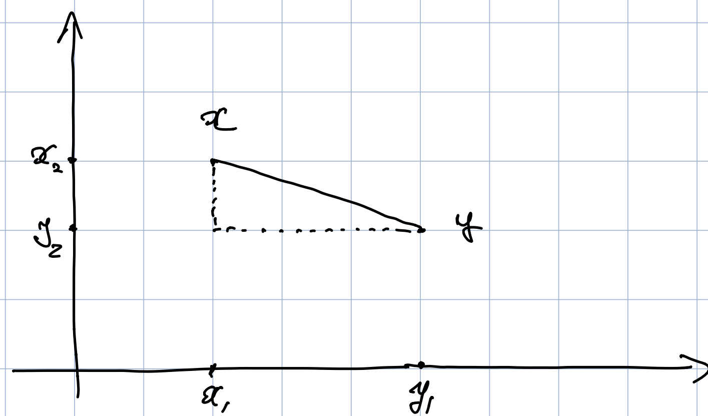
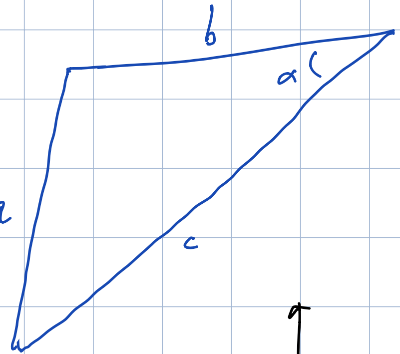
Geometria 4/3/20



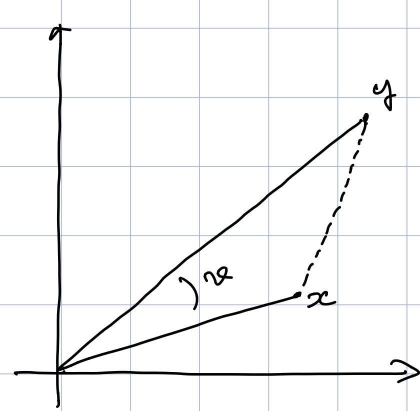
$$d(x,y) = \sqrt{(y_1-x_1)^2+(y_2-x_2)^2}$$

$$d(0,x) = \sqrt{x_1^2+x_2^2}$$

Distanza ⟶ angoli : Carnot



$$a^2 = b^2 + c^2 - 2bc \cdot \cos(\alpha)$$



$$\cos(\vartheta) = \frac{d(0,x)^2 + d(0,y)^2 - d(x,y)^2}{2\, d(0,x) \cdot d(0,y)}$$

$$= \frac{(x_1^2 + x_2^2) + (y_1^2 + y_2^2) - \left((y_1 - x_1)^2 + (y_2 - x_2)^2\right)}{2 \sqrt{x_1^2 + x_2^2} \cdot \sqrt{y_1^2 + y_2^2}}$$

$$= \frac{x_1 y_1 + x_2 y_2}{\sqrt{x_1^2 + x_2^2} \cdot \sqrt{y_1^2 + y_2^2}}$$

Def: chiamo <u>prodotto scalare canonico</u> di $\mathbb{R}^2$ la funzione

$$\langle \cdot | \cdot \rangle_{\mathbb{R}^2} : \mathbb{R}^2 \times \mathbb{R}^2 \longrightarrow \mathbb{R}$$

$$\langle x | y \rangle_{\mathbb{R}^2} = x_1 y_1 + x_2 y_2 .$$

funzione con argomento <u>due vettori</u> e valore <u>scalare</u>

Def: chiamo inoltre norma associata

$$\| \cdot \|_{\mathbb{R}^2} : \mathbb{R}^2 \longrightarrow \mathbb{R}$$

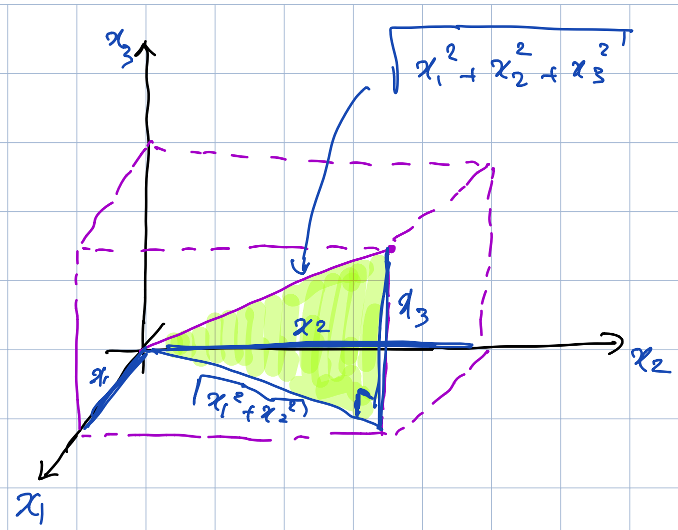
$$\| x \|_{\mathbb{R}^2} = \sqrt{\langle x | x \rangle_{\mathbb{R}^2}} \qquad \left(\langle x | x \rangle_{\mathbb{R}^2} = x_1^2 + x_2^2 \right)$$

