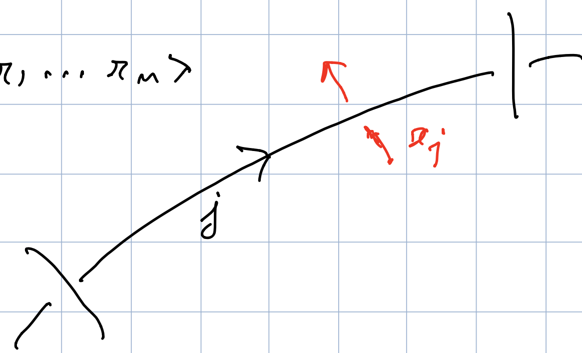
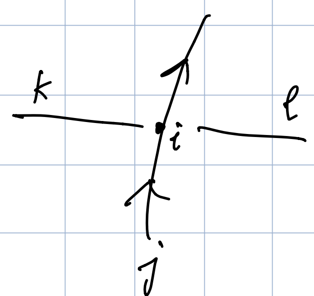
Teoria dei Nodi 4/4/19

$\pi_1(E(K)) = \langle x_1 \dots x_n \mid r_1 \dots r_n \rangle$

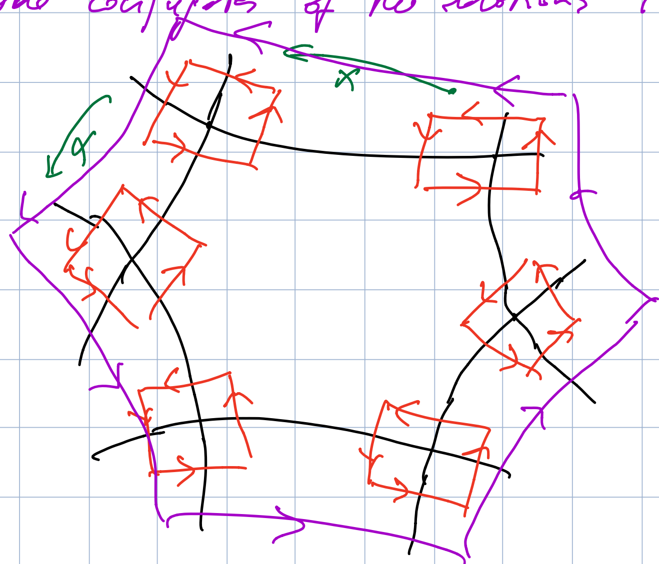


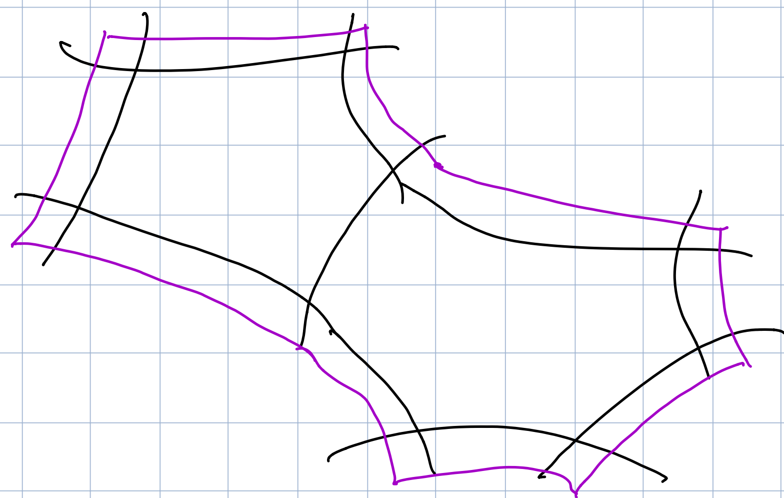


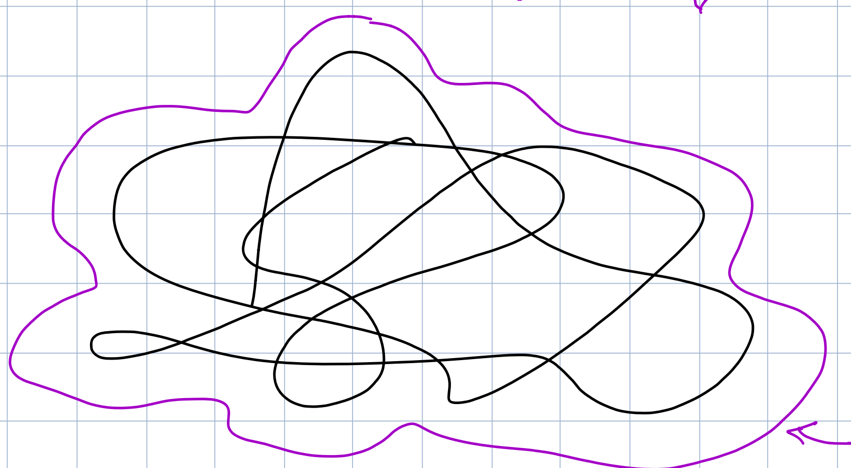
$x_j x_k = x_\ell x_j$

$r_i = x_j x_k x_j^{-1} x_\ell^{-1}$

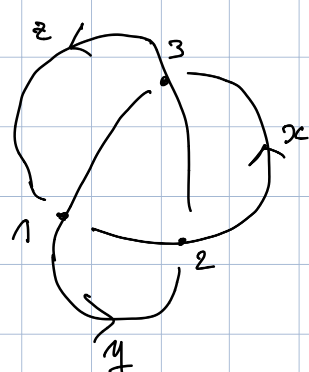
Fact: one relation may be dismissed: because product of suitable conjugates of the relations is 1.







product of conjugates of all relations is trivial in $\pi_1(E(K))$



