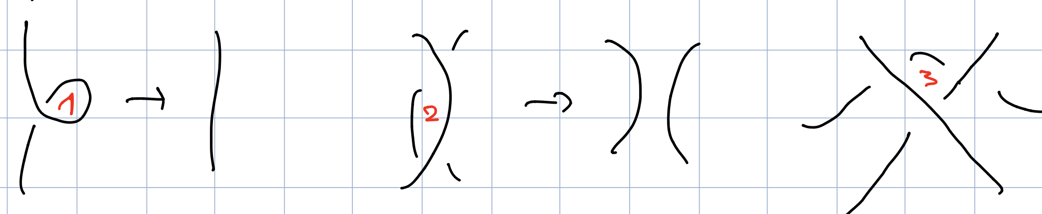
Teoria dei nodi 14/3/2019

Q: Reidemeister moves give algorithm to classify knots?

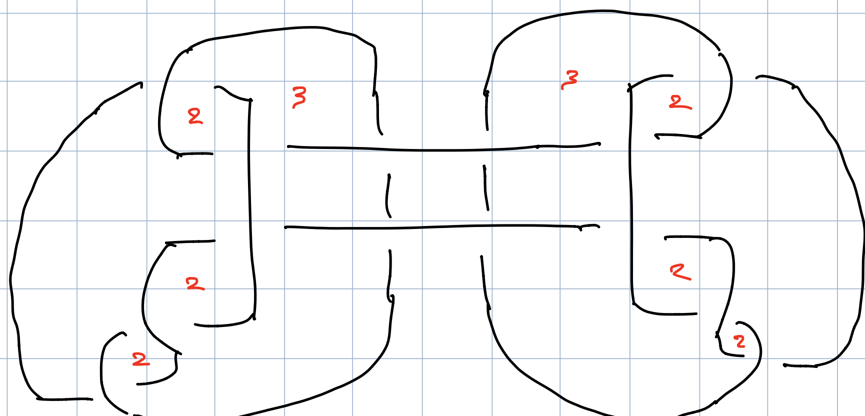
Yes if: $\exists f: \mathbb{N}\times\mathbb{N}\to\mathbb{N}$ s.t. given D_1, D_2 with c_1, c_2 crossings they represent same knot iff related by R-moves involving diagrams with $\leq f(c_1,c_2)$ crossings.

Partial results like this known but not in general.

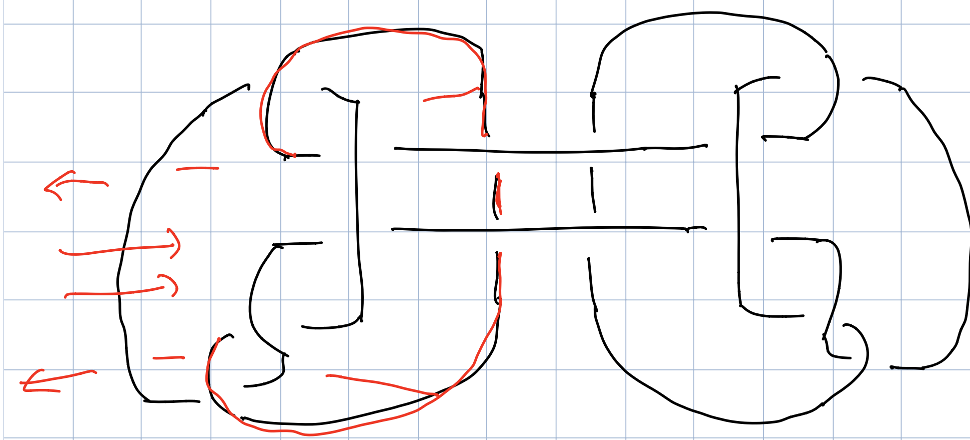
Remark: a non-increasing R-move involves a region (component of $S^2 \setminus$ diagram) with ≤ 3 corners:

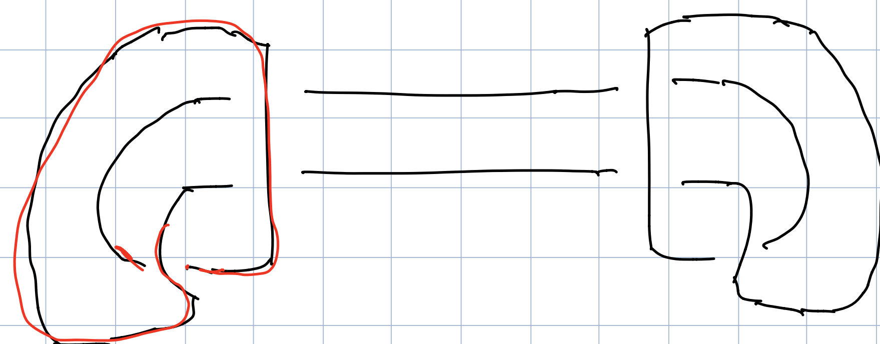


Ex:



No non-increasing R-move applies





Trivial knot

$L \subset S^3$ link $\qquad E(L) = S^3 \setminus \mathring{U}(L)$

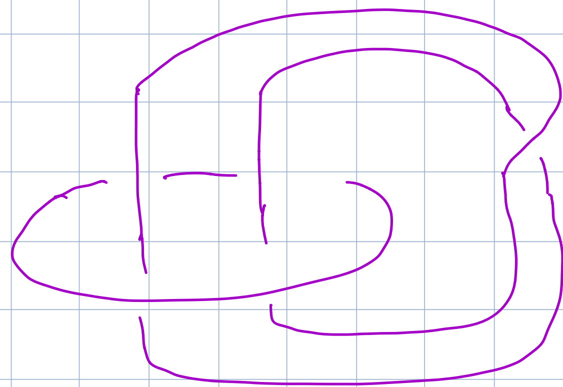
3-manifold with $\partial E(L) = \bigcup$ tori.

