Knot theory 30/4/2019

Polynomial invariants of links

- Alexander polynomial

  Strength: – well related to algebraic topological invariants (genus, Seifert surfaces, coverings, fundamental group of the link complement)

  Weakness: – it is not the most powerful to distinguish knots (e.g., many non-trivial knots have trivial Alexander polynomial; the Alexander polynomial cannot distinguish the right-hand trefoil from the left-hand trefoil).

Alexander polynomial and colorings.

An n-coloring of a diagram is a coloring of each overarc by an element in $\mathbb{Z}/_{n\mathbb{Z}} = \mathbb{Z}_n$ such that, at each crossing,

$$a+b=2c \pmod n$$

Colourings in $\mathbb{Z}/3\mathbb{Z}$ are just the 3-colourings introduced at the very beginning of the course.

Def.: Let K be a knot with Alexander polynomial $\Delta(K)$. The determinant of K is

$$\det K = \left|\Delta(K)(-1)\right| \in \mathbb{Z}$$

(it is well-defined).

Theorem: Let K be a knot, and let $p \in \mathbb{Z}$ be a prime number. Then K admits a non-constant p-coloring $\Longleftrightarrow$ p divides $\det K$.

Corollary: p-colourings are controlled by the Alexander polynomial (indeed, the exact number of p-colourings can be deduced from the collection of all the Alexander ideals).

Proof: p-colorings are just the solutions of an $n \times n$ square linear system over $\mathbb{Z}_p$, where $n = \#$ overarcs $= \#$ crossings of a fixed diagram of K. If x_i is the variable associated to the i-th arc, and at a crossing the i-th arc is overcrossing the j-th and k-th arcs, then the corresponding equation is

$$2x_i - x_j - x_k = 0$$

Since p is prime, $\mathbb{Z}_p$ is a field. We always have a 1-dimensional space of solutions, generated by $(1, 1, \dots, 1)$. Admitting a non-constant coloring is equivalent to asking the rank of the system to be $\leq n-2$, i.e. every $(n-1)\times(n-1)$ minor
determinant of the
should vanish.

From the Wirtinger presentation for $\pi_1(E(K))$ we get a presentation matrix for the unreduced Alexander module of K. To the crossing

$\Longrightarrow x_i x_j x_i^{-1} x_k^{-1} = 1$

$\Longrightarrow$ gives the following relation for the unreduced Alexander module:

$$(1-t)\tilde{s}_i + t\tilde{s}_j - \tilde{s}_k$$

If we set $t=-1$, this gives

$$2\tilde{s}_i - \tilde{s}_j - \tilde{s}_k = 0$$

Now, in order to compute $\Delta(K)(t)$, we need to remove one random row and one random column of the presentation matrix for the unreduced Alexander module, and take the determinant.

We have showed that by setting $t=-1$, the square presentation matrix for the Alexander polynomial corresponding to the Wirtinger presentation becomes the square matrix of the coefficients of the system defining p-colorings.

(In fact, to its transpose).

Therefore, the linear system has rank $\leq n-2$

$\Longleftrightarrow$ $\Delta(K)(-1)$ is $=0$ in $\mathbb{Z}_p$, i.e.

$p \mid \Delta(K)(-1) \Longleftrightarrow p \mid \det K$

Infinitely many knots with trivial Alexander polynomial

Recall that, if C is a knot, and $K \subseteq T$ is a knot in the standard solid torus T, then we can define a knot K' by embedding T over a regular neighborhood T' of C in S^3,

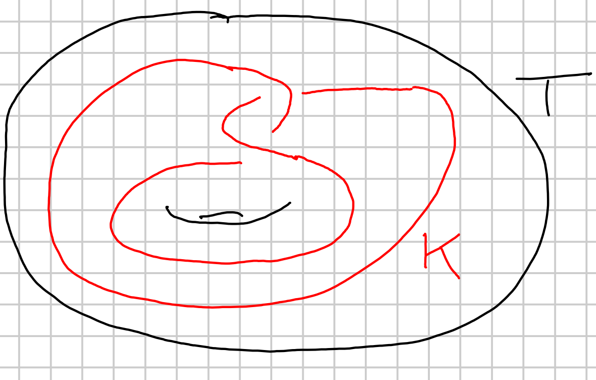
$$e: T \hookrightarrow T'$$

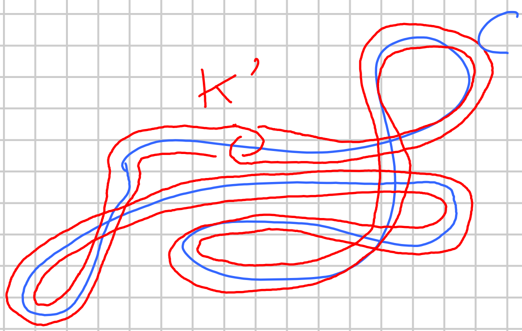
and setting

$$K' = e(K).$$

We require e to send the longitude on ∂T to the longitude in $\partial T'$.