<u>Dunque</u>:

$$d(0,x) = \| x \|_{\mathbb{R}^2}$$

$$d(x,y) = \| x - y \|_{\mathbb{R}^2}$$

$$\cos(\vartheta(x,y)) = \frac{\langle x | y \rangle_{\mathbb{R}^2}}{\| x \|_{\mathbb{R}^2} \cdot \| y \|_{\mathbb{R}^2}} .$$



In $\mathbb{R}^n$: prodotto scalare canonico

$$\langle \cdot | \cdot \rangle_{\mathbb{R}^n} : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$$

$$\langle x|y\rangle_{\mathbb{R}^n} = x_1y_1 + x_2y_2 + \cdots + x_ny_n$$

$$= \underbrace{\underbrace{{}^t \underbrace{x}_{n\times 1}}_{1\times n} \cdot \underbrace{y}_{n\times 1}}_{1\times 1}$$

$$(x_1\ x_2\ \cdots\ x_n)\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}$$

$$= x_1y_1 + x_2y_2 + \cdots + x_ny_n$$

norma associata:

$$\|x\|_{\mathbb{R}^n} = \sqrt{\langle x|x\rangle_{\mathbb{R}^n}}$$

$$d(x,y) = \|x-y\|_{\mathbb{R}^n}$$

$$\cos(\vartheta(x,y)) = \frac{\langle x|y\rangle_{\mathbb{R}^n}}{\|x\|_{\mathbb{R}^n} \cdot \|y\|_{\mathbb{R}^n}}.$$

Def: se V è spazio vettoriale su $\mathbb{R}$ chiamo una
$$f : V \times V \longrightarrow \mathbb{R}$$
su $\mathbb{C}$ è un po' diverso

- bilineare se è lineare in ciascun argomento fissato l'altro
$$f(\lambda_1 v_1 + \lambda_2 v_2, w) = \lambda_1 f(v_1, w) + \lambda_2 f(v_2, w) \quad \forall \ldots \text{ sinistra}$$
$$f(w, \lambda_1 v_1 + \lambda_2 v_2) = \lambda_1 f(w, v_1) + \lambda_2 f(w, v_2) \quad \forall \ldots \text{ destra}$$

- simmetrica se $f(w,v) = f(v,w) \quad \forall v,w$

- definita positiva se $f(v,v) > 0 \quad \forall v \neq 0$

- prodotto scalare se è bilineare, simmetrica, def. pos.

Prop: $\langle \cdot | \cdot \rangle_{\mathbb{R}^n}$ è un prod. scal.

Dim: $\langle x | y \rangle_{\mathbb{R}^n} = {}^t x \cdot y$

lin. sin.
$$\begin{aligned} \langle \lambda_1 x_1 + \lambda_2 x_2 | y \rangle &= {}^t(\lambda_1 x_1 + \lambda_2 x_2) \cdot y \\ &= (\lambda_1 \cdot {}^t x_1 + \lambda_2 \cdot {}^t x_2) \cdot y \\ &= \lambda_1 \cdot {}^t x_1 \cdot y + \lambda_2 \cdot {}^t x_2 \cdot y \\ &= \lambda_1 \langle x_1 | y \rangle + \lambda_2 \langle x_2 | y \rangle \end{aligned}$$

lin. dx
$$\begin{aligned} \langle y | \lambda_1 x_1 + \lambda_2 x_2 \rangle &= {}^t y \cdot (\lambda_1 x_1 + \lambda_2 x_2) \\ &= \lambda_1 {}^t y \, x_1 + \lambda_2 {}^t y \, x_2 \\ &= \lambda_1 \langle y | x_1 \rangle + \lambda_2 \langle y | x_2 \rangle \end{aligned}$$

simm: $\langle x|y\rangle = {}^t x y = x_1 y_1 + \dots + x_n y_n$

$= y_1 x_1 + \dots + y_n x_n = {}^t y \cdot x = \langle y|x\rangle$

def. pos: $\langle x|x\rangle = x_1^2 + \dots + x_2^2 > 0 \quad \forall x \neq 0.$ $\boxtimes$