$E(L)/_{\text{homeo}}$ is invariant of L.

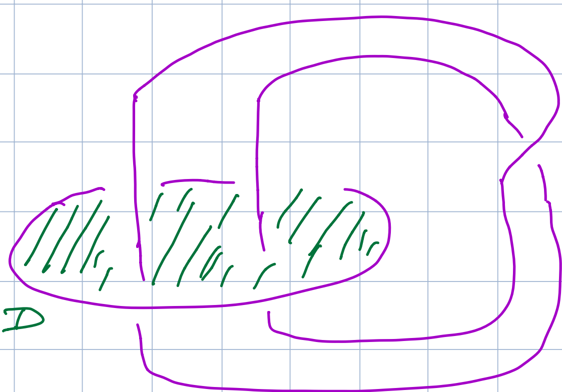
Thm (Gordon–Luecke): for knot K, complete invariant.

False for links:

$E(L) = \text{"}S^3 \setminus L\text{"}$
open manifold that compactifies adding tori



two unknots

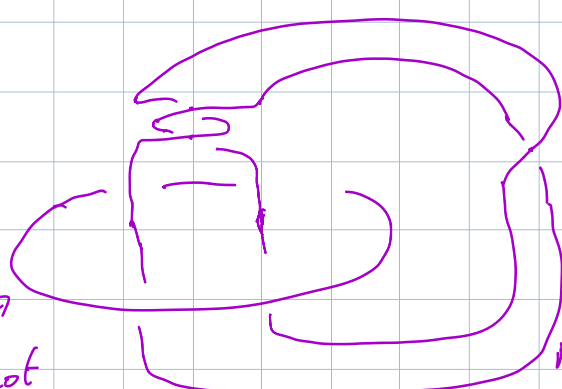


D

D open 2-disc with 2 missing points

Can cut $E(L)$ along D; give full 2π-twist + glue back

Exterior remains same



unknot

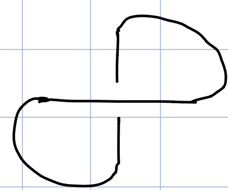
trefoil

different link with same exterior.

Crossing number: $c(L) = \min\{n : \exists D$ diagram of L with n crossings$\}$

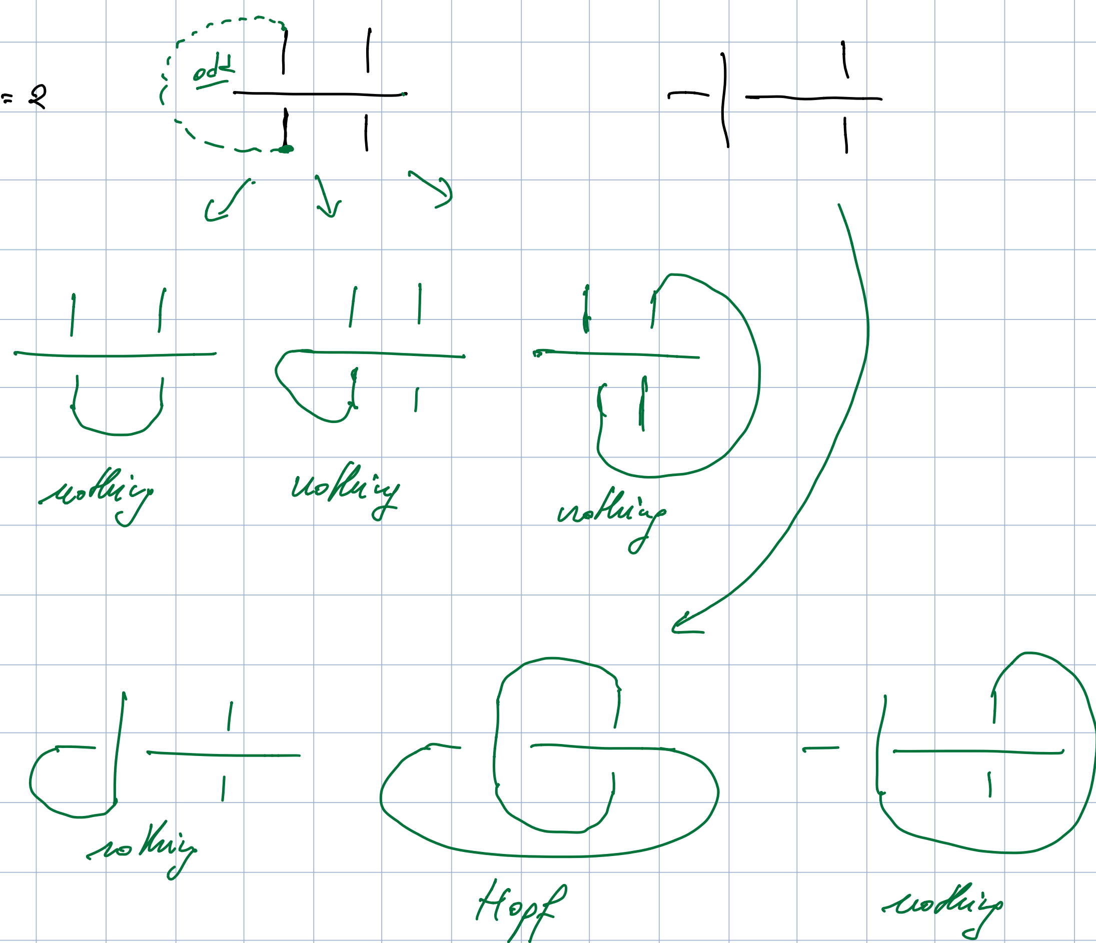
Exercise: classify links without split unknot components with $c \leq 4$.

