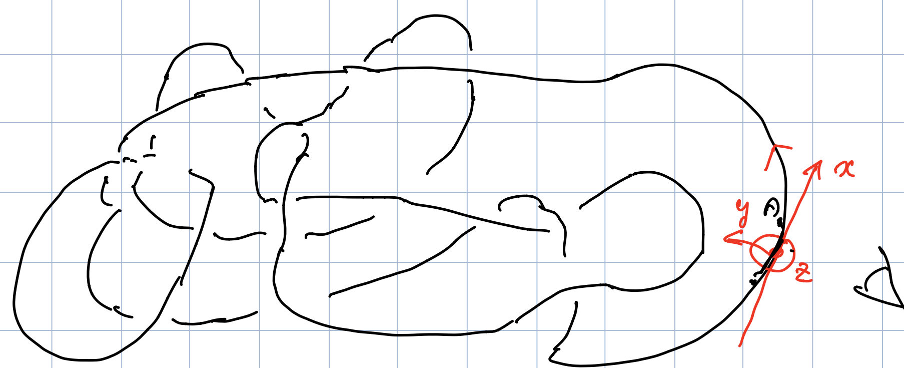
$$\text{links in } \mathbb{R}^3/S^3 \Big/ \text{isotopy} \longleftrightarrow \text{diagrams} \Big/ R_{I}, R_{II}, R_{III}$$

Prop: every diagram is trivialized by crossing switches.

Pf. trivial: isotopic to O . . . O

links: easy if true for knots.

Knots:



switch so the first time I visit a crossing I do from above

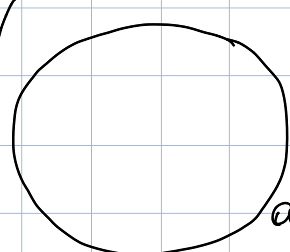
To distinguish knots/links:

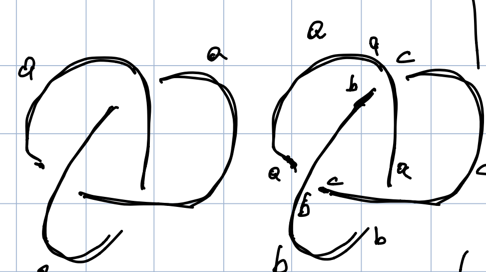
invariant $\{\text{diagrams}\} \longrightarrow\ ?$

unchanged under R_I, R_{II}, R_{III}

Def: given letters a, b, c a 3-coloring of a diagram D is a coloring in a, b, c of the arcs of D s.t. at every crossing  the colors should be all the same or all different.

Def: $c_3(D) = \#\{3\text{-colorings}\}/S_3$

$c_3\Big($ 

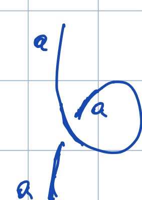
$\Big) = 1$ $\quad c_3\Big($ 

$\Big) = 2$

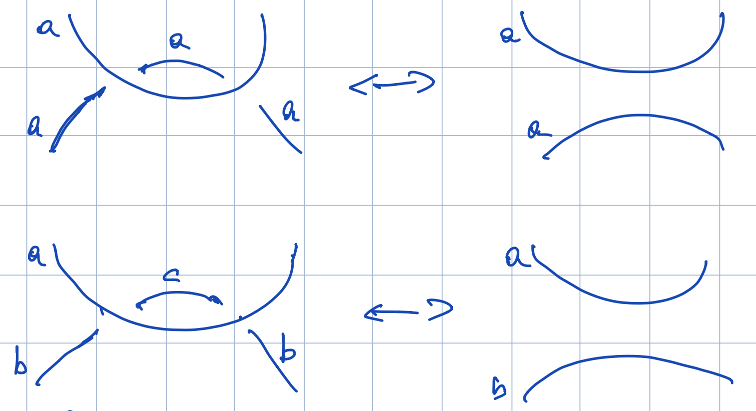
Thm: c_3 is invariant under R_*.

Pf: I exhibit a natural bijection between $\{3\text{-colorings}\}$ before/after move.