Esempi: (1) $V = C^0([0,1], \mathbb{R})$

$$\langle x|y\rangle = \int_0^1 x(t)\cdot y(t)\, dt$$

lin. sin: $$\langle \alpha x + \beta y|z\rangle = \int_0^1 (\alpha x + \beta y)(t)\cdot z(t)\, dt$$

$$= \int_0^1 \big(\alpha\cdot x(t) + \beta\cdot y(t)\big)\cdot z(t)\, dt$$

$$= \int_0^1 \big(\alpha\cdot x(t)\cdot z(t) + \beta\cdot y(t)\cdot z(t)\big)\, dt$$

$$= \alpha\cdot \int_0^1 x(t)\cdot z(t)\, dt + \beta\cdot \int_0^1 y(t)\cdot z(t)\, dt$$

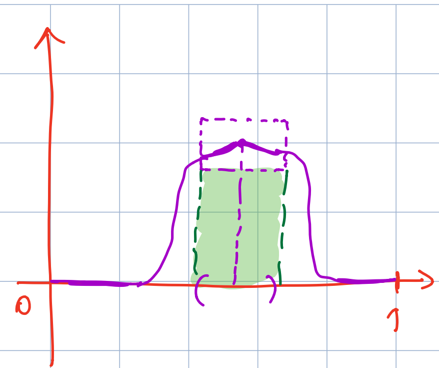
$$= \alpha\cdot\langle x|z\rangle + \beta\cdot\langle y|z$$

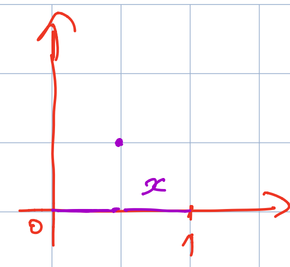
lin. dx. analoga [In generale: lin. sin + simm. $\Rightarrow$ lin. dx]

simm $\langle y|x\rangle = \int_0^1 y(t)\cdot x(t)\, dt = \int_0^1 x(t)\cdot y(t)\, dt = \langle x|y\rangle.$

def. pos. $\langle x|x\rangle = \int_0^1 x(t)^2\cdot dt > 0$ se $x \neq 0$

idea :





$$\int_0^1 x(t)\,dt = 0$$

(1.1) $V = C^0([-\pi,\pi],\mathbb{R})$ $\qquad \langle x|y\rangle = \int_{-\pi}^{\pi} x(t)\cdot y(t)\,dt$

$$\langle \cos|\sin\rangle = \int_{-\pi}^{\pi}\cos(t)\cdot\sin(t)\,dt = \frac{1}{2}\int_{-\pi}^{\pi}\sin(2t)\,dt = -\frac{1}{4}\cos(2t)\Big|_{-\pi}^{\pi} = 0$$

scoperto : cos, sin sono perpendicolari tra loro rispetto a $\langle .|.\rangle$
(uso <u>perpendicolare</u> in qualsiasi V con $\langle .|.\rangle$
e scrivo $v \perp w$, se $\langle v|w\rangle = 0$)

(2) $V = M_{m\times m}(\mathbb{R})$ $\qquad \langle A|B\rangle = tr\Big(\underbrace{\underbrace{{}^tA}_{m\times m}\cdot\underbrace{B}_{m\times m}}_{\underbrace{m\times m}_{m\times m}}\Big)$

lin. sin: $\langle \lambda A+\mu B|C\rangle = tr\left({}^t(\lambda A+\mu B)\cdot C\right)$

$$= tr\left((\lambda\cdot{}^tA+\mu\cdot{}^tB)\cdot C\right)$$
$$= tr\left(\lambda\cdot{}^tA\cdot C+\mu\cdot{}^tB\cdot C\right)$$
$$= \lambda\cdot tr({}^tA\,C)+\mu\cdot tr({}^tB\cdot C)$$
$$= \lambda\cdot\langle A|C\rangle+\mu\cdot\langle B|C\rangle$$

lin. dx analoga (e segue da simm).

