Knot theory 2/5/2019

Titolo nota 02/05/2019

Recall that the Alexander polynomial comes with an ambiguity: $\Delta_L(t)$ is defined only up to $\pm t^n$.

However, there exists a well-defined Conway normalization of Δ_L.
When L is a knot, it is the unique representative of Δ_K s.t.

1. $\Delta_K(1) = 1$ (not $= -1$)
2. $\Delta_K(t) = \Delta_K(t^{-1})$

For example, $\Delta_{trefoil}(t) = t^2 - t + 1$ (hence, $|\det(\text{trefoil})| = 3$), and the Conway normalization of $\Delta_{trefoil}$ is

$$t^{-1} - 1 + t = \Delta_{trefoil}(t)$$

Exercise: One obtains the Conway normalization of Δ_K (K knot) by setting

$$\Delta_K(t) = \det\left(t^{\frac{1}{2}} A - t^{-\frac{1}{2}} A^t\right),$$

where A is a Seifert matrix for some Seifert surface for K.

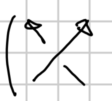
The formula given in the Exercise also gives the well-defined Conway normalization for links. This means that

$$\det\left(t^{\frac{1}{2}}A - t^{-\frac{1}{2}}A^t\right)$$

is always independent of the choice of A.
However, this normalization lies in $\mathbb{Z}[t^{\frac{1}{2}}, t^{-\frac{1}{2}}]$ (for knots, it is in $\mathbb{Z}[t, t^{-1}]$).
It is still symmetric in $t^{\frac{1}{2}}$, $t^{-\frac{1}{2}}$, but the determination of the sign cannot be obtained by looking at $\Delta_L(1)$, since $\Delta_L(1)=0$ if L is not a knot.

We are not proving this because it follows (also) from the theory of the HOMFLY polynomial.

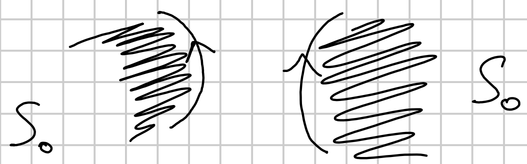
Theorem: The Conway normalisation of Δ satisfies:

1. $\Delta_{\text{unknot}}(t) = 1$
2. $\Delta($  $) - \Delta($  $) = (t^{-\frac{1}{2}} - t^{\frac{1}{2}})\,\Delta($  $)$

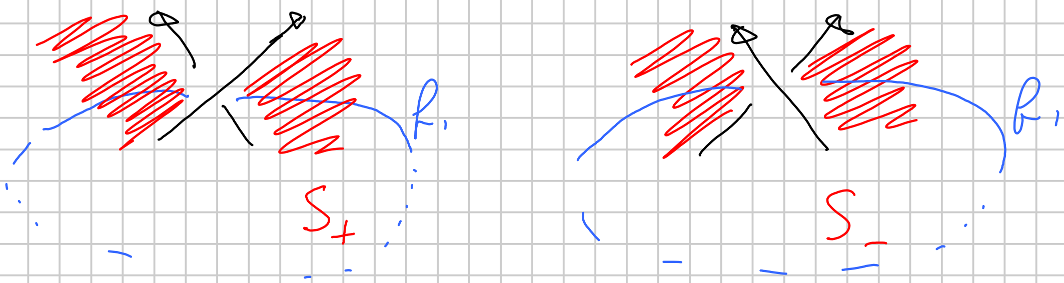
From 1 and 2, it follows that $\Delta(L)=0$ for L the trivial link with $n \geq 2$ components.

Proof: We need to prove (2).

Let S_0 be a Seifert surface for L_0, constructed via the Seifert algorithm.



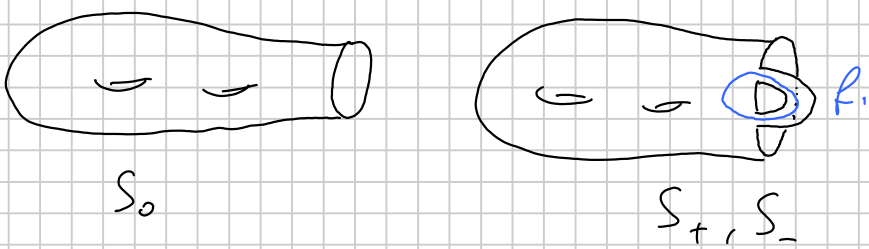
We can construct Seifert surfaces for L_+, L_- by adding bands:



(S_+ and S_- are indeed oriented).

If $\ell_2, \dots, \ell_n$ is a basis for $H_1(S_0)$ we obtain bases for $H_1(S_+)$, $H_1(S_-)$ just by adding a loop ℓ_1 as in the picture.

In fact, abstractly the situation is as follows:



(this is in the case of a knot; the case of links is similar).