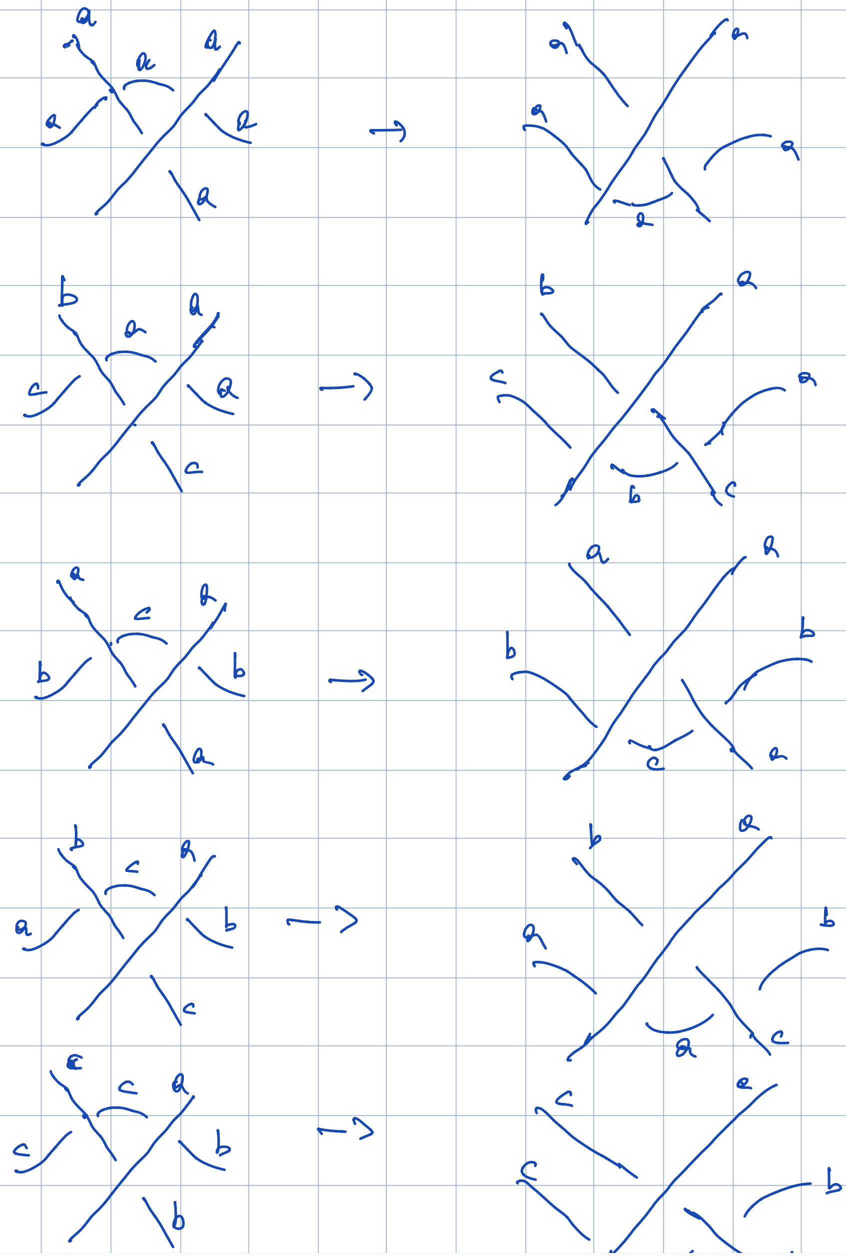
R_I 

$\longleftrightarrow$ 

R_{II}


a
a
a
a
a
a
a
c
a
b
b
a
b

R_{III}


a
a
a
a
a
a
a
a
a
a
a
a
a
b
a
a
c
a
c
b
a
c
b
a
b
c
b
c
a
b
a
b
a
c
a
b
b
b
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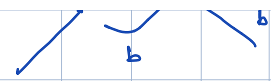
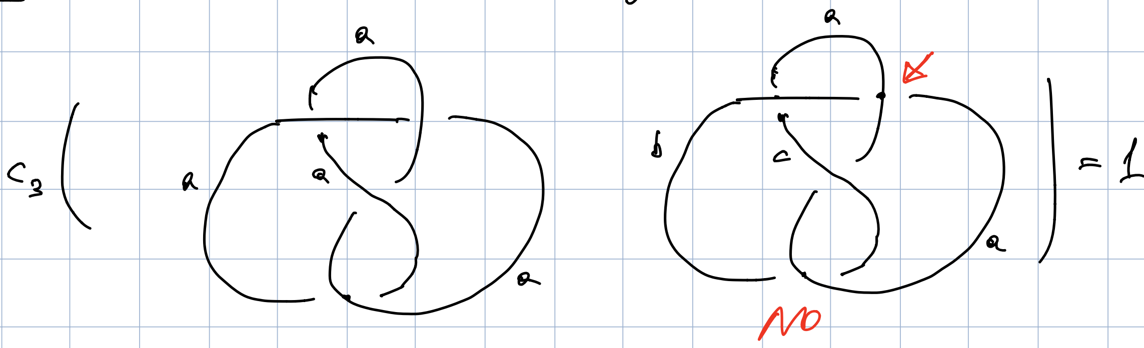
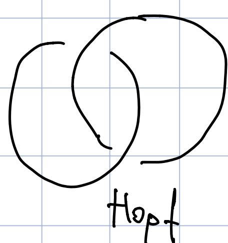
b
c
c
a
c
b