1. $yx = zy$
2. $xz = yx$
3. $zy = xz$

$1: \quad z = yxy^{-1}$

2: $\quad xyxy^{-1} = yx \qquad xyxy^{-1}x^{-1}y^{-1}$

3. $\quad yxy^{-1}y = xyxy^{-1} \qquad xyxy^{-1}$

same relation...

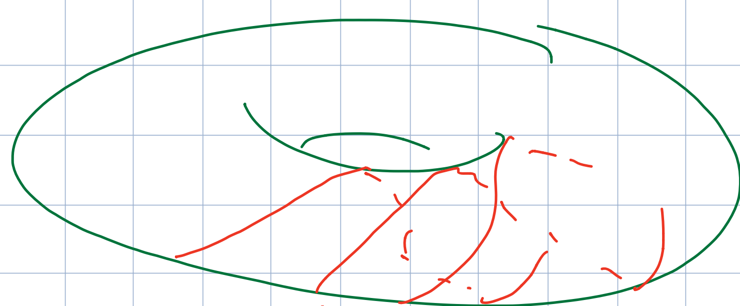
$\pi_1(E(T_{2,3})) = \langle x, y \mid xyx = yxy \rangle \qquad a = xy \qquad b = yxy$

$a^3 = xyxyxy \qquad b^2 = yxyyxy$

$\langle a, b \mid a^3 = b^2 \rangle$ exercise: equivalent presentation.

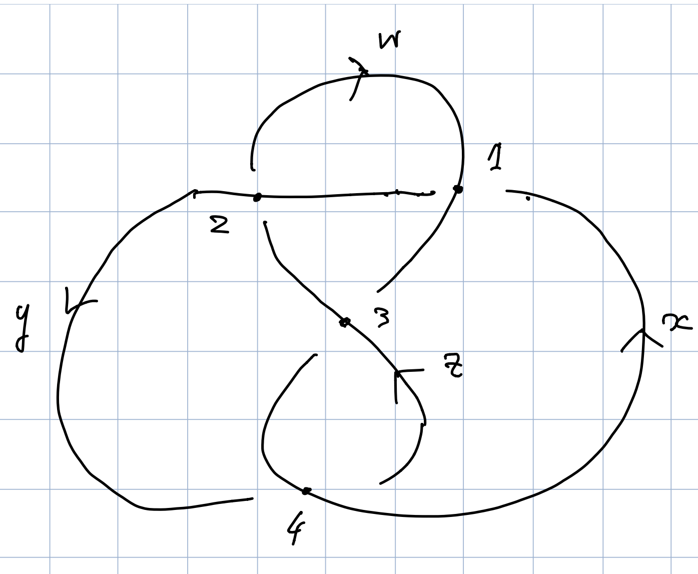
Prop: $\pi_1(E(K_{p,q})) = \langle a, b \mid a^p = b^q \rangle$