K' is a satellite of C (with pattern K).





Prop.: Suppose K is null-homologous in T (as in the picture). Then

$$\Delta(K') = \Delta(K)$$

Corollary: If K is as in the picture, then $\Delta(K) = 1$, hence $\Delta(K') = 1$.
In this case, K' is called the Whitehead double of C. We proved that a satellite of a non-trivial knot is non-trivial, hence there are lots of examples of non-trivial knots with $\Delta = 1$.

Lemma 1: $e : T \to T'$ longitude-preserving embedding of the standard solid torus T, let γ_1, γ_2 be disjoint simple oriented loops in T.
Then $lk(\gamma_1, \gamma_2) = lk(e(\gamma_1), e(\gamma_2))$.

Proof: Let $N = N(\gamma_1)$ be a regular neighborhood of γ_1 inside T. By looking at the Mayer-Vietoris sequence for

$$T \setminus N, \quad N, \quad T \quad \text{we get}$$

$$\underset{\substack{\| \\ 0}}{H_2(T)} \to \underset{\substack{\| \\ \mathbb{Z}^2}}{H_1(\partial N)} \to \underset{\substack{\| \\ \mathbb{Z}}}{H_1(N)} \oplus \underset{\substack{\uparrow \\ \cong \mathbb{Z}^2}}{H_1(T \setminus \mathring{N})} \to \underset{\substack{\| \\ \mathbb{Z}}}{H_1(T)} \to H_0(\partial N)$$

$$H_1(T) \to 0 \to H_0(\partial N)$$

$\Rightarrow H_1(T \setminus \mathring{N})$ is generated by a meridian m of γ_1 and the longitude ℓ of the core of T.
Therefore, $[\gamma_2] = lk(\gamma_1, \gamma_2) \cdot [m] + n \cdot [\ell]$

in $H_1(T \setminus \mathring{N})$

(because $[\ell] = 0$ in $H_1(S^3 \setminus N)$).

Therefore,

$$[e(\gamma_2)] = lk(\gamma_1, \gamma_2)\,[e(m)] + n \cdot [e(\ell)]$$

in $H_1(T' \setminus N(e(\gamma_1)))$

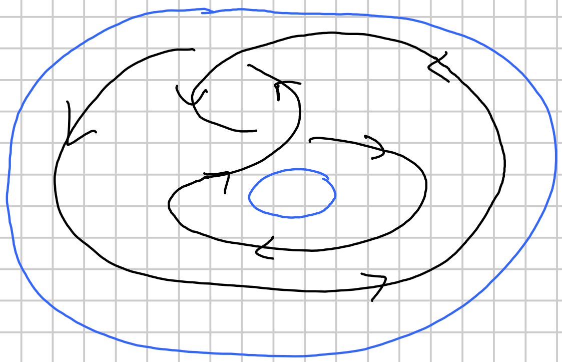
Since $e(\ell)$ is a longitude for T', $[e(\ell)] = 0$ in $H_1(S^3 \setminus e(\gamma_1))$. $\Longrightarrow$

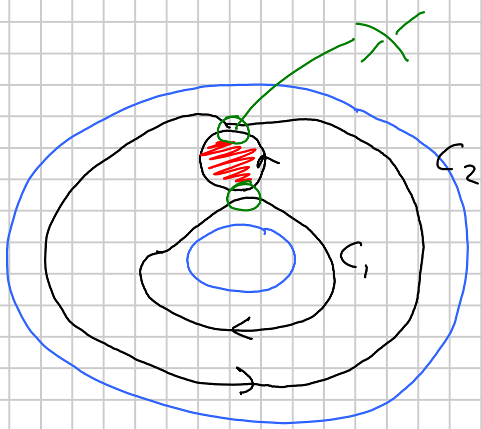
$$[e(\gamma_2)] = lk(\gamma_1, \gamma_2)\,[e(m)] \quad \text{in} \quad H_1(S^3 \setminus e(\gamma_1))$$

Since $e(m)$ is a meridian of $e(\gamma_1)$, we conclude that $lk(e(\gamma_1), e(\gamma_2)) = lk(\gamma_1, \gamma_2)$.

Lemma 2: $K \subseteq T$ null-homologous. Then there exists a Seifert surface S for K contained in T.