$c = 1$

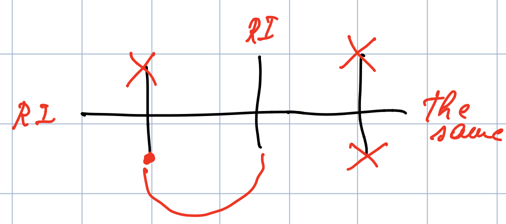


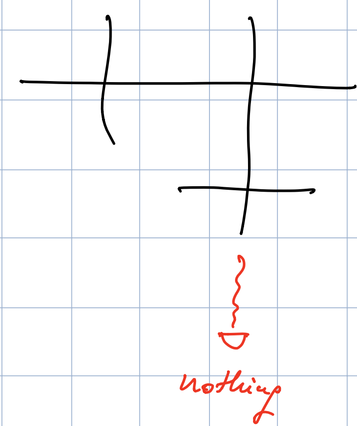
nothing

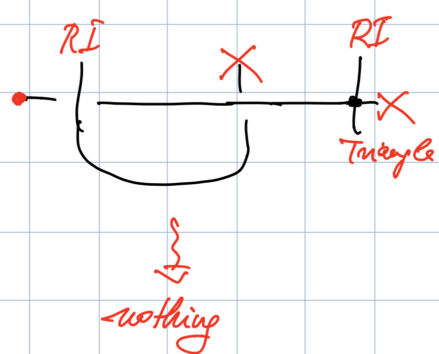
$c = 2$



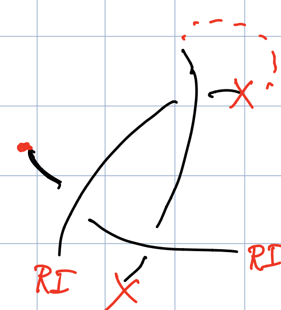
$c=3$ case 1 : no triangle . Take maximal tree in D

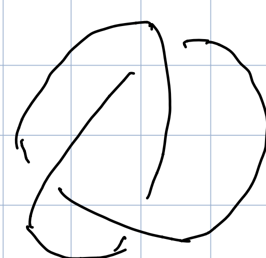






case 2 : triangle





trefoil

$c=4$ ...

Fact: done by computer with lots of invariants used for $c \leq 19$ (maybe more).

Unknotting number for L

(1) $\min\{n :\ \exists\ D$ diagram of L s.t. switching n crossings of D get diagram of unlink with same ... $\}$

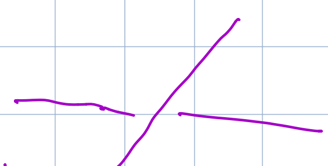
(2) $\min\{n :\ \exists\ D_1, D_1', \dots, D_n, D_n'$ s.t. D_1 represents L, $D_i \rightsquigarrow D_i'$ switch of 1 crossing, $D_i' \rightsquigarrow D_{i+1}$ R-moves, D_n' represents unlink $\}$

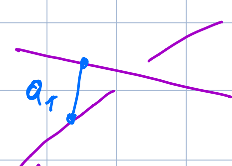
(3) $\min\{n :\ \exists\ (L_t)_{t\in[0,1]}$ s.t. $L_0 = L$, $L_1 =$ unlink

L_t link except for $t = t_1, \dots, t_n$ and
L_{t_i} has only one transverse double point $\}$

Obvious: $(1) \geq (2) \geq (3)$. Fact $(3) \geq (1) \Rightarrow$ all same.

Reason: exceptional times for $(L_t)_{t\in[0,1]}$ $0 < t_1 < \dots < t_n < 1$.