[**Prop:** $X \in \mathcal{M}_{m\times n}$ $\quad Y \in \mathcal{M}_{n\times k}$ $\Rightarrow$

$$\underbrace{{}^t(\underbrace{\underbrace{X}_{m\times n}\cdot\underbrace{Y}_{n\times k}}_{m\times k})}_{k\times m} = \underbrace{\underbrace{{}^tY}_{\underbrace{}_{n\times k}\,k\times n}\cdot\underbrace{{}^tX}_{\underbrace{}_{m\times n}\,n\times m}}_{k\times m}$$

Dim:

$$\left({}^t(X\cdot Y)\right)_{ij} = (X\cdot Y)_{ji} = \sum_{l=1}^{n}(X)_{jl}\cdot(Y)_{li}$$

$$\left({}^tY\cdot{}^tX\right)_{ij} = \sum_{l=1}^{n}({}^tY)_{il}\cdot({}^tX)_{lj} = \sum_{l=1}^{n}(Y)_{li}\,(X)_{jl} \qquad \boxed{/\!/}$$]

Simm:

$$\langle B|A\rangle = tr\left({}^tB\cdot A\right) = tr\left({}^t({}^tB\cdot A)\right) = tr\left({}^tA\cdot B\right) = \langle A|B\rangle$$

def. pos:

$$\langle A|A\rangle = tr\left({}^tA\,A\right) = \sum_{i=1}^{n}\left({}^tA\cdot A\right)_{ii}$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{m}({}^tA)_{ij}\cdot(A)_{ji}$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{m}(A)_{ji}^{2}$$ = la somma dei quadrati di tutte le entrate di A.

> 0 per $A \neq 0$.

V sp. vett. su $\mathbb{R}$; $\quad f: V \times V \to \mathbb{R}$

- bil se lin. sin/dx
- simm
- def. pos. se $f(v,v) > 0 \quad \forall v \neq 0$.

Q: quali sono su $\mathbb{R}^n$?

Def: data $A \in M_{n\times n}(\mathbb{R})$ pongo $\langle \cdot | \cdot \rangle_A : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$

$$\langle x|y\rangle_A = {}^t x \cdot A \cdot y$$

Oss: per $A = I_n$ ottengo $\langle \cdot | \cdot \rangle_{\mathbb{R}^n}$.

$$\boxed{\underbrace{\underbrace{\underbrace{{}^t x}_{n\times 1}}_{1\times n} \cdot \underbrace{\underbrace{A}_{n\times n} \cdot \underbrace{y}_{n\times 1}}_{n\times 1}}_{1\times 1}}$$

Prop: •) $\langle \cdot | \cdot \rangle_A$ è bilineare

•) ogni bilineare su $\mathbb{R}^n$ è del tipo $\langle \cdot | \cdot \rangle_A$.

Dim: •) sin:
$$\begin{aligned}\langle \lambda x + \mu y | z \rangle_A &= {}^t(\lambda x + \mu y) \cdot A \cdot z \\ &= (\lambda\, {}^t x + \mu\, {}^t y) \cdot A \cdot z \\ &= \lambda\, {}^t x A z + \mu\, {}^t y A z \\ &= \lambda \cdot \langle x | z \rangle_A + \mu \cdot \langle y | z \rangle_A\end{aligned}$$

dx ... analoga

•) data $f: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ bilineare cerco A t.c. $f = \langle \cdot | \cdot \rangle_A$.

Se tale A esiste, osservo che

$$\langle e_i | e_j \rangle_A = {}^t e_i \cdot A \cdot e_j = \begin{array}{c}(0 \cdots 0\,1\,0 \cdots 0)\\ \quad\ \ i \end{array} \underbrace{\begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \dots & a_{nn} \end{pmatrix} \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix}}\begin{array}{c} \\ j \\ \\ \end{array}$$

$$= \begin{array}{c}(0 \dots 0\,1\,0 \dots 0)\\ \quad i \end{array} \begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{nj} \end{pmatrix} = a_{ij}$$

dunque per forza $a_{ij} = f(e_i, e_j)$.