Pf: $K_{p,q} \subset T \qquad T = \partial T_1 = \partial T_2$



$K_{p,q}$

$\square$



1. $wx = yw$
2. $yz = wy$
3. $zx = wz$
4. $xz = yx$

2. $w = yzy^{-1}$
1. $x = w^{-1}yw = yz^{-1}yzy^{-1}$

<u>Exercise</u>: check that replacing x, w in 3,4 get twice

$$y^{-1}z^{-1}yzy^{-1} = z^{-1}yzy^{-1}z^{-1} \qquad a = y^{-1} \quad b = z^{-1}$$

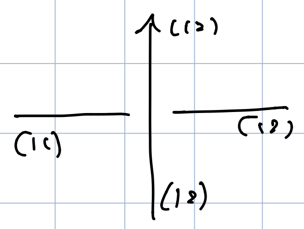
$$\pi_1(E(\text{fig-}8)) = \langle a, b \mid a \cdot [b, a^{-1}] = [b, a^{-1}] \cdot b \rangle .$$

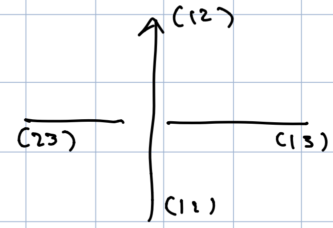
3-coloring : coloring of overarcs with three colours s.t. at each crossing all equal or all different + all colors used.

<u>Prop</u>: $\exists$ 3-coloring of diagram of K
$\Longleftrightarrow \exists\, \phi : \pi_1(E(K)) \twoheadrightarrow S_3$

<u>Proof</u>: $(\Longrightarrow)$ 3-coloring with colours a, b, c
Define ϕ on $x_1, \dots, x_n$ & check relations:

$$\phi(x_i) = \begin{cases} (12) \\ (23) \\ (13) \end{cases} \quad \text{if colour } \begin{matrix} a \\ b \\ c \end{matrix}$$





$$(12)(12) = (12)(12) \qquad \underbrace{(12)\cdot(23)}_{=(123)} = \underbrace{(13)\cdot(12)}_{=(123)}$$

surjective because $(12),(23),(13)$ generate $\mathfrak{S}_3$.

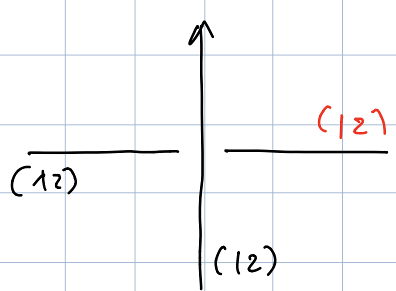
$\Leftarrow$ Suppose $\exists\, \phi : \pi_1(\mathcal{C}(K)) \to \mathfrak{S}_3$

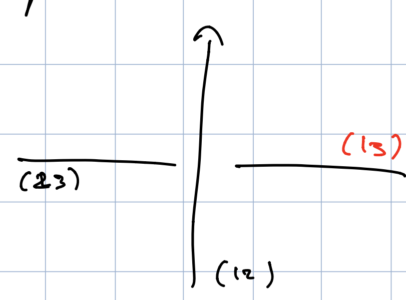
$$x_j x_k = x_\ell x_j \implies \underline{\phi(x_j)}\cdot\phi(x_k) = \phi(x_\ell)\cdot\underline{\phi(x_j)}$$

