Algebra Lineare 13/11/19

$\boxed{5.1.2}$ Trovare Ker/Im + formula dim.

$\boxed{m}$ $f : \underbrace{\mathbb{R}_{\leq d}[t] \to \mathbb{R}_{\leq d}[t]}_{\dim = d+1}, \quad f(p(t)) = p'(t)$

$$f\left(\sum_{k=0}^{\infty} a_k t^k\right) = \sum_{k=0}^{\infty} (k+1) a_{k+1} t^k$$

$\text{Ker}(f) = \text{Span}(1) \qquad \dim(\text{Ker}) = 1$

$\text{Im}(f) = \mathbb{R}_{\leq d-1}[t] \qquad \dim(\text{Im}) = d$

$q(t) \in \mathbb{R}_{\leq d-1}[t] \; ; \; q(t) = f(p(t)) \qquad p(t) = \int_0^t p(s)\,ds$

$\boxed{o}$ $f : \overbrace{\mathbb{R}^4}^{4} \to \overbrace{\mathbb{R}_{\leq 2}[t]}^{3}$

non iniettiva: $\dim(\text{Ker}) \geq 4 - 3 = 1$

$$f(\underline{x}) = \underbrace{(x_1 + 3x_2 - x_4)} + \underbrace{(x_2 + 2x_3 - x_4)}\,t + \underbrace{(x_1 - 6x_3 + 2x_4)}\,t^2$$

$$x \in \text{Ker}(f) \iff \begin{cases} x_4 = x_1 + 3x_2 \\ x_2 + 2x_3 - x_1 - 3x_2 = 0 \\ x_1 - 6x_3 + 2x_1 + 6x_2 = 0 \end{cases} \qquad \begin{cases} x_1 = -2x_2 + 2x_3 \\ -6x_2 + 6x_3 - 6x_3 + 6x_2 = 0 \\ x_4 = -2x_2 + 2x_3 + 3x_2 \end{cases}$$

$$\begin{cases} x_1 = -2x_2 + 2x_3 \\ x_4 = x_2 + 2x_3 \end{cases}$$

$$\text{Ker}(f) = \text{Span}\left(\begin{pmatrix} -2 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 1 \\ 2 \end{pmatrix} \right) \qquad \dim(\text{Ker} f) = 2$$

Formula dim: $\dim(\text{Im}(f)) = 4 - 2 = 2$: infatti:

$\mathrm{Im}(f) = \mathrm{Span}\left(1+t^2,\ 3+t,\ 2t-6t^2,\ -1-t+2t^2\right)$

$\checkmark$ $\checkmark$ $-6\mathrm{I}+2\mathrm{II}$ $\times$ $2\mathrm{I}-\mathrm{II}$ $\times$

OK : $\dim = 2$.

$\boxed{5.1.4}$ data $f: \underbrace{\{x\in\mathbb{R}^8 : x_3 - 2x_5 + 3x_8 = 0\}}_{\dim = 7} \to \underbrace{\mathbb{R}_{\leq 10}[t]}_{\dim = 11}$ lin

con $f(\underbrace{e_1+2e_3+e_5}_{v_1}) = f(\underbrace{3e_3-e_7-e_8}_{v_2}) = \underbrace{\sqrt{5}-7t^3}_{\neq 0}$; $\dim(\mathrm{Im}(f)) = \ldots ?$

$\sqrt{5}-7t^3 \in \mathrm{Im}(f)$

$0 \neq v_1 - v_2 \in \mathrm{Ker}$

$\Downarrow$

$\dim(\mathrm{Ker}) \geq 1$

$\Downarrow$

$\dim(\mathrm{Im}) \geq 1$

$\dim(\mathrm{Ker}) + \dim(\mathrm{Im}) = 7$

$\dim(\mathrm{Ker})$	$\dim(\mathrm{Im})$
0	7
1	6
2	5
3	4
4	3
5	2
6	1
7	0

R: $1 \leq \dim(\mathrm{Im}(f)) \leq 6$

Att: dovrei far vedere esempi per tutti i casi

$\boxed{5.1.5}$ data $f: \underbrace{\mathbb{R}^5}_{5} \to \underbrace{\mathbb{R}^8}_{8}$ lin. con $\dim(\mathrm{Ker}(f)) = 2$

e sttosp. $Z \subset \mathbb{R}^8$ t.c. $Z \cap \mathrm{Im}(f) = \{0\}$; $\dim(Z) = \ldots ?$

$\dim(\mathrm{Im}(f)) = 3$

$\underbrace{\dim(Z\cap \mathrm{Im}(f))}_{0} + \underbrace{\dim(Z+\mathrm{Im}(f))}_{3\leq ? \leq 8} = \underbrace{\dim(Z)}_{?} + \underbrace{\dim(\mathrm{Im}(f))}_{3}$

$R: 0 \leq \dim(Z) \leq 5$

Bisognerebbe fare esempi.....