Pertanto pongo $a_{ij} = f(e_i, e_j)$, $A = (a_{ij})_{\substack{i=1\dots n\\ j=1\dots n}}$

e verifico che $f = \langle \cdot | \cdot \rangle_A$, cioè:

- sono entrambe $\mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$ ✓
- $f(x,y) = \langle x | y \rangle_A \quad \forall x, y$:

$$f(x,y) = f\left(\sum_{i=1}^n x_i e_i , \sum_{j=1}^n y_j e_j \right)$$

$$\underset{\text{lin. sin.}}{=} \sum_{i=1}^n x_i \underbrace{f\left(e_i , \sum_{j=1}^n y_j e_j\right)}$$

$$\underset{\text{lin dx}}{=} \sum_{i=1}^n x_i \sum_{j=1}^n y_j \cdot \underbrace{f(e_i, e_j)}_{a_{ij}}$$

$$\underbrace{= \sum_{i=1}^n \underbrace{x_i}_{({}^t x)_{1i}} \cdot \underbrace{\sum_{j=1}^n a_{ij} \cdot y_j}_{(A \cdot y)_{i1}}}_{{}^t x \cdot A \cdot y} \quad ✓$$

$\boxtimes$

Prop: $\langle \cdot | \cdot \rangle_A$ è simm $\Leftrightarrow$ A simm

Dim: $\langle \cdot | \cdot \rangle_A$ simm $\Leftrightarrow$ $\langle y|x\rangle_A = \langle x|y\rangle_A \quad \forall x,y$

$\Leftrightarrow {}^t y \cdot A \cdot x = {}^t x \cdot A \cdot y \quad \forall x,y$

$\Leftrightarrow {}^t({}^t y \cdot A \cdot x) = {}^t x \cdot A \cdot y \quad \forall x,y$

$\Leftrightarrow {}^t x \cdot {}^t A \cdot y = {}^t x \cdot A \cdot y \quad \forall x,y$

$\Leftrightarrow {}^t x \cdot (A - {}^t A) \cdot y = 0 \quad \forall x,y$

$\Leftarrow$ A simm

$\Rightarrow$ ponendo $x = e_i$ $y = e_j$ trovo $(A - {}^t A)_{ij} = 0 \quad \forall i,j \Rightarrow A = {}^t A$. ■

Es: $A = \begin{pmatrix} 3 & -2 \\ -2 & 5 \end{pmatrix}$

è 2×2 simmetrica $\Rightarrow$ definisce $\langle \cdot | \cdot \rangle_A$ bilin. simm. su $\mathbb{R}^2$

$$\langle x|y\rangle_A = (x_1, x_2)\begin{pmatrix} 3 & -2 \\ -2 & 5 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = (x_1\ x_2)\begin{pmatrix} 3y_1 - 2y_2 \\ -2y_1 + 5y_2 \end{pmatrix}$$

$$= 3x_1y_1 - 2x_1y_2 - 2x_2y_1 + 5x_2y_2$$

È def. pos.? Dato $x \neq 0$ ho sempre $\langle x|x\rangle_A > 0$?

$$\langle x|x\rangle_A = 3x_1^2 - 4x_1x_2 + 5x_2^2$$
$$= 2x_1^2 + x_2^2 + x_1^2 - 4x_1x_2 + 4x_2^2$$
$$= 2x_1^2 + x_2^2 + (x_1 - 2x_2)^2 \qquad \underline{\text{sì}}$$

Dunque $\langle\cdot|\cdot\rangle_{\begin{pmatrix}3 & -2\\ -2 & 5\end{pmatrix}}$ è un prodotto scalare.

Esercizio: trovare tutti i vettori di $\mathbb{R}^2$ ortogonali a $\begin{pmatrix}4\\-3\end{pmatrix}$ e di norma 1 rispetto a $\langle\cdot|\cdot\rangle_A$.

[In $\mathbb{R}^2$ con $\langle\cdot|\cdot\rangle_{\mathbb{R}^2}$ sono $\pm\frac{1}{5}\begin{pmatrix}3\\4\end{pmatrix}$]

Voglio: $\begin{pmatrix}a\\b\end{pmatrix}$ che sia $\perp$ a $\begin{pmatrix}4\\-3\end{pmatrix}$ risp. a $\langle\cdot|\cdot\rangle_A$:

$$(a\ b)\begin{pmatrix}3 & -2\\ -2 & 5\end{pmatrix}\begin{pmatrix}4\\-3\end{pmatrix} = (a\ b)\begin{pmatrix}18\\-23\end{pmatrix} = 18a - 23b$$

$$\Rightarrow \begin{pmatrix}23\\18\end{pmatrix}$$

$$\left\|\begin{pmatrix}23\\18\end{pmatrix}\right\|_A^2 = (23\ 18)\begin{pmatrix}3 & -2\\ -2 & 5\end{pmatrix}\begin{pmatrix}23\\18\end{pmatrix} = (23\ 18)\begin{pmatrix}33\\64\end{pmatrix}$$

$$= 1911$$

$$\pm\frac{1}{\sqrt{1911}}\cdot\begin{pmatrix}23\\18\end{pmatrix}$$