$\implies \phi(x_k)$ same parity as $\phi(x_\ell)$

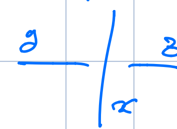
$\implies$ all $\phi(x_i)$ have the same parity

since ϕ onto they're transpositions





$\implies$ use transpositions as colours. $\square$

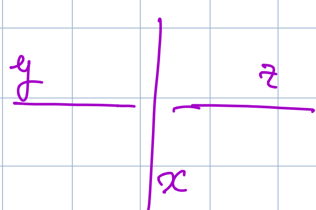
Prop: $\exists$ 3-coloring of $K \iff \exists f : \{\text{overarcs}\} \to \mathbb{Z}/3$
s.t. $f(x)+f(y)+f(z) \equiv 0 \ (3)$
at each 

Proof: $a, b, c \in \mathbb{Z}/3$

$a+b+c = 0 \iff$ all equal or all different.

$1+0+2 = 0$ $\boxed{}$

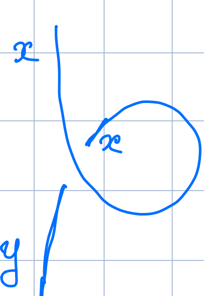
Def: given $p \in \mathbb{N}$ I call p-coloring of diagram a coloring of overarcs in $\mathbb{Z}/p$ s.t.



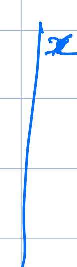
$y+z = 2x \ (p)$

Prop: $\#\{p\text{-colorings}\}/\mathbb{Z}_p$ invariant.

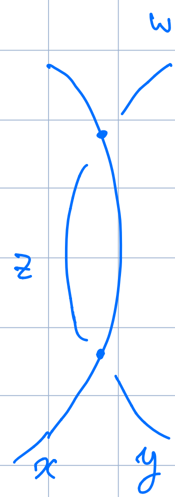
Proof: show invariance under R's :

R_I 

$x+y = 2x \ (p)$
$y = x$

$\longleftrightarrow$ 

R_{II}

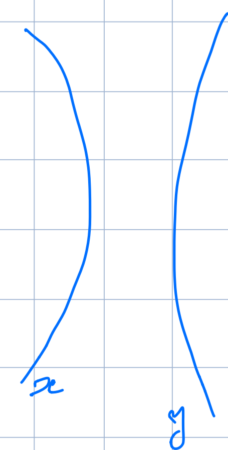


$y+z=2x$

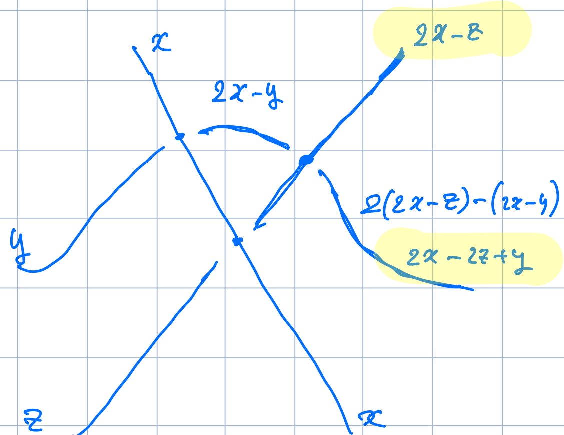
$z+w=2x$

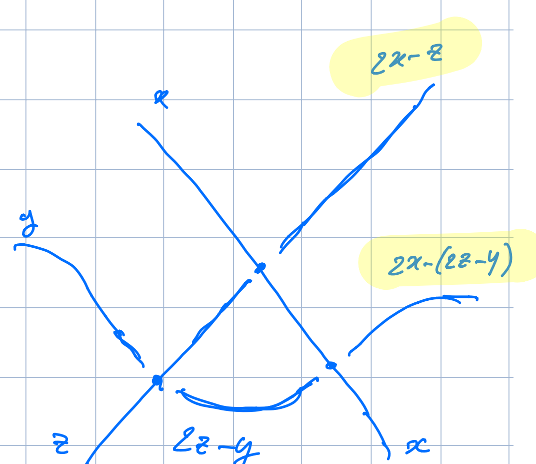
$w=y$

$\longleftrightarrow$



R_{III}.