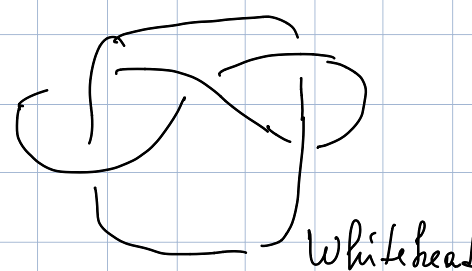


<u>Rem</u>: c_3 cannot show chirality. (Exercise)

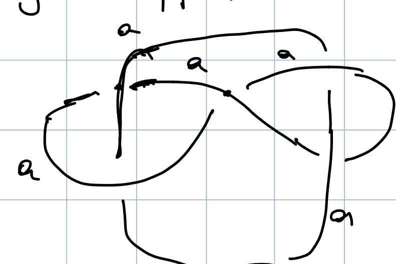


<u>Fact</u>:

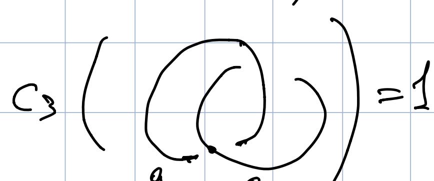


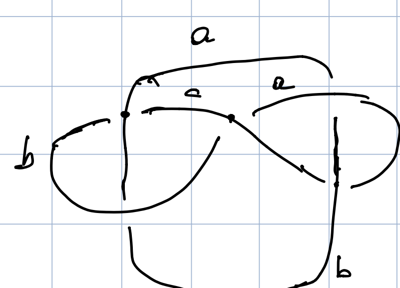


c_3 shows they are non-trivial
(easy for Hopf; also easy Hopf $\neq$ Whitehead)



$c_3(\bigcirc\ \bigcirc) = 2$



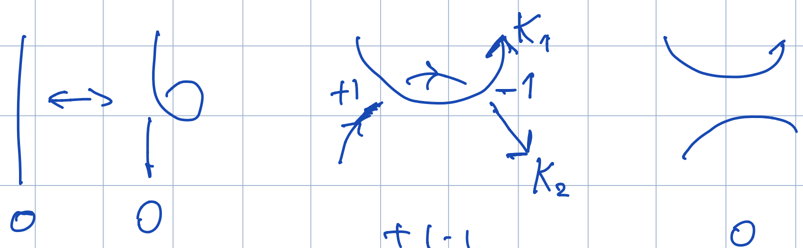


$c_3 = 1$

Def: if $\vec{K}_1, \vec{K}_2$ are oriented knots, $K_1 \cap K_2 = \emptyset$ their linking number is

$$\sum (+1) + \sum (-1)$$

Fact: invariant under R_*:



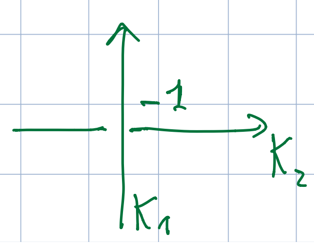
R_{III}: easy

Prop: $lk(\vec{K}_2, \vec{K}_1) = lk(\vec{K}_1, \vec{K}_2)$

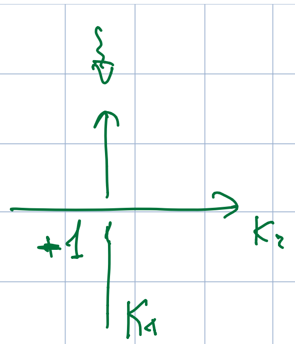
Proof: enough to show that

$$lk(\vec{K}_2, \vec{K}_1) - lk(\vec{K}_1, \vec{K}_2)$$

unchanged by crossing switches.