$\boxed{5.3.3}$ Verificare $(A\cdot B)\cdot C = A\cdot(B\cdot C)$

$$\left(\begin{pmatrix} 4 & -1 \\ 2 & 3 \\ 1 & 5 \end{pmatrix} \cdot \begin{pmatrix} 2 & 3 \\ -4 & 7 \end{pmatrix}\right) \cdot \begin{pmatrix} 1 & 0 & -5 \\ 6 & -1 & 4 \end{pmatrix}$$

$$\begin{pmatrix} 12 & 5 \\ -8 & 27 \\ -18 & 38 \end{pmatrix} \begin{pmatrix} 1 & 0 & -5 \\ 6 & -1 & 4 \end{pmatrix} = \begin{pmatrix} 42 & -5 & -40 \\ 154 & -27 & 148 \\ 210 & -38 & 242 \end{pmatrix}$$

$$\begin{pmatrix} 4 & -1 \\ 2 & 3 \\ 1 & 5 \end{pmatrix} \cdot \left(\begin{pmatrix} 2 & 3 \\ -4 & 7 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & -5 \\ 6 & -1 & 4 \end{pmatrix}\right)$$

$$\begin{pmatrix} 4 & -1 \\ 2 & 3 \\ 1 & 5 \end{pmatrix} \cdot \begin{pmatrix} 20 & -3 & 2 \\ 38 & -7 & 48 \end{pmatrix} = \begin{pmatrix} 42 & -5 & -40 \\ 154 & * & * \\ * & * & * \end{pmatrix}$$

$\boxed{5.3.4}$ Esibire basi di $\mathrm{Ker}(A), \mathrm{Im}(A)$

$A \in M_{m\times n}$ vista come $A: \mathbb{R}^n \to \mathbb{R}^m$

$\boxed{a}$ $\begin{pmatrix} 3 & -2 & 8 \\ 4 & 1 & 6 \end{pmatrix}$

$\mathrm{Ker}(A)$: $\begin{cases} 3x - 2y + 8z = 0 \\ 4x + y + 6z = 0 \end{cases}$ $\begin{cases} y = -4x - 6z \\ 3x + 8x + 12z + 8z = 0 \end{cases}$

$\begin{cases} 11x + 20z = 0 \\ y = -4x - 6z \end{cases}$

$\mathrm{Ker}(A) = \mathrm{Span}\begin{pmatrix} 20 \\ -14 \\ -11 \end{pmatrix}$

$\dim = 1.$

$$\mathrm{Im}(A) = \mathrm{Span}\left(\begin{pmatrix}3\\4\end{pmatrix}\begin{pmatrix}-2\\1\end{pmatrix}\begin{pmatrix}8\\6\end{pmatrix}\right) \qquad \dim = 2.$$

$\boxed{c}$ $$A = \begin{pmatrix} 2 & -3 & 1 & -4 \\ -3 & 5 & -1 & 7 \\ -1 & 3 & 1 & 5 \end{pmatrix} \qquad \mathbb{R}^4 \to \mathbb{R}^3$$

$$\mathrm{Ker}(A): \begin{cases} x = 3y + z + 5w \\ 6y + 2z + 10w - 3y + z - 4w = 0 \\ -9y - 3z - 15w + 5y - z + 7w = 0 \end{cases}$$

$$\begin{cases} 3y + 3z + 6w = 0 \\ -4y - 4z - 8w = 0 \\ x = 3y + z + 5w \end{cases} \qquad \begin{cases} y = -z - 2w \\ x = -2z - w \end{cases}$$

$$\mathrm{Ker}(A) = \mathrm{Span}\left(\begin{pmatrix}-2\\-1\\1\\0\end{pmatrix}\begin{pmatrix}-1\\-2\\0\\1\end{pmatrix}\right) \qquad \dim = 2.$$

$$\mathrm{Im}(A) = \mathrm{Span}\left(\begin{pmatrix}2\\-3\\-1\end{pmatrix}\begin{pmatrix}-3\\5\\3\end{pmatrix}\begin{pmatrix}1\\-1\\1\end{pmatrix}\begin{pmatrix}-4\\7\\5\end{pmatrix}\right) \qquad \dim = 2$$

$2 \cdot I + II$ $\qquad$ ...

$\boxed{5.3.5}$ Fissate $A \in M_{m\times n}$ $\quad B \in M_{k\times h}$

$f: M_{n\times k} \to M_{m\times h}$ $\quad f(X) = A\cdot X\cdot B$.

Provare che f lin. e provare che $\mathrm{Ker}(f) =$

$= \{X : X(\mathrm{Im}(B)) \subset \mathrm{Ker}(A)\}$

Def ok: $$\underset{m\times n}{A} \cdot \underset{n\times k}{X} \cdot \underset{k\times h}{B}$$

Lin : $\overbrace{\left(A\cdot(k_1X_1+k_2X_2)\right)\cdot B}^{m\times h} = \left(k_1\cdot A\cdot X_1 + k_2\cdot A\cdot X_2\right)\cdot B$

$$= k_1\cdot A\cdot X_1\cdot B + k_2\cdot A\cdot X_2\cdot B$$

$$\mathrm{Ker}(f) = \{X\in M_{m\times k} : f(X) = 0 \in M_{m\times h}\}$$

$$= \{X\in M_{m\times k} : A\cdot X\cdot B = 0 \in M_{m\times h}\}$$

$$= \{X\in M_{m\times k} : (A\cdot X\cdot B : \mathbb{R}^m \longrightarrow \mathbb{R}^h) = 0\}$$

$$= \{X\in M_{m\times k} : A\cdot X\cdot (B\cdot u) = 0 \quad \forall u\in \mathbb{R}^m\}$$

$$= \{X\in M_{m\times k} : A(X(w)) = 0 \quad \forall w\in \mathrm{Im}(B)\}$$

$$= \{X\in M_{m\times k} : X(w)\in \mathrm{Ker}(A) \quad \forall w\in \mathrm{Im}(B)\}$$

$$= \{X\in M_{m\times k} : X(\mathrm{Im}(B)) \subset \mathrm{Ker}(A)\}.$$