$\blacksquare$

Fact: interesting only for odd p.

$p=2$ $\quad y+z=0$, $y=z$ $\Longrightarrow$ constant colourings

$p=2q$ $\quad y+z=2x$ $\Longrightarrow$ all colours have same parity all even: $y+z \equiv 2x \quad (2q)$

$$2Y+2Z \equiv 2X \quad (2q)$$

$$Y+Z \equiv X \quad (q)$$

halfing colours get q-coloring

all odd $y+z = 2x \quad (2q)$

$$(2Y+1)+(2Z+1) \equiv 2(X+1) \quad (2q)$$

$$Y+Z \equiv X \quad (q).$$

$$\mathbb{F}(x_1,\dots,x_m) = \{\text{words in letters } x_1^{\pm 1},\dots,x_m^{\pm 1}\} \Big/ w\cdot x_i^{\pm 1}\cdot x_i^{\mp 1}\cdot u \sim w\cdot u$$

group: $[w]\cdot[u] = [w\cdot u]$

$$e = [\emptyset]$$

$$[x_{i_1}^{\alpha_1}\cdots x_{i_p}^{\alpha_p}]^{-1} = [x_{i_p}^{-\alpha_p}\cdots x_{i_1}^{-\alpha_1}].$$

Rem: if G is group, $g_1,\dots,g_m \in G$ then

$\exists!\ \phi: \mathbb{F}(x_1,\dots,x_m) \to G$ homomorphism

s.t. $\phi(x_i) = g_i$

reason: $\phi(x_{i_1}^{\alpha_1}\cdots x_{i_p}^{\alpha_p}) = g_{i_1}^{\alpha_1}\cdots g_{i_p}^{\alpha_p}$.

Prop: $\mathbb{F}(x_1,\dots,x_n) \cong \mathbb{F}(y_1,\dots,y_m) \implies n=m$

Pf:

$$\#\{\underset{\text{homo}}{\phi:\mathbb{Z}[x_1,\dots,x_n] \to \mathbb{Z}/2}\} = 2^n. \quad \boxtimes$$

$r_1, \dots, r_k \in \mathbb{F}(x_1, \dots, x_n)$

$N(r_1, \dots, r_k)$ = smallest normal subgroup containing $r_1, \dots, r_k$

<u>Fact</u>: $N(r_1, \dots, r_k) = \{$ all products of conjugates of $r_1^{\pm 1} \dots r_k^{\pm 1} \}$

$\supset$ obvious

$\subset$ must see that RHS is subgroup, contain $r_1, \dots, r_k$ + normal :

$$u \cdot \left(\left(w_1 r_{i_1}^{\alpha_1} w_1^{-1} \right) \dots \left(w_p r_{i_p}^{\alpha_p} w_p^{-1} \right) \right) \cdot u^{-1}$$

$$(uw_1) r_{i_1}^{\alpha_1} (uw_1)^{-1} \cdot (uw_2)^{-1} \dots$$

Finitely presented group:

$$\langle x_1, \dots, x_n \mid r_1, \dots, r_k \rangle := \frac{\mathbb{F}(x_1, \dots, x_n)}{N(r_1, \dots, r_k)}$$

How to check if two FPG's are the same?

<u>Fact</u>: $\nexists$ no Turing machine telling whether $\langle x_1 \dots x_n \mid r_1 \dots r_k \rangle$ is the trivial group.

Tietze moves :

I : $\langle x_1,\dots,x_n \mid r_1,\dots,r_k \rangle \rightsquigarrow \langle x_1,\dots,x_n \mid r_1,\dots,r_k, r\rangle$

$r \in N(r_1,\dots,r_k)$

II : $\langle x_1,\dots,x_n \mid r_1,\dots,r_k \rangle \rightsquigarrow \langle x_1,\dots,x_n, y \mid r_1 \dots r_k\, y^{-1}w \rangle$

$w \in \mathbb{F}(x_1,\dots,x_n)$

Rem : give isomorphic groups.