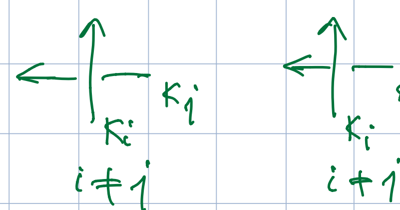
	$lk(K_2, K_1)$	$lk(K_1, K_2)$
	0	-1



$+1$ $\qquad\qquad\qquad$ 0

$\square$

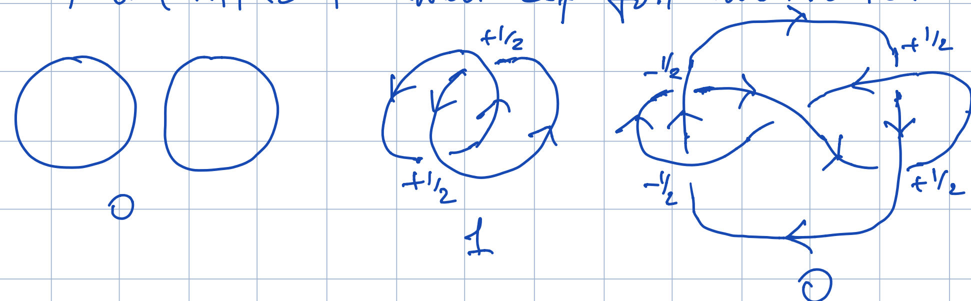
Consequence: $\quad lk(K_1, K_2) = \sum (+1) + \sum (-1)$



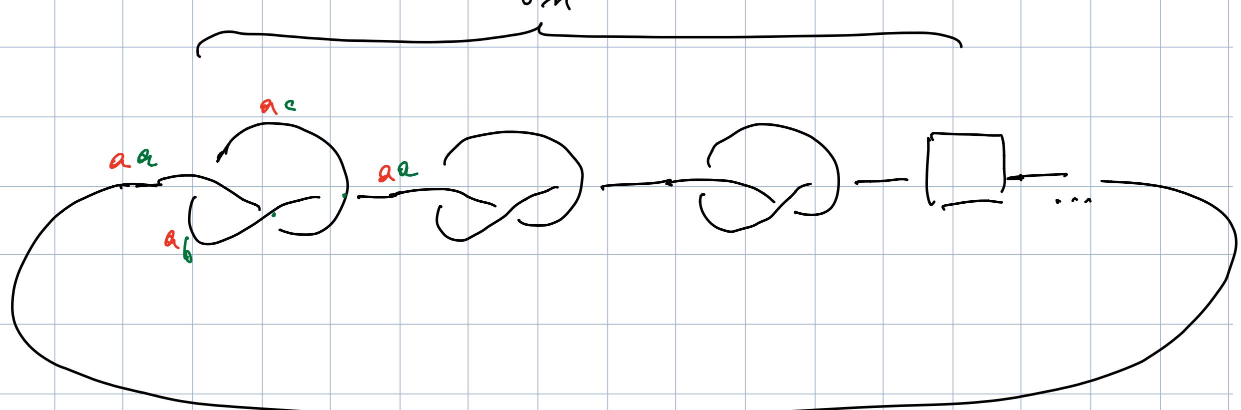
Rem: $lk(\vec{K}_1, -\vec{K}_2) = -lk(\vec{K}_1, \vec{K}_2)$

$\Rightarrow$ $|lk(K_1, K_2)|$ well-def for unoriented

$|lk|$:

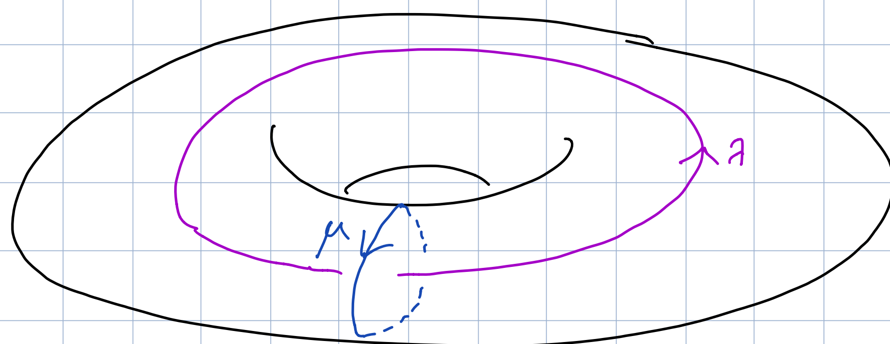


Fact: $\exists$ ∞ many isotopy classes of knots:



$c_3(T)$ increases with n

<u>Torus knots</u> : $\mathbb{T} \subset S^3$ trivially embedded solid torus.