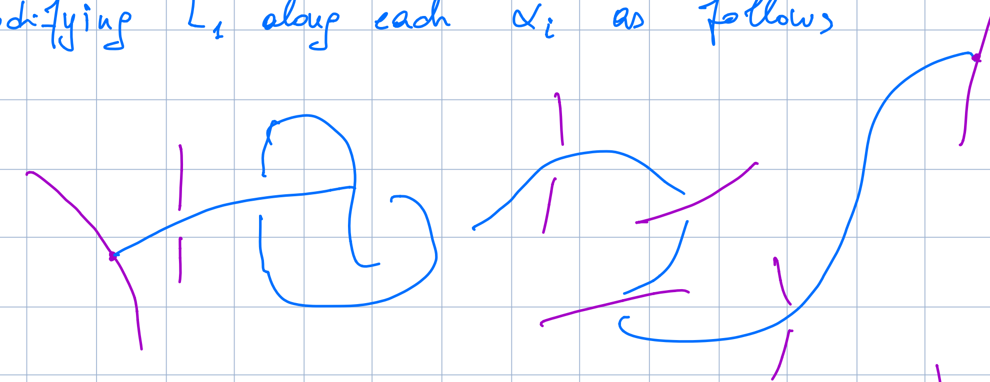

$L_{t_1-\varepsilon}$

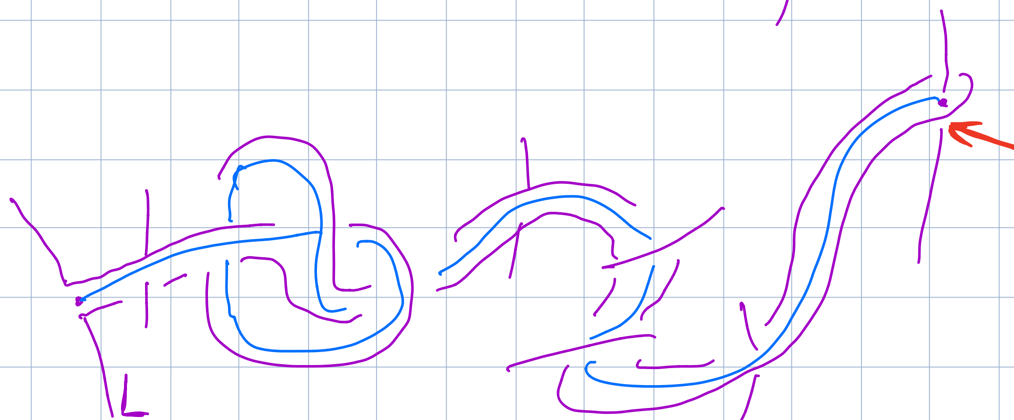


$L_{t_1+\varepsilon}$

Since have isotopy $L_{t_1+\varepsilon} \longrightarrow L_{t_2-\varepsilon}$ is ambient isotopy
have isotopy $L_{t_1+\varepsilon} \cup a_1 \longrightarrow L_{t_2-\varepsilon} \cup \tilde{a}_1$; add a_2

similarly ... all the way; in the end have L_t = unlink $L_1 \cup \alpha_1 \cup \dots \cup \alpha_m$ s.t. modifying L_1 along each α_i as follows

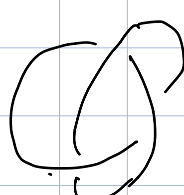




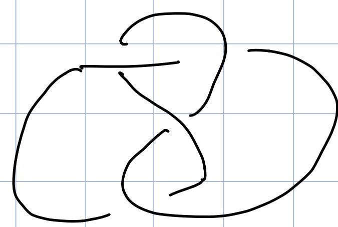
I get back L.

Now take projection of $L_1 \cup \alpha_1 \cup \dots \cup \alpha_m$ and change it as in previous picture.
Now switching the m crossings indicated by red arrow get unlink.

Examples:



$u = 1$

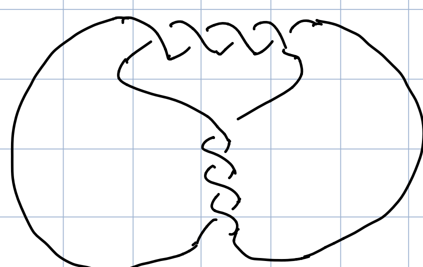


$u = 1$

k crossings

$u = 1$

Not easy to exhibit K's with $u(K) > 1$.

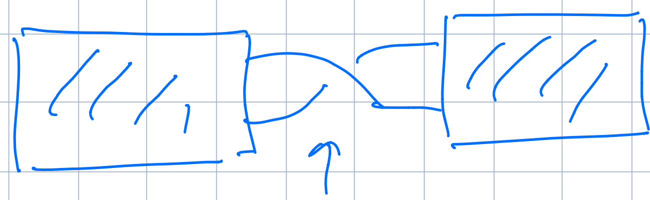


$u = 2$
uses big things.

<u>Def</u>: diagram D is <u>alternating</u> if following any component any undercrossing is followed by overcrossing and conversely:

<u>Rem</u>: no R_{II}, R_{III} non-increasing applies to D alternating.

Def: a crossing of D is nugatory if

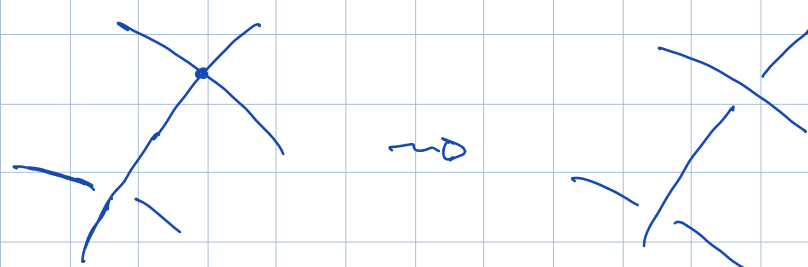


can be eliminated.