By computing the Seifert matrices associated to S_0, S_+, S_-, if A_0 is the matrix associated to S_0 (w.r.t. $f_2, \dots, f_n$), and A_+, A_- are the ones associated to S_+, S_- (w.r.t. $f_1, f_2 \dots, f_n$), then

$$A_+ = \left(\begin{array}{c|c} N & a \\ \hline b & A_0 \end{array}\right) \qquad A_- = \left(\begin{array}{c|c} N-1 & a \\ \hline b & A_0 \end{array}\right)$$

Now

$$\Delta_{L_+} = \det\left(t^{\frac{1}{2}} A_+ - t^{-\frac{1}{2}} A_+^t\right) =$$

$$= \det \left(\begin{array}{c|c} (t^{\frac{1}{2}} - t^{-\frac{1}{2}}) . N & t^{\frac{1}{2}} a - t^{-\frac{1}{2}} b^t \\ \hline t^{\frac{1}{2}} b - t^{-\frac{1}{2}} a^t & t^{\frac{1}{2}} A_0 - t^{-\frac{1}{2}} A_0^t \end{array}\right)$$

$$\Delta_{L_-} = \det \left(\begin{array}{c|c} (t^{\frac{1}{2}} - t^{-\frac{1}{2}}) (N-1) & \text{the same as for } L_+ \\ \hline \text{the same as for } L_+ & \text{the same as for } L_+ \end{array}\right)$$

By multinearity of the determinant (used on the 1st column) we have

$$\Delta_{L_+} - \Delta_{L_-} = \det \left(\begin{array}{c|c} t^{\frac{1}{2}} - t^{-\frac{1}{2}} & * \\ \hline 0 & t^{\frac{1}{2}} A_0 - t^{-\frac{1}{2}} A_0^t \end{array}\right) =$$

$$= (t^{\frac{1}{2}} - t^{-\frac{1}{2}}) \det(t^{\frac{1}{2}} A_0 - t^{-\frac{1}{2}} A_0^t) =$$

$$= (t^{\frac{1}{2}} - t^{-\frac{1}{2}}) \cdot \Delta_{L_0}.$$

THE HOMFLY POLYNOMIAL

H, O, M, F, L, Y are the initials of the mathematicians who discovered the polynomial.
M + L and independently H + O + F + Y.

- Jones discovered the Jones polynomial via a very deep theory involving representations, mathematical physics, . . .
- Kauffman discovered a combinatorial (skein) formula for the Jones polynomial
- Right after that, many mathematicians worked on looking for the "most general" polynomial invariant coming from skein computations.

Theorem: There exists a well-defined map

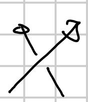
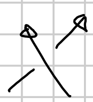
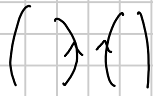
$$P : \{\text{oriented links in } S^3\}/_{\text{isotopy}} \longrightarrow \mathbb{Z}[\ell^{\pm 1}, m^{\pm 1}]$$

such that:

① $P(\text{unknot}) = 1$

② $\ell P(L_+) + \ell^{-1} P(L_-) + m P(L_0) = 0$

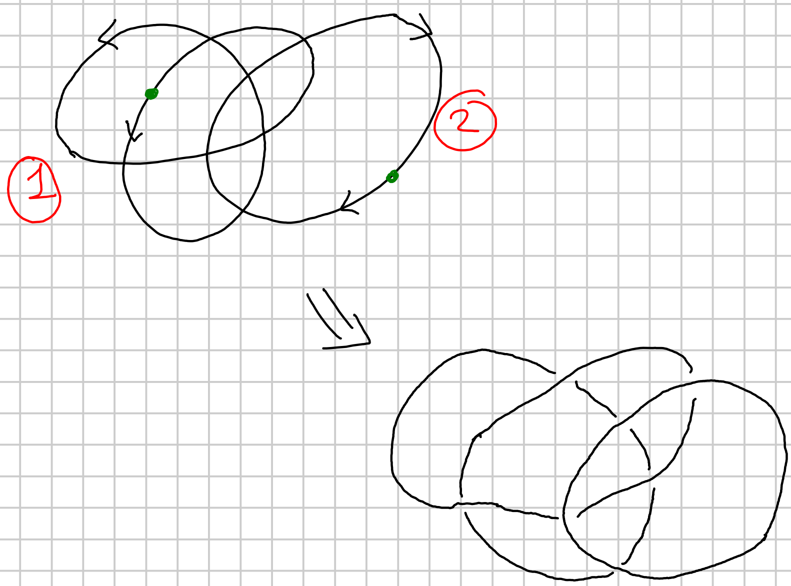
i.e.

$\ell P($  $) + \ell^{-1} P($  $) + m P($  $) = 0$

Moreover, conditions 1 and 2 determine P on all links (see the proof of "uniqueness" for the Alexander-Conway polynomial).

