Algebra Lineare 3/12/19

$E \subset V$ sttsp. aff se $E = x_0 + W$, W sttsp. vett.

$V = \mathbb{R}^n$ present. param: $E = \{x_0 + t_1 w_1 + \dots + t_k w_k : t_1, \dots, t_k \in \mathbb{R}\}$

present. cart.: $E = \{x : Ax = b\}$.

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Rette nel piano: $E \subset \mathbb{R}^2$ $\dim(E) = 1$

param: $E : \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + t \cdot \begin{pmatrix} a \\ b \end{pmatrix}$ a, b non entrambi 0

cart: $E : \alpha x + \beta y = \gamma$ α, β non entrambi 0 $(\alpha^2 + \beta^2 > 0)$

Passaggi:

param $\to$ cart: $bx - ay = bx_0 - ay_0$

cart $\to$ param: $\begin{pmatrix} \gamma/\alpha \\ 0 \end{pmatrix} + t \cdot \begin{pmatrix} \beta \\ -\alpha \end{pmatrix}$

uno dei due ok

$\begin{pmatrix} 0 \\ \gamma/\beta \end{pmatrix}$

sempre ok $\longrightarrow$ $\dfrac{\gamma}{\alpha^2+\beta^2}\begin{pmatrix}\alpha \\ \beta\end{pmatrix}$

Es: $7x-5y=2 \rightsquigarrow \begin{pmatrix}1\\1\end{pmatrix}+t\cdot\begin{pmatrix}5\\7\end{pmatrix}$

$$\begin{pmatrix}\sqrt{2}\\ -\frac{2}{5}+\frac{7}{5}\sqrt{2}\end{pmatrix}+t\cdot\begin{pmatrix}-55\pi\\ -77\pi\end{pmatrix}$$

ok

Es: $\begin{pmatrix}4\\-13\end{pmatrix}+t\begin{pmatrix}21\\-12\end{pmatrix} \rightsquigarrow 4x+7y=-75$

$(16-91)$

Rette in $\mathbb{R}^3$: $(\dim = 1)$

param: $\begin{pmatrix}x_0\\y_0\\z_0\end{pmatrix}+t\cdot\begin{pmatrix}a\\b\\c\end{pmatrix}$ $\quad a,b,c$ non tutti nulli

cart: $\begin{cases}\alpha_1 x+\beta_1 y+\gamma_1 z=\delta_1\\ \alpha_2 x+\beta_2 y+\gamma_2 z=\delta_2\end{cases}$ $\quad \operatorname{rank}\begin{pmatrix}\alpha_1 & \beta_1 & \gamma_1\\ \alpha_2 & \beta_2 & \gamma_2\end{pmatrix}=2$

[R-C : $\exists$ soluz]

cioè una delle 3 sottomatr. 2x2 ha det $\neq 0$

(In gen: $A\in M_{k\times m}$, $\operatorname{rank}(A)=k \Longrightarrow A\cdot x=b$ ha soluz.)

param $\rightsquigarrow$ cart.

$$\begin{cases} bx - ay = bx_0 - ay_0 \\ cx - az = cx_0 - az_0 \\ cy - bz = cy_0 - bz_0 \end{cases}$$

Sembrano 3 equaz. ma: $\det\begin{pmatrix} b & -a & 0 \\ c & 0 & -a \\ 0 & c & -b \end{pmatrix}$

$$= 0+0+0-0-abc+abc = 0$$

$\Rightarrow$ rank (incompleta) ≤ 2 ma ho sottomatr 2×2 con det a^2, b^2, c^2 almeno uno $\neq 0 \Rightarrow$ rank $= 2$

Ne scrivo 3 perché una posso sempre scartarla ma non so quale.