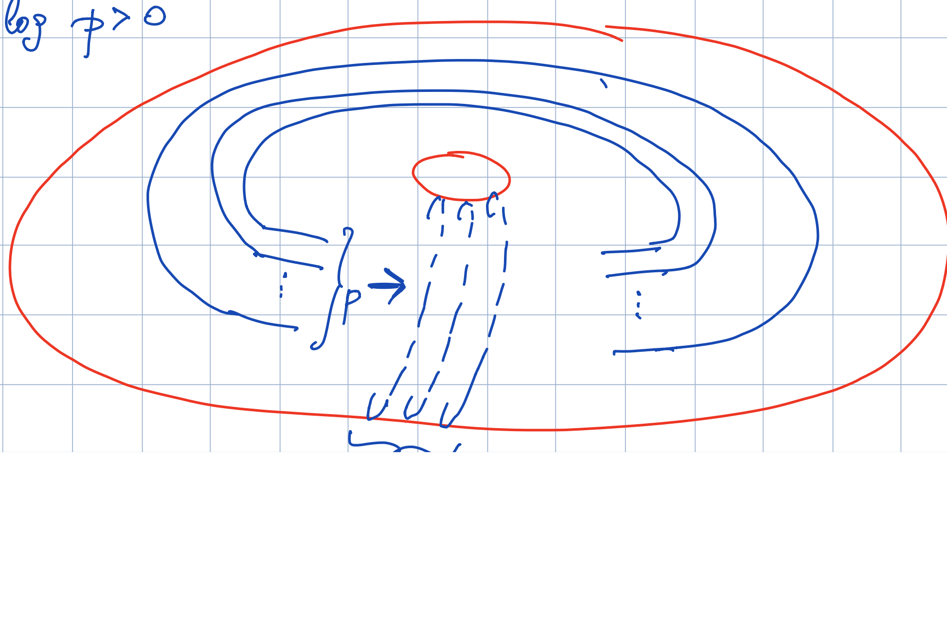


λ = core with some orientation μ = meridian wrapping positively around λ

λ also denotes a parallel copy pushed on $\partial\mathbb{T}$

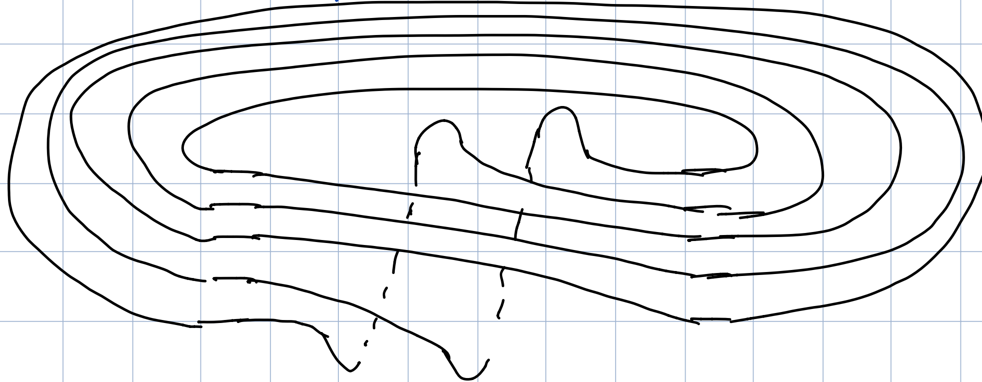
$p, q \in \mathbb{Z}$ coprime : $K_{p,q}$ = curve on $\partial\mathbb{T}$ homologous to $\pm(p\cdot\lambda + q\cdot\mu)$ (not oriented)

wlog $p > 0$



q

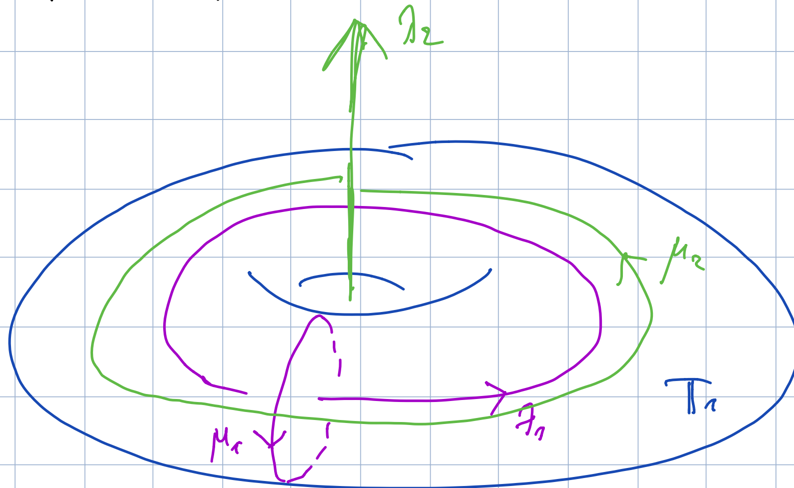
join these bending right for $q>0$, left for $q<0$

$K_{5,2} =$ 

For $GCD(p,q) = k > 0$ this gives k parallel copies of knot $K_{p/k,\, q/k}$

Prop: $K_{q,p}$ isotopic to $K_{p,q}$.