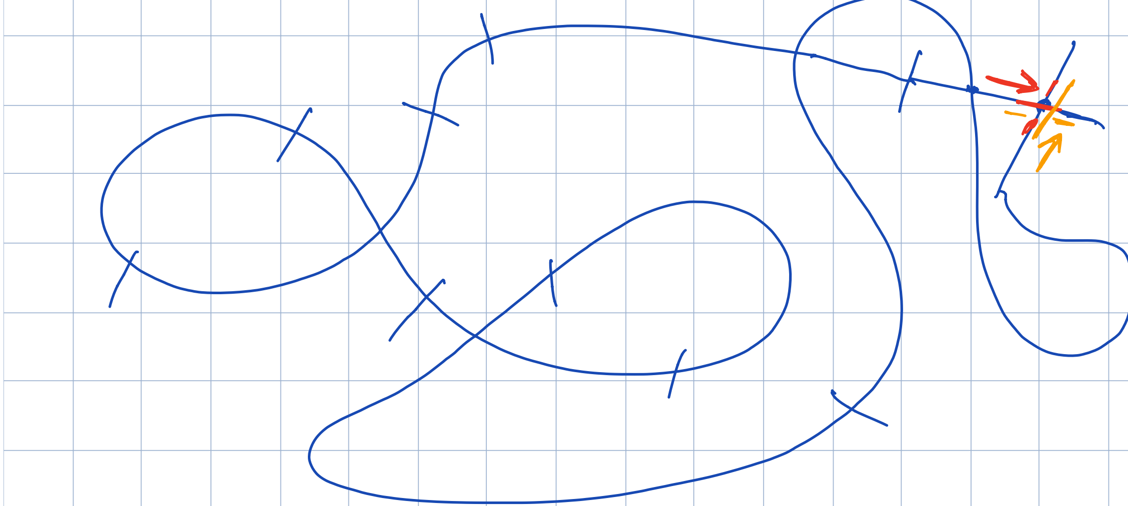
Thm (Tait): if D is alternating and has no nugatory crossing then D realizes the crossing number.

Prop: any D can be turned into alternating projection by crossing switches.

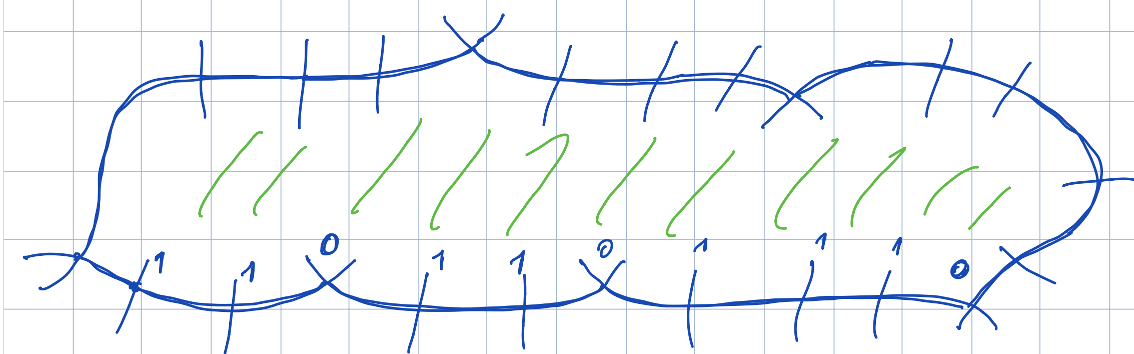
Proof: first turn all crossings into double pts
Restore one crossing randomly ; proceed one double point at a time using an already restored neighbour:

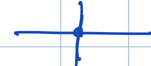


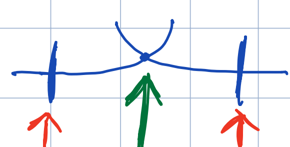
Must show that no contradiction arises: otherwise I have a cycle at a double point whose ends prescribe opposite things:



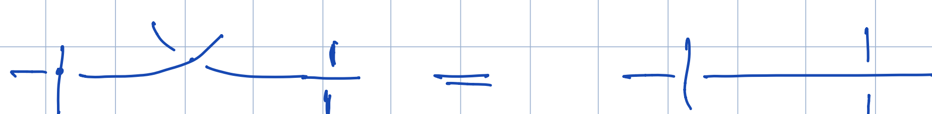
wlog can assume cycle is simple $\Rightarrow$ bounds a top. disc.