<u>Es</u>: $\begin{pmatrix} 7 \\ 3 \\ -1 \end{pmatrix} + t\cdot\begin{pmatrix} 4 \\ 6 \\ -9 \end{pmatrix} \rightsquigarrow \begin{cases} 3x - 2y = 15 \\ 3y + 2z = 7 \end{cases}$

$$\begin{matrix} 3 & -2 & 0 \\ 0 & 3 & 2 \end{matrix}$$

cart $\rightsquigarrow$ param:

$$\begin{cases} \alpha_1 x + \beta_1 y + \gamma_1 z = \delta_1 \\ \alpha_2 x + \beta_2 y + \gamma_2 z = \delta_2 \end{cases}$$

Uno dei det 2×2 di $\begin{pmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \end{pmatrix}$ è $\neq 0$.

<u>giacitura</u>: mi serve un vettore $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ che risolva il sistema

$$\begin{cases} \alpha_1 x + \beta_1 y + \gamma_1 z = 0 \\ \alpha_2 x + \beta_2 y + \gamma_2 z = 0. \end{cases}$$

Noto che

$$\det\begin{pmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \end{pmatrix} = \det\begin{pmatrix} \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \end{pmatrix} = 0$$

$$\alpha_1 \cdot \underbrace{\det\begin{pmatrix} \beta_1 & \gamma_1 \\ \beta_2 & \gamma_2 \end{pmatrix}}_{a} + \beta_1 \cdot \Big(- \underbrace{\det\begin{pmatrix} \alpha_1 & \gamma_1 \\ \alpha_2 & \gamma_2 \end{pmatrix}}_{b}\Big) + \gamma_1 \cdot \underbrace{\det\begin{pmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{pmatrix}}_{c} = 0$$

$$\alpha_2 \cdot \underbrace{\det\begin{pmatrix} \beta_1 & \gamma_1 \\ \beta_2 & \gamma_2 \end{pmatrix}}_{a} + \beta_2 \cdot \Big(- \underbrace{\det\begin{pmatrix} \alpha_1 & \gamma_1 \\ \alpha_2 & \gamma_2 \end{pmatrix}}_{b}\Big) + \gamma_2 \cdot \underbrace{\det\begin{pmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{pmatrix}}_{c} = 0$$

Dunque

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} \beta_1\gamma_2 - \beta_2\gamma_1 \\ -(\alpha_1\gamma_2 - \alpha_2\gamma_1) \\ \alpha_1\beta - \alpha_2\beta_1 \end{pmatrix}$$

Es:

$$\begin{cases} 7x - 5y + 2z = 0 \\ 4x + 9y - 3z = 0 \end{cases}$$

$$\begin{pmatrix} 7 & -5 & 2 \\ 4 & 9 & -3 \end{pmatrix} \qquad \text{Soluz:} \quad \begin{pmatrix} -3 \\ 29 \\ 83 \end{pmatrix}$$

check:

$-21 - 145 + 166 = 0$ ✓

$-12 + 261 - 249 = 0$ ✓

Soluz. part. di

$$\begin{cases} \alpha_1 x + \beta_1 y + \gamma_1 z = \delta_1 \\ \alpha_2 x + \beta_2 y + \gamma_2 z = \delta_2 \end{cases}$$

So che uno dei 3 determinanti della $\left(\vdots \; \vdots \; \vdots \right)$ togliendo

I, II o III col è $\neq 0$; corrisp. se pongo $x=0$, $y=0$ o $z=0$ trovo un sist. 2×2 con soluz. unica.