PROOF

Let D be a diagram. Choose an ordering of the components of D (by which I mean the components corresponding to the components of the link it represents). Choose also a point (not on a crossing) on each component of D. We say that D is ascending (w.r.t. these choices) if by starting at the chosen basepoints (in the order prescribed by the ordering of the components), every crossing is first visited by an underpass.



Fact: Any link represented by an ascending diagram is the unlink $\bigcirc \; \bigcirc \; \bigcirc \; \bigcirc$.

Let $\mathbb{D}_n$ be the set of diagrams with at most n crossings, up to equivalence (i.e. orientation preserving homeos of the plane, or isotopy of the plane).

Set
$$\mu = -\frac{\ell+\ell^{-1}}{m} = -m^{-1}(\ell+\ell^{-1})$$

We will prove by induction on n the following:

Claim: We can define $P : \mathbb{D}_n \to \mathbb{Z}[\ell^{\pm 1}, m^{\pm 1}]$ so that:

1. $\ell P(D_+) + \ell^{-1} P(D_-) + m P(D_0) = 0 \qquad (*)$
   $\forall D_+, D_-, D_0 \in \mathbb{D}_n$
2. P is invariant w.r.t. Reidemeister moves that involve diagrams with at most n crossings.
3. If D is ascending (w.r.t. any choice) then $P(D) = \mu^{(\#D - 1)}$

For $n=0$, we only have trivial diagrams of trivial links. They are all ascending, and on each of them we define $P(D) = \mu^{(\#D-1)}$.
Hence 3 is satisfied, and 1, 2 are empty.

So let $n \geq 1$. We define P as follows.

Choose orderings of the components and basepoints. Let D be given, and let D_α be the ascending diagram corresponding to D. Thus D is obtained from D_α by changing some crossings. I order these crossings, and at each of them I use the skein relation $(*)$ to pass from D_α to D, knowing that $P(D_\alpha) = \mu^{(\#D - 1)}$, and the value of P on any diagram with $< n$ crossings.

We first check that this definition does not depend on the ordering of crossings I need to change to pass from D_α to D. It is sufficient to check that I can switch two consecutive crossings. For simplicity, suppose that these crossings are positive in D and negative in D_α.

We denoted the involved diagrams by D_{++} (the final one), D_{--} (the initial one) D_{+-}, D_{+0}, D_{-+}, D_{--}, D_{0+}, D_{0-}, D_{00}.

If I pass from D to D_α by looking first at the first crossing, I get

$$\ell P(D_{++}) + \ell^{-1} P(D_{-+}) + m P(D_{0+}) = 0$$

$$P(D_{++}) = \ell^{-1}\left(-\ell^{-1}P(D_{-+}) - m\,P(D_{0+})\right)$$

Now I look at the second crossing to compute $P(D_{-+})$, and I get

$$\ell P(D_{-+}) + \ell^{-1}P(D_{--}) + m\,P(D_{-0}) = 0 \implies$$

$$P(D_{-+}) = \ell^{-1}\left(-\ell^{-1}P(D_{--}) - m\,P(D_{-0})\right)$$

By replacing this equality in the expression above,

$$P(D_{++}) = \ell^{-1}\left(-\ell^{-2}\left(-\ell^{-1}P(D_{--}) - m\,P(D_{-0})\right) - m\,P(D_{0+})\right)$$

By switching the order of the operations, I get

$$P(D_{++}) = \ell^{-1}\left(-\ell^{-2}\left(-\ell^{-1}P(D_{--}) - m\,P(D_{0-})\right) - m\,P(D_{+0})\right)$$

Hence I must prove

$$+\ell^{-2}\cancel{m}P(D_{-0}) - \cancel{m}P(D_{0+}) = +\ell^{-2}\cancel{m}P(D_{0-}) - \cancel{m}P(D_{+0})$$

i.e.

$$P(D_{+0}) + \ell^{-2}P(D_{-0}) = P(D_{0+}) + \ell^{-2}P(D_{0-})$$

i.e.

$$\ell P(D_{+0}) + \ell^{-1}P(D_{-0}) = \ell P(D_{0+}) + \ell^{-1}P(D_{0-}).$$

Now $D_{+0}, D_{-0}, D_{0+}, D_{0-}, D_{00}$ all have $< n$ crossing, so I may apply induction to apply the skein formula, which gives that both sides of the equality are equal to

$-m\,P(D_{00})$

Next things to check:

1- This definition of P respects the skein formula
2- It does not depend by the choice of basepoints
3- It is invariant w.r.t. Reidemeister moves
4- It does not depend on the choice of the ordering of the components
5- It respects the normalisation on every ascending diagram.

1 is almost obvious since we proved the independence of the definition of P from the order on crossings. It holds by definition, because if D, D' are the diagrams we are looking at to check the skein formula, then at the crossing involved one of them coincides with D_α (say D'). Then, in order to compute $P(D)$ I may start by applying the skein relation at the crossing I am looking at, hence the skein relation holds by definition.