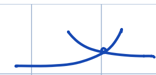
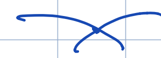


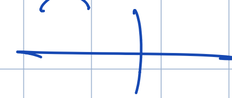
let's count how many mod 2 strands enter the region
$\Rightarrow$ total number of  is even; notice that

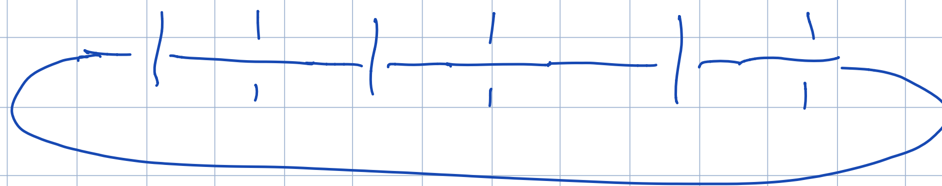


Green crossing imposes the same prescriptions on red ones as if it didn't exist:



$\Rightarrow$ can ignore  and 

$\Rightarrow$ left with <u>even</u> number of 



$\Rightarrow$ no contradiction. $\blacksquare$

<u>Wirtinger presentation</u> of $\pi_1(E(L))$.

$$G = \langle x_1, \dots, x_m \mid r_1, \dots, r_m \rangle$$

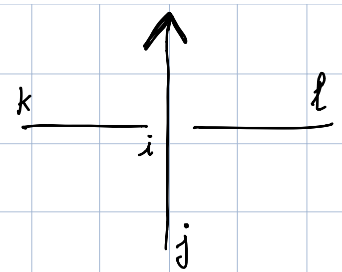
means $$G = \frac{\text{free group in the letters } x_1, \dots, x_m}{\text{smallest normal subgroup containing } r_1, \dots, r_m}$$

r_j word in the letters $x_1^{\pm 1}, \dots, x_m^{\pm 1}$

<u>Thm</u>: 1) if $D = D_1 \sqcup D_2$ (split projection)
then $\pi_1(E(D)) = \pi_1(E(D_1)) * \pi_1(E(D_2))$

2) if D connected and without circular overarcs
then $\pi_1(E(D)) = \langle x_1, \dots, x_m \mid r_1, \dots, r_m \rangle$
where $m = \#$ overarcs $= \#$ crossings

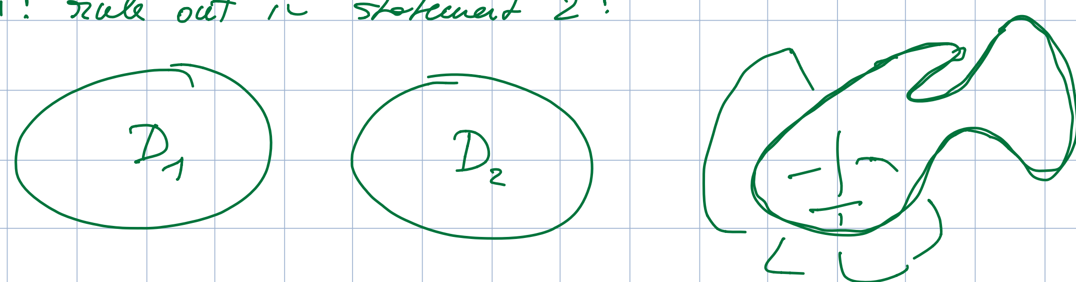
x_j oriented overarcs (wrt global arbitrary orientation)
r_i relation associated to i-th crossing



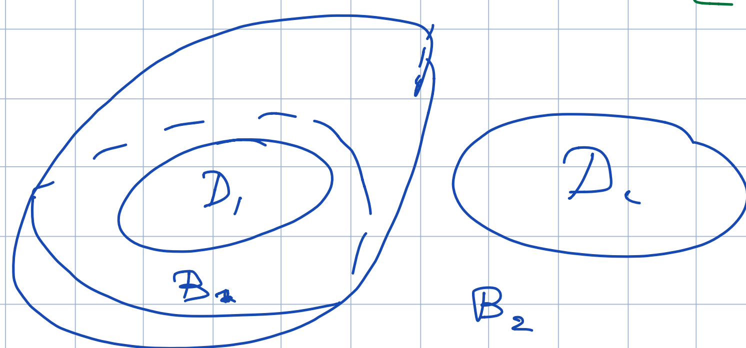
$$r_i : \quad x_j x_k = x_\ell x_j$$
$$r_i = x_j x_k x_j^{-1} x_\ell^{-1}$$

Moreover one relation can be omitted.

Rem: rule out in statement 2:



Proof.



$$B_1 \cap B_2 = S^2$$

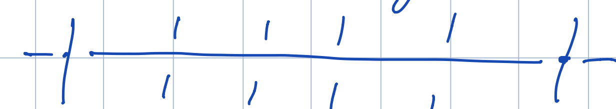
$$\pi_1(E(D_1)) = \pi_1(B_1 \setminus D_1)$$
$$S^3 \setminus D_1 = (B_1 \setminus D_1) \cup_{\underbrace{S^2}_{\pi_1 = 1}} \underbrace{B_2}_{\pi_1 = 1}$$

$$S^3 \setminus (D_1 \cup D_2) = (B_1 \setminus D_1) \cup_{S^2} (B_2 \setminus D_2)$$

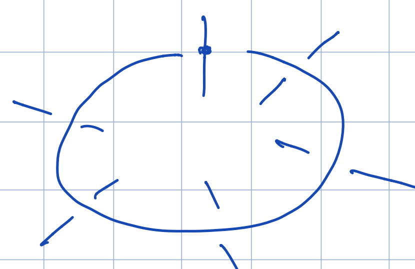
$\pi_1 = 1$

2) Must show that if there is no circular overarc then # overarcs = # crossings. Because:

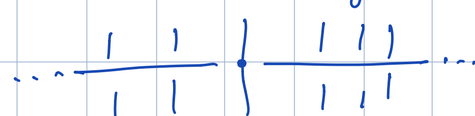
- all overarc ends at two crossings distinct



except

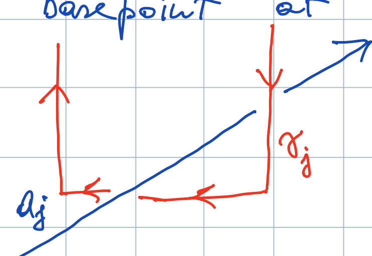


- at each crossing two overarcs originate distinct

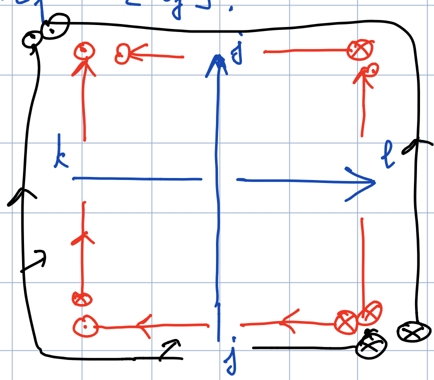


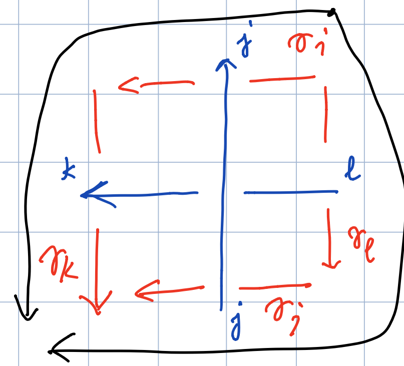
except same situation.

Let's denote the overarcs by $a_1, \dots, a_m$; take $\pi_1(E(D))$ with basepoint at ∞; take γ_j the loop:



$x_j = [\gamma_j]$.

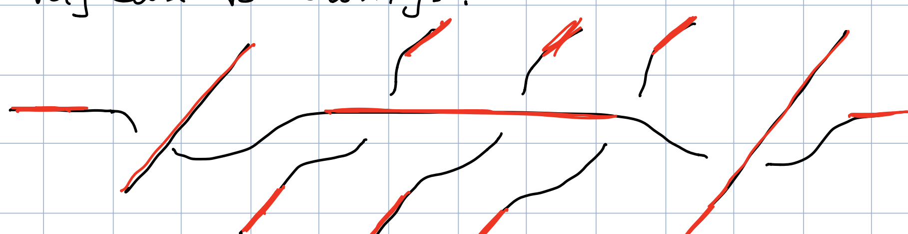


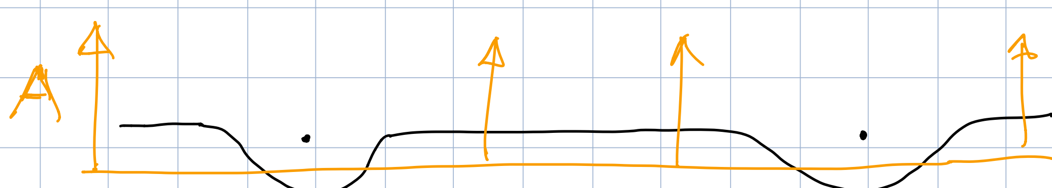


$x_j x_k = x_\ell x_j$ $\qquad$ $x_j x_k = x_\ell x_j$

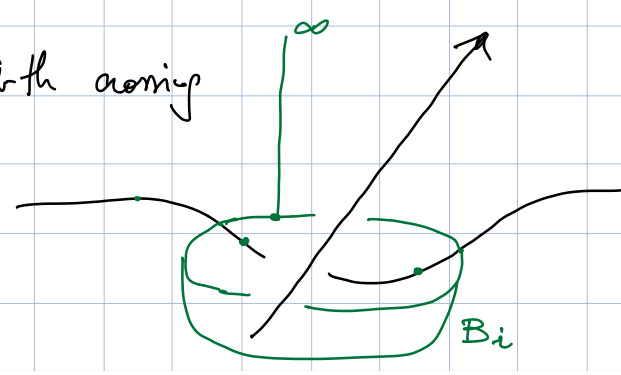
Use Van Kampen expressing $S^3 \setminus D = A \cup B_1 \cup \dots \cup B_n \cup C$

I assume the overarcs are all at the same height except very close to crossings:



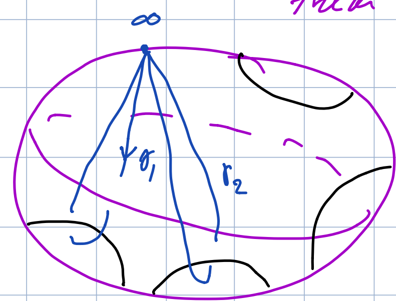


B_i small piece near i-th crossing



Hemisphere with upper disc on plane boundary A minus one arc.