Es:
$$\begin{cases} 7x - 5y + 2z = 4 \\ 4x + 9y - 3z = 10 \end{cases}$$

$$\rightsquigarrow \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + t \cdot \begin{pmatrix} -3 \\ 29 \\ 83 \end{pmatrix}$$

oppure trovo soluz part. con $y=0$: $\begin{cases} 7x + 2z = 4 \\ 4x - 3z = 10 \end{cases}$

$$\frac{1}{29}\begin{pmatrix} 32 \\ 0 \\ -54 \end{pmatrix} + t \cdot \begin{pmatrix} +3\sqrt{2} \\ -29\sqrt{2} \\ -83\sqrt{2} \end{pmatrix}$$

anche lui ok

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Piani in $\mathbb{R}^3$ $(\dim = 2)$

param: $\begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} + t_1 \begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix} + t_2 \cdot \begin{pmatrix} a_2 \\ b_2 \\ c_2 \end{pmatrix}$ $\quad$ $\operatorname{rank}\begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \\ c_1 & c_2 \end{pmatrix} = 2$

cioè uno dei 3 det 2×2 è $\neq 0$.

cart: $\alpha x + \beta y + \gamma z = \delta$. $\quad$ α, β, γ non tutti zero.

cart $\rightsquigarrow$ par: $\underbrace{\begin{pmatrix} \delta/\alpha \\ 0 \\ 0 \end{pmatrix}}_{\text{oppure}} + \underbrace{t_1 \begin{pmatrix} \beta \\ -\alpha \\ 0 \end{pmatrix} + t_2 \begin{pmatrix} \gamma \\ 0 \\ -\alpha \end{pmatrix} + t_3 \cdot \begin{pmatrix} 0 \\ \gamma \\ -\beta \end{pmatrix}}_{o.}$

$$\begin{pmatrix}0\\ \delta/\beta\\ 0\end{pmatrix},\ \begin{pmatrix}0\\ 0\\ \delta/\gamma\end{pmatrix},\ \frac{\delta}{\alpha^2+\beta^2+\gamma^2}\begin{pmatrix}\alpha\\ \beta\\ \gamma\end{pmatrix}$$

sembrano 3 ma rank $\begin{pmatrix}\beta & \gamma & 0\\ -\alpha & 0 & \gamma\\ 0 & -\alpha & -\beta\end{pmatrix} = 2$

$\Rightarrow$ ne scarto uno.

par $\rightarrow$ cart
$$\begin{pmatrix}x_0\\ y_0\\ z_0\end{pmatrix} + t_1\begin{pmatrix}a_1\\ b_1\\ c_1\end{pmatrix} + t_2\cdot\begin{pmatrix}a_2\\ b_2\\ c_2\end{pmatrix}$$

$$0 = \det\begin{pmatrix}a_1 & a_1 & a_2\\ b_1 & b_1 & b_2\\ c_1 & c_1 & c_2\end{pmatrix} = \underbrace{\det\begin{pmatrix}b_1 & b_2\\ c_1 & c_2\end{pmatrix}}_{\alpha}\cdot a_1 - \underbrace{\det\begin{pmatrix}a_1 & a_2\\ c_1 & c_2\end{pmatrix}}_{\beta}\cdot b_1 + \underbrace{\det\begin{pmatrix}a_1 & a_2\\ b_1 & b_2\end{pmatrix}}_{\gamma}\cdot c_1$$

$$0 = \det\begin{pmatrix}a_2 & a_1 & a_2\\ b_2 & b_1 & b_2\\ c_2 & c_1 & c_2\end{pmatrix} = \underbrace{\det\begin{pmatrix}b_1 & b_2\\ c_1 & c_2\end{pmatrix}}_{\alpha}\cdot a_2 - \underbrace{\det\begin{pmatrix}a_1 & a_2\\ c_1 & c_2\end{pmatrix}}_{\beta}\cdot b_2 + \underbrace{\det\begin{pmatrix}a_1 & a_2\\ b_1 & b_2\end{pmatrix}}_{\gamma}\cdot c_2$$

$$(b_1c_2 - b_2c_1)\cdot x - (a_1c_2 - a_2c_1)\cdot y + (a_1b_2 - a_2b_1)\cdot z = \dots$$

stesso con x_0, y_0, z_0

Es: $\begin{pmatrix}11\\ -2\\ 7\end{pmatrix} + t\cdot\begin{pmatrix}2\\ 5\\ -9\end{pmatrix} + s\cdot\begin{pmatrix}4\\ -3\\ 6\end{pmatrix}$