Thm : $\langle x_i \mid r_p \rangle \cong \langle y_j \mid s_l \rangle$ iff related by sequence of Tietze moves $\mathrm{I}^{\pm 1}, \mathrm{II}^{\pm 1}$.

Lem : $\rho : \mathbb{F}(x_i, y_j) \longrightarrow \mathbb{F}(x_i)$ homomorphism $\rho(x_i) = x_i$

$\phi : \mathbb{F}(x_i) \longrightarrow \langle x_i \mid r_p \rangle$ projection

$\Longrightarrow \operatorname{Ker}(\phi \circ \rho) = N(r_p,\ y_j^{-1} \cdot \rho(y_j))$

$$\begin{tikzcd} \mathbb{F}(x_i, y_j) \arrow[r, "\rho"] & \mathbb{F}(x_i) \arrow[r, "\phi"] \arrow[l, bend left] & \langle x_i \mid r_p \rangle \end{tikzcd}$$

Proof : $\boxed{\supset}$ $\operatorname{Ker}(\phi \circ \rho)$ is normal subgroup :

enough to show $r_p,\ y_j^{-1} \cdot \rho(y_j)$ belong to it :

$$(\phi \circ \rho)(r_p) = \phi(\rho(\underset{\substack{\Uparrow \\ \mathbb{F}(x_i)}}{r_p})) = \phi(r_p) = 1$$

$$(\phi \circ \rho)(y_j^{-1} \cdot \rho(y_j)) = \phi(\rho(y_j)^{-1} \cdot \underset{\substack{\| \\ \rho}}{\rho^2}(y_j)) = \phi(1) = 1 .$$

$\boxed{\subseteq}$ Set $H = N(r_\rho,\ y_j^{-1}\cdot \rho(y_j))$ and consider:

$$\gamma : \mathbb{F}(x_i, y_j) \longrightarrow \mathbb{F}(x_i, y_j)/H$$
$$\eta = \gamma|_{\mathbb{F}(x_i)}$$

$$\begin{tikzcd}
\mathbb{F}(x_i, y_j) \arrow[r, "\rho"] \arrow[dr, "\gamma"'] & \mathbb{F}(x_i) \arrow[r, "\phi"] \arrow[d, "\eta"] & \langle x_i \mid r_\rho \rangle \\
& \mathbb{F}(x_i, y_j)/H &
\end{tikzcd}$$

Claim that triangle commutes: enough to see on x_i, y_j:

$\boxed{x_i}$ $\eta(\rho(x_i)) = \eta(x_i) = \gamma(x_i)$ ✓

$\boxed{y_j}$ $\gamma(y_j) = \eta(\rho(y_j))$

$\iff \gamma(y_j^{-1})\cdot \eta(\underbrace{\rho(y_j)}_{\in \mathbb{F}(x_i)}) = 1 \in \mathbb{F}(x_i, y_j)/H$

$\iff \gamma(y_j^{-1})\cdot \gamma(\rho(y_j)) = 1$

$\iff \gamma(y_j^{-1}\cdot \rho(y_j)) = 1 \in \mathbb{F}(x_i, y_j)/H$

true because $y_j^{-1}\cdot \rho(y_j) \in H$

Now for $u \in \mathbb{F}(x_i, y_j)$ I have

$$\gamma(u\cdot \rho(u)^{-1}) = (\eta\circ\rho)(u\cdot\rho(u)^{-1})$$
$$= \eta(\rho(u))\cdot \eta(\underset{\rho}{\rho^2}(u)^{-1}) = 1$$

$\Longrightarrow \quad u \cdot \rho(u) \in \operatorname{Ker}(\gamma) = H.$

If $\quad u \in \operatorname{Ker}(\phi \circ \rho) \quad \Longrightarrow \quad \rho(u) \in \operatorname{Ker}(\phi) = N(r_p) \subset H$

$\Longrightarrow \quad u \in H. \qquad \blacksquare$

Proof of Thm:

