Alg. Lin

5.4.4 (b) $f: \mathbb{R}^2 \longrightarrow \mathbb{R}^3$

$$f(x) = \begin{pmatrix} x_1 + 2x_2 \\ 3x_1 - x_2 \\ 5x_1 + 4x_2 \end{pmatrix}$$

$$\mathcal{B} = \left(\underset{\substack{\shortparallel \\ b_1}}{\begin{pmatrix} 2 \\ -1 \end{pmatrix}}, \underset{\substack{\shortparallel \\ b_2}}{\begin{pmatrix} 3 \\ 5 \end{pmatrix}} \right)$$

sono base
perché sono 2
vettori $\neq 0$, non
uno mult. di altro
in $\mathbb{R}^2$

$$\mathcal{C} = \left(\underset{c_1}{\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}}, \underset{c_2}{\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}}, \underset{c_3}{\begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}} \right)$$

$$\det \begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & 2 \\ 1 & 1 & 0 \end{pmatrix} = (-1) \begin{vmatrix} 1 & 2 \\ 1 & 0 \end{vmatrix} + (-1) \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} =$$

$$= -(-2) - 1 = 1 \neq 0$$

$\mathcal{C}$ è base di $\mathbb{R}^3$

$$f(b_1) = \begin{pmatrix} 0 \\ 7 \\ 6 \end{pmatrix} = \alpha_1 c_1 + \alpha_2 c_2 + \alpha_3 c_3$$

$$f(b_2) = \begin{pmatrix} 13 \\ 4 \\ 35 \end{pmatrix} = \beta_1 c_1 + \beta_2 c_2 + \beta_3 c_3$$

$(\alpha_2 = \alpha_3)$

$$\begin{cases} 0 = 0\alpha_1 + \alpha_2 - \alpha_3 \\ 7 = \alpha_1 + 2\alpha_3 \\ 6 = \alpha_1 + \alpha_2 \end{cases} \qquad \alpha_2 = \alpha_3$$

$$II \rightarrow \quad 7 = \alpha + 2\alpha_2$$

$$6 = \alpha_1 + \alpha_2 