Proof: The Seifert algorithm applied to a diagram of the knot produces some discs that could intersect ∂T.





C_1 and C_2 bound discs that are NOT contained in T. But each such disc intersect ∂T in a parallel copy of the longitude of T. The union of these copies is homologous to the knot. Since the knot is null-homologous, this implies that we have $2n$ longitudes, n of which are positively oriented, and n of which are negatively oriented. I can consider a pair of consecutive longitudes with opposite sign, and modify the Seifert surface by removing the corresponding discs outside T and by adding an annulus on ∂T (or just inside). The condition on signs ensures that the resulting surface is orientable. We can go on till we get rid of all the discs outside T.

Proof of the Theorem: let $K \subseteq T$ be null-homologous, and let $S \subseteq T$ be a Seifert surface for K. If $e: T \longrightarrow T'$ is as in the definition of satellite knot, $e(S)$ is a Seifert surface for $K' = e(K)$. A basis for $H_1(S)$ is taken by e to a basis for $H_1(e(S))$

and the corresponding Seifert forms coincide because e preserves the linking numbers of curves in T and T' (Lemma 1).

Hence $\Delta(K) = \Delta(K')$.

Skein relations

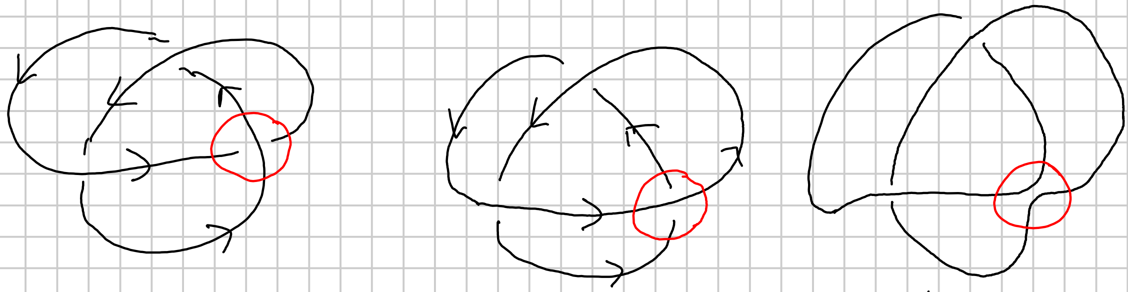
Theorem: The (Conway normalisation of) the Alexander polynomial is completely characterised by the following properties:

① $\Delta(\bigcirc) = \Delta(\text{unknot}) = 1$

② $\Delta(L_+) - \Delta(L_-) = \left(t^{-\frac{1}{2}} - t^{\frac{1}{2}}\right)\Delta(L_0)$

$$\Delta(L_+) - \Delta(L_-) = \left(t^{-\frac{1}{2}} - t^{\frac{1}{2}}\right)\Delta(L_0)$$

(L_+, L_-, L_0 are links defined by diagrams which coincide outside a small neighbourhood of a crossing of L_+ and L_-)



L_- $\qquad$ L_+ $\qquad$ L_0

Uniqueness: Recall that, if D is a diagram, the unknotting number of D is the least number of crossing changes we need to obtain a diagram of the trivial link.

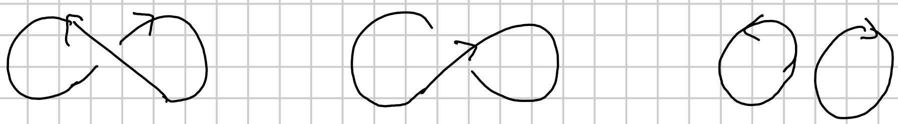
To each diagram we associate

$$(\# \text{ crossings of } D, \ \text{unknotting number of } D) \in \mathbb{N}^2$$

(of course each such pair is of the form (m,n), $n \leq m$). We order diagrams in lexicographical order. By operating on a suitably chosen crossing of D, we can replace one of D_+, D_-, D_0 with the other two diagrams by going strictly down in the lexicographical order. In a finite number of steps we reduce the computation of the polynomial to a certain number of trivial knots / links, for which we know

$$\Delta(\circ) = 1, \qquad \Delta(\text{unlink with} \geq 2 \text{ components}) = 0$$

deduced from the skein relation



L_+ $\qquad$ L_- $\qquad$ L_0

By assuming Δ is a link invariant

$\Delta(L_+) = \Delta(L_-) = 1$ (property ①)

$0 = 1 - 1 = \Delta(L_+) - \Delta(L_-) = (-t^{\frac{1}{2}} + t^{-\frac{1}{2}})\Delta(L_0)$

$\Rightarrow \Delta(L_0) = 0$