Suppose $f : \langle x_i \mid r_p \rangle \longrightarrow \langle y_j \mid s_\ell \rangle$ isomorphism. Set $g = f^{-1}$. Choose liftings F, G under projections ϕ, ψ.

$$
\begin{tikzcd}
\mathbb{F}(x_i) \arrow[r, shift left, "F"] \arrow[d, "\phi"'] & \mathbb{F}(y_j) \arrow[l, shift left, "G"] \arrow[d, "\psi"] \\
\langle x_i \mid r_p \rangle \arrow[r, shift left, "f"] & \langle y_j \mid s_\ell \rangle \arrow[l, shift left, "g"]
\end{tikzcd}
$$

Define: $\quad \rho : \mathbb{F}(x_i, y_j) \longrightarrow \mathbb{F}(x_i) \qquad \begin{array}{l} \rho(x_i) = x_i \\ \rho(y_j) = G(y_j) \end{array}$

$\qquad\qquad \sigma : \mathbb{F}(x_i, y_j) \longrightarrow \mathbb{F}(y_j) \qquad \begin{array}{l} \sigma(y_j) = y_j \\ \sigma(x_i) = F(x_i) \end{array}$

Get a commutative diagram:

$$
\begin{tikzcd}
 & \mathbb{F}(x_i, y_j) \arrow[dl, "\rho"'] \arrow[dr, "\sigma"] & \\
\mathbb{F}(x_i) \arrow[rr, shift left, "F"] \arrow[d, "\phi"'] & & \mathbb{F}(y_j) \arrow[ll, shift left, "G"] \arrow[d, "\psi"] \\
\langle x_i \mid r_p \rangle \arrow[rr, shift left, "f"] & & \langle y_j \mid s_\ell \rangle \arrow[ll, shift left, "g"]
\end{tikzcd}
$$

Tietze moves :

$$\langle x_i \mid r_p \rangle \xrightarrow{\text{II}} \langle x_i, y_j \mid r_p,\ y_j^{-1}\cdot \rho(y_j) \rangle$$

$$\langle y_j \mid s_\ell \rangle \longrightarrow \langle x_i, y_j \mid s_\ell,\ x_i^{-1}\cdot \sigma(x_i) \rangle$$

Lem: $\operatorname{Ker}(\phi \circ \rho) = N(r_p, y_j^{-1}\cdot \rho(y_j))$

but $\phi\circ\rho = g\circ\psi\circ\sigma$ & g isomorphism

$\Rightarrow \operatorname{Ker}(\psi\circ\sigma) = N(r_p, y_j^{-1}\cdot\rho(y_j))$

Now:

$$(\psi\circ\sigma)(s_\ell) = \psi(s_\ell) = 1$$

$$(\psi\circ\sigma)(x_i^{-1}\cdot\sigma(x_i)) = \psi(\sigma(x_i)^{-1})\cdot\psi(\underset{\sigma}{\underset{\|}{\sigma^2}}(x_i)) = 1$$

$$\Rightarrow s_\ell,\ x_i^{-1}\cdot\sigma(x_i) \in N(r_p, y_j^{-1}\cdot\rho(y_j))$$

So I have: Tietze I move

$$\langle x_i, y_j \mid r_p, y_j^{-1}\cdot\rho(y_j)\rangle \xrightarrow{\text{I}} \langle x_i, y_j \mid r_p, y_j^{-1}\rho(y_j), s_\ell, x_i^{-1}\sigma(x_i)\rangle$$

By symmetry also have

$$\langle x_i, y_j \,/\, r_e,\ x_i^{-1}\sigma(x_i)\rangle \xrightarrow{\ I\ } \langle x_i, y_j \,/\, r_{P},\ y_j^{-1}\rho(y_j),\ r_e,\ x_i^{-1}\sigma(x_i)\rangle$$

$\blacksquare$

Next theme: study position of surfaces in S^3 wrt. knots & links.