Pf:



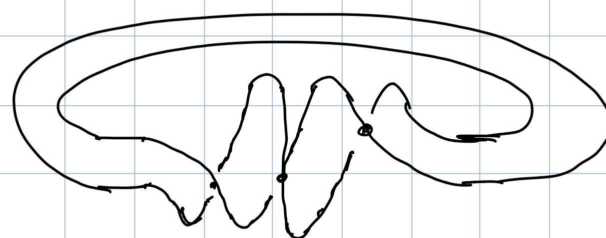
$S^3 \setminus T_1 = T_2$

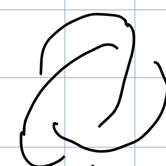
$\lambda_2 = \mu_1$ (pushed on ∂T)
$\mu_2 = \lambda_1$ (")

But T_2 and T_1 are isotopic.

Rem: $K_{p,-q}$ = mirror of $(K_{p,q})$

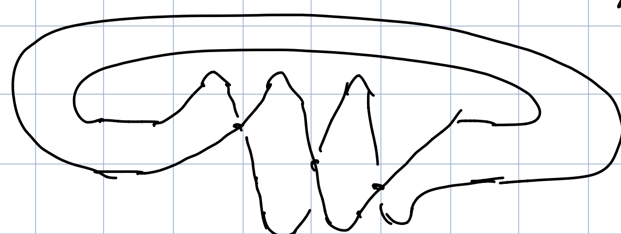
$K_{3,2} = K_{2,3}$

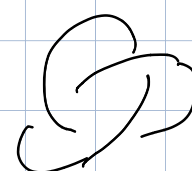




right-handed trefoil

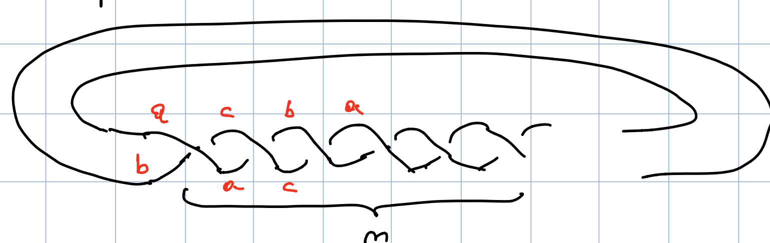
$K_{3,-2} = K_{2,-3}$



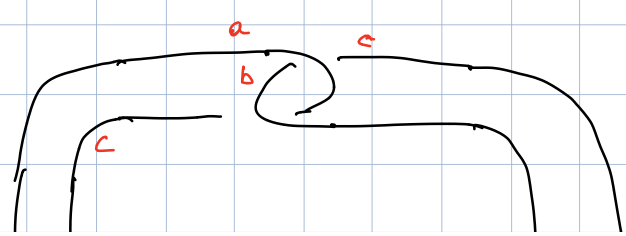


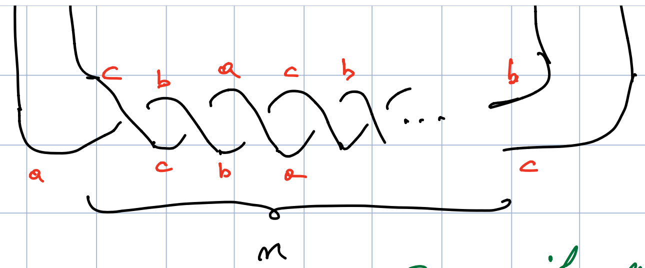
left-handed trefoil

Computation of $c_3(K_{2,m})$ $\quad m \in \mathbb{Z}$

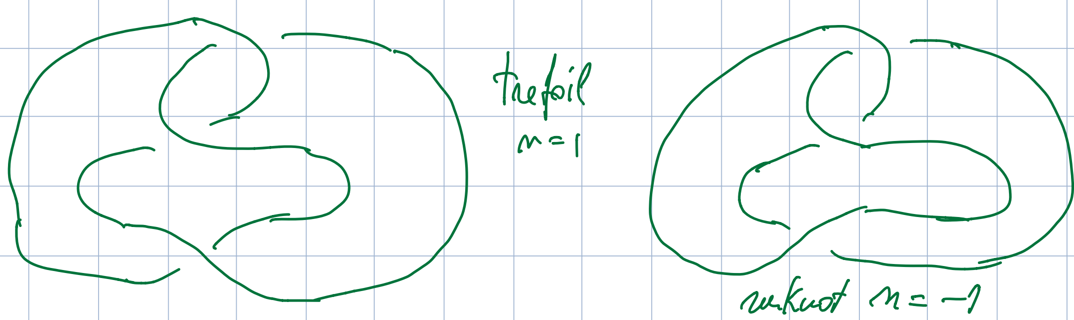


$$\Rightarrow c_3(K_{2,m}) = \begin{cases} 2 & \text{if } m \equiv 0 \ (3) \\ 1 & \text{otherwise} \end{cases}$$