$3x - 48y - 26 = -53$

$33 + 96 - 182$

check: $6 - 240 + 234 = 0$ ✓

$12 + 144 - 156 = 0$ ✓

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$\boxed{6.2.2}$ Data $A = (v_1, \dots, v_n)$ e B data trovare $\det(B)$ sapendo $\det(A)$:

(a) $n = 2$, $\det(A) = -3$, $B = (3v_1 - 2v_2, -5v_1 + 4v_2)$

I: $\det(B) = 3\det(v_1, -5v_1 + 4v_2) - 2\det(v_2, -5v_1 + 4v_2)$

$$= 3\cdot(-5)\underbrace{\det(v_1, v_1)}_{0} + 3\cdot 4\cdot \underbrace{\det(v_1, v_2)}_{-3}$$
$$-2\cdot(-5)\underbrace{\det(v_2, v_1)}_{+3} - 2\cdot 4 \underbrace{\det(v_2, v_2)}_{0}$$

$$= -36 + 10 = -6$$

II: $B = (3v_1 - 2v_2, -5v_1 + 4v_2) = \underbrace{(v_1, v_2)}_{A}\cdot\underbrace{\begin{pmatrix} 3 & -5 \\ -2 & 4 \end{pmatrix}}_{M}$

$$\det(B) = \det(A\cdot M) = \det(A)\cdot\det(M) = (-3)\cdot 2 = -6$$

(d) $n = 4$, $\det(A) = 2$, $B = (v_3 - v_1, v_1 - 3v_4, 2v_2 + v_4, v_3 - 2v_4)$

I: usando lin. nelle col. $\longrightarrow$ 16 addendi ma posso evitare di scrivere quelli nulli

$\det(B) =$

- $+1 \cdot \det(v_3,$
  - $+1 \cdot +1 \cdot \det(v_3, v_1$
    - $+1 \cdot +1 \cdot 2 \cdot \det(v_3, v_1, v_2$
      - 0
      - $+1 \cdot +1 \cdot 2 \cdot -2 \cdot \det(v_3, v_1, v_2, v_4)$
    - $+1 \cdot +1 \cdot +1 \cdot \det(v_3, v_1, v_4$
      - 0
      - 0
  - $+1 \cdot -3 \cdot \det(v_3, v_4$
    - $+1 \cdot -3 \cdot \det(v_3, v_4, v_2$
      - 0
      - 0
    - 0
- $-1 \cdot \det(v_1,$
  - $-1 \cdot +1 \cdot \det(v_1, v_1$
  - $-1 \cdot -3 \cdot \det(v_1, v_4$
    - $-1 \cdot -3 \cdot 2 \cdot \det(v_1, v_4, v_2$
      - $-1 \cdot -3 \cdot 2 \cdot +1 \cdot \det(v_1, v_4, v_2, v_3)$
      - 0
    - 0

$$= -4 \cdot \underbrace{\det(v_3, v_1, v_2, v_4)}_{\det(A)} + 6 \underbrace{\det(v_1, v_4, v_2, v_3)}_{\det(A)} = 4$$

3 1 2 4
1 3 2 4
1 2 3 4

1 4 2 3
1 2 4 3
1 2 3 4

$$B = (v_3 - v_1, v_1 - 3v_4, 2v_2 + v_4, v_3 - 2v_4)$$

II: $B = (v_1, v_2, v_3, v_4) \cdot \begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & -3 & 1 & -2 \end{pmatrix}$

$$\det(B) = \underbrace{\det(A)}_{=2} \cdot \det\begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & -3 & 1 & -2 \end{pmatrix} \quad 4$$

$$\det\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & -3 & 1 & -2 \end{pmatrix}$$

$$= (-1) \cdot (-2) \cdot (-2+3) = 2$$