<u>Morse theory in a nutshell</u>:

$M^{(m)}$ abstract mfld. $\quad x \in M$

$$T_x M = \{ D : C^\infty(M) \to \mathbb{R} \text{ linear},\ D(f \cdot g) = D(f)\cdot g(x) + f(x)\cdot D(g) \}$$

$\alpha : \mathbb{R} \to M \qquad \alpha(0) = x$

$\alpha'(0) \in T_x M \qquad \alpha'(0)(f) = \frac{d}{dt}\Big|_{t=0} (f \circ \alpha)$

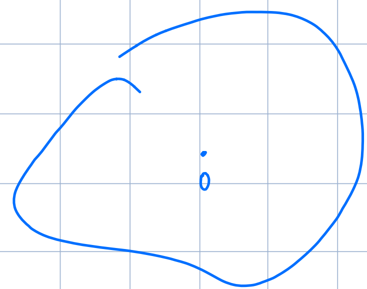
$u : M \to N \qquad d_x u : T_x M \to T_{u(x)} N$

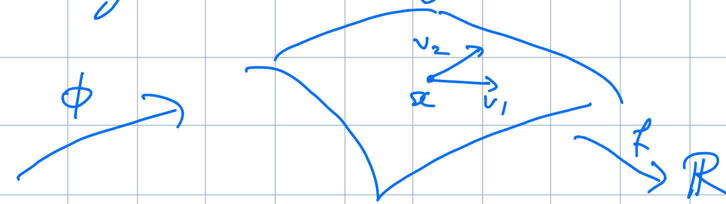
$(d_x u)(v)(f) = v(f \circ u)$

Prop: if $f : M^{(m)} \to \mathbb{R}$ and $d_x f = 0$

then $H_x f : T_x M \times T_x M \to \mathbb{R}$

well-defined using an arbitrary chart ϕ:





$$(H_x f)(v_1, v_2) = \left(H_0(f\circ\phi)\right)\left((d_0\phi)^{-1}(v_1), (d_0\phi)^{-1}(v_2)\right)$$

Proof: if use two charts

$$\begin{tikzcd}
(\mathbb{R}^n, 0) \arrow[dr, "\phi"] & & \\
 & (M, x) \arrow[r, "f"] & \mathbb{R} \\
(\mathbb{R}^n, 0) \arrow[uu, "\phi^{-1}\circ\psi = h"] \arrow[ur, "\psi"'] & &
\end{tikzcd}$$

$T_x M$ is identified to $\mathbb{R}^n$ via $(d_0\phi)^{-1}, (d_0\psi)^{-1}$
these identifications differ by $d_0 h$: enough to show that $H_0(f\circ\phi)$ $H_0(f\circ\psi)$ differ by the natural action of $d_0 h$ on sym. bil. forms:

In fact:

$$\frac{\partial^2(f\circ\psi)}{\partial y_i \partial y_j} = \frac{\partial^2(f\circ\phi\circ h)}{\partial y_i \partial y_j} = \frac{\partial}{\partial y_i} \cdot \frac{\partial((f\circ\phi)\circ h)}{\partial y_j}$$

$$= \frac{\partial}{\partial y_i}\left(\sum_k \left(\frac{\partial(f\circ\phi)}{\partial x_k}\circ h\right)\cdot \frac{\partial h_k}{\partial y_j}\right)$$

$$= \sum_{k,\ell} \frac{\partial^2 (f\circ\phi)}{\partial x_k \partial x_\ell} \cdot \frac{\partial h_\ell}{\partial y_i} \cdot \frac{\partial h_k}{\partial y_j} + \sum_k \underbrace{\frac{\partial (f\circ\phi)}{\partial x_k}}_{=0} \frac{\partial^2 h_k}{\partial y_i \partial y_j}$$

because $d_x f = 0$

$$H_0(f\circ\psi) = ({}^t d_0 h) \cdot d_0(f\circ\phi) \cdot (d_0 h)$$

natural action of $d_0 h$ on forms.

$\boxtimes$

<u>Def</u>: $f : M \to \mathbb{R}$ Morse function if $H_x f$ is non-deg. at all pts where $d_x f = 0$ (critical points).

<u>Fact</u>: {Morse functions} is an open dense subset of $C^\infty(M, \mathbb{R})$.