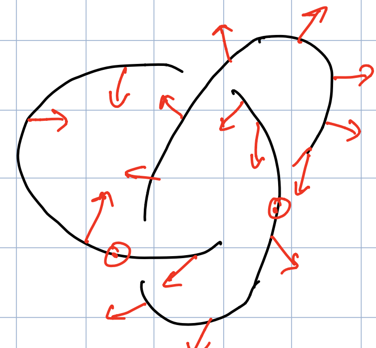
$$c_3(\uparrow) = \begin{cases} 2 & \text{if } n \equiv 1 \ (3) \\ 1 & \text{otherwise} \end{cases}$$



Framed knots and links (Smooth viewpoint)

A framing on a knot K is any of the following objects up to appropriate notion of isotopy:

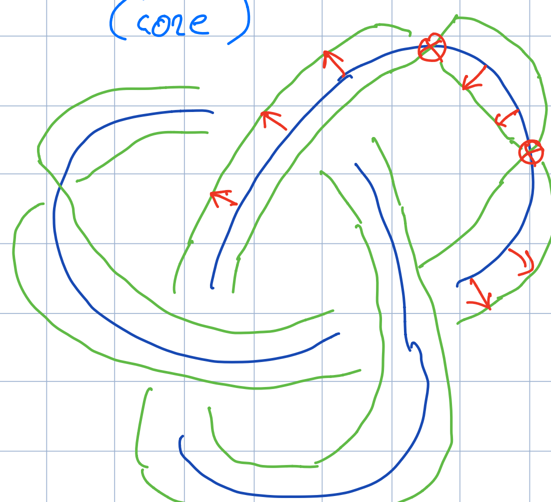
- nowhere-zero nowhere tangent vector field along K
- nowhere-zero normal vector field along K
- triple of vectors t, n, b along K



with t = tangent, n normal, $b = t \wedge n$

- Trivialization of normal bundle to K.

- embedding of a cylinder $S^1 \times [-1,1]$ in S^3 s.t. $K = S^1 \times \{0\}$ (core)

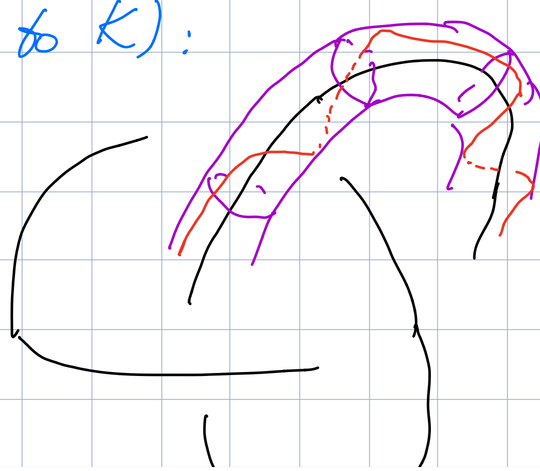


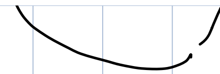
- embedding of cylinder $S^1 \times [0,1]$ in S^3 so that $K = S^1 \times \{0\}$.

$\left(\begin{aligned} U(K) &= \text{regular neighbourhood of } K \\ &= \text{solid torus with core } K \end{aligned} \right)$

Always

- choose a longitude on $\partial U(K)$ (curve parallel to K):





• parametrization $D^2 \times S^1 \longrightarrow U(K)$

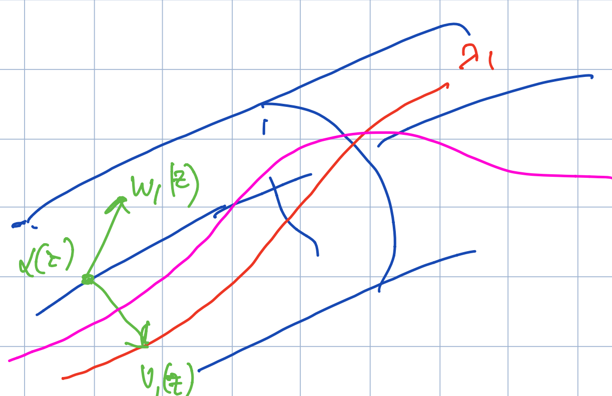
Prop: the set of framings on K is in a natural way an affine space over $\mathbb{Z}$.