$C = \overline{S^3 \setminus (A \cup B_1 \cup \dots \cup B_m)}$ actually should flatten them all to apply VK.

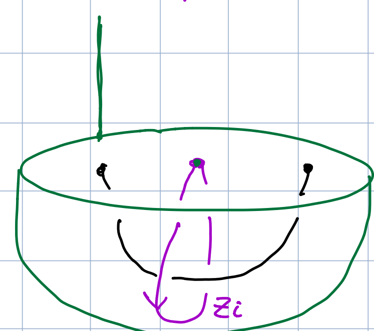
$\pi_1(A) = \pi_1\Bigg($  $\Bigg)$

$= $ free group generated by $x_1, \dots, x_m$.

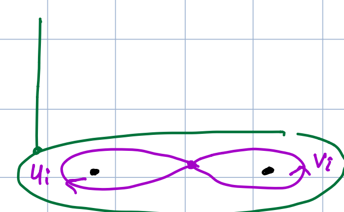
$\pi_1(C_i) = 1$

must see effect of attaching B_i to A

(we show it corresponds to introducing relation r_i).



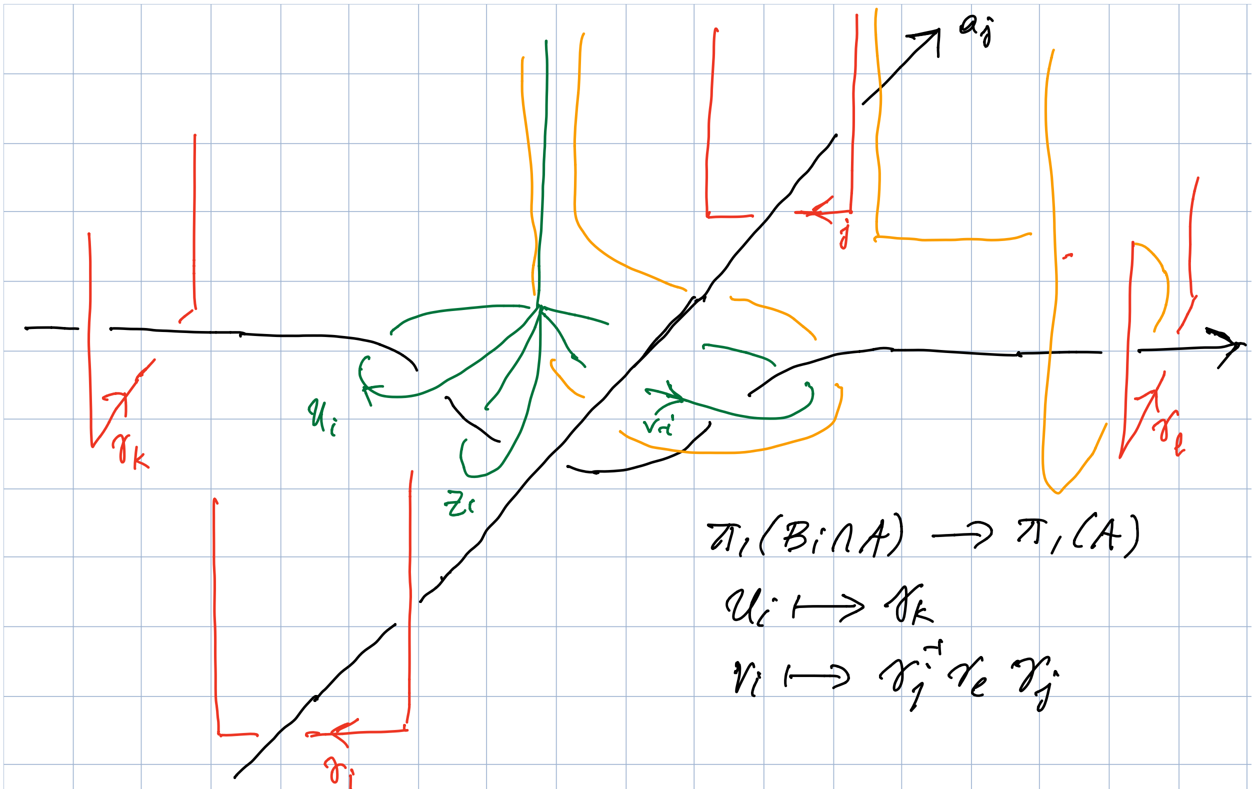
B_i



$\pi_1(B_i) = \mathbb{Z} = \langle z_i \rangle$

$\pi(B_i \cap A) = \mathbb{Z} * \mathbb{Z}$
$= \langle u_i, v_i \rangle$

$$\pi_1(B_i \cap A) \to \pi_1(B_i)$$
$$u_i \longmapsto z_i$$
$$v_i \longmapsto z_i$$



$$\pi_1(B_i \cap A) \longrightarrow \pi_1(A)$$

$$u_i \longmapsto \gamma_k$$

$$v_i \longmapsto \gamma_j^{-1} \gamma_l \gamma_j$$

$$\pi_1(B_1 \cup A) = \langle x_1, \dots, x_n \mid \gamma_k = \gamma_j^{-1} \gamma_l \gamma_j \rangle . \qquad \square$$