edges are considered to belong to both the green and purple trees. Each time the cutter breaks up one of the trees the connector repairs it with an edge of the other tree. In the end, the painted edges will form a spanning tree for the entire subgraph. In the example the vertices A and B are connected after 8 moves.

Hamidoune and Las Vergnas (1988) show that if the conditions of the theorem hold, then cut wins the Misère version of the Shannon switching game.

9. Vertices vs. edges

These games share elegant formulations, the ability to vary the scale of the game, and the property that the first mover has a winning strategy.[7] It is straightforward to look at a completed game board and identify the winner. Most importantly, these games all satisfy the No-Draw Theorem. In spite of these similarities, there is an essential difference. In Bridg-It and Shannon's Switching Game, simple algorithms describe the winning strategy. The results of Gross and Lehman are beautiful and surprising, but once they are known, Bridg-It and Shannon are – like tic–tac-toe – no longer interesting to play. On the other hand, the existence of a winning strategy follows from non-constructive arguments for Nash, Milnor, and Shapley. It seems unlikely that there is a polynomial-time procedure that implements a winning strategy. These games remain interesting to play because while we know that the first mover should win, we do not know how to find a winning strategy. Is there a fundamental property that characterizes games in it is computationally easy to describe winning strategies from those which it is not? Lacking an answer to this question, we conclude with an observation. In the hard-to-solve games[8] of Nash, Milnor, and Shapley, the players can be viewed as alternately choosing vertices of a graph. In the easy-to-solve games of Bridg-It and Shannon, the players choose edges. The same dichotomy exists for a pair of basic problems in graph theory. It is hard to determine if a graph has a (Hamiltonian) tour that visits each vertex exactly once, but it is easy to determine if a graph has a (Eulerian) tour that visits each edge exactly once.

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[7] The simple strategy-stealing idea that proves that first movers have a winning strategy fails in Misère versions of the games.
[8] Even and Tardon (1976) show that the variation of the Shannon Switching Game in which players pick vertices is complete in polynomial space.

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