Proof: use longitude viewpoint. λ_1, λ_2 longitudes on $\partial U(K)$
Choose parametrization of K, $\alpha : S^1 \to K$
(includes orientation).
λ_1 gives basis of normal bundle v_1, w_1 s.t.

trivialization of $\partial U(K)$ as

$$S^1 \times \underset{\substack{\parallel \\ [0,2\pi]/_{0=2\pi}}}{S^1} \ni (z,\vartheta) \mapsto \alpha(z) + \cos(\vartheta)\, v_1(z) + \sin(\vartheta)\, w_1(z)$$

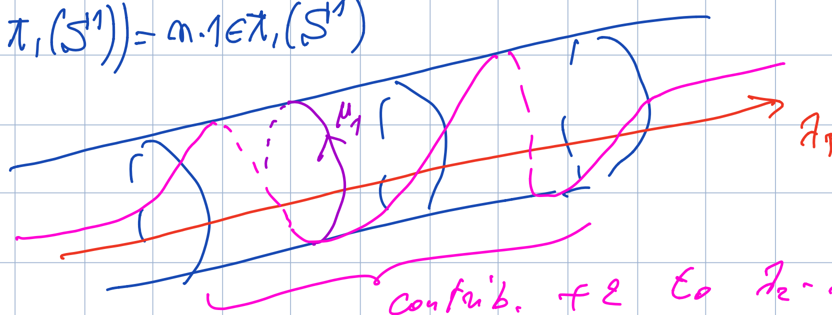
s.t. $\lambda_1 =$ image of $\alpha(z) + v_1(z)$



$$\lambda_2(z) = \alpha(z) + \cos(\theta(z))\cdot v_1(z) + \sin(\theta(z))\cdot w_1(z)$$

def: $\lambda_2 - \lambda_1 = \deg\left(S^1 \ni z \longmapsto e^{i\theta(z)} \in S^1\right)$

$\deg(f : S^1 \to S^1) = n$.
if $f_*(1 \in \pi_1(S^1)) = n \cdot 1 \in \pi_1(S^1)$



Facts: indep. of orientation

changing α to $\tilde{\alpha}(z) = \alpha(\bar{z})$) opposite orientation
gives
$$\lambda_2 - \lambda_1 = \deg\left(S^1 \ni z \longmapsto e^{-i\theta(\bar{z})} \in S^1\right)$$
same as before.

- well-def / isotopy of λ_1, λ_2

- knowing λ_1 and $\lambda_2 - \lambda_1$ determines λ_2 / isotopy because curves on $\partial U(T)$ are determined by their homology class.

<u>Fact</u>: there is a natural 0-framing

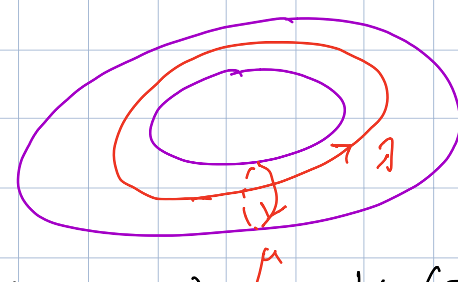
so $\{\text{framings on } K\} = \mathbb{Z}$ naturally.

$U(K)$ = reg. neighborhood of K

$E(K) = S^3 \setminus U(K)$ (K unknot, $E(K)$ = solid torus)

$S^3 = U \cup E \qquad U \cap E = T$

M-V:



$$\begin{array}{ccccccc} H_2(S^3) & \to & H_1(T) & \to & H_1(U) \oplus H_1(E) & \to & H_1(S^3) \\ \| & & \| & & \| & & \| \\ 0 & & \mathbb{Z}_\lambda \oplus \mathbb{Z}_\mu & & \mathbb{Z}_\lambda & & 0 \end{array}$$

$\Longrightarrow H_1(E) = \mathbb{Z}_\mu$

$j : T \longrightarrow E \qquad j_*(\lambda) = n \cdot \mu$

$\Longrightarrow \lambda_0 = \lambda - n \cdot \mu$

$j_*(\lambda_0) = 0$.

Basepoint λ_0 for $\{\text{framings}\} = \{\text{longitudes}\}$ on K

= unique longitude that is 0 in $H_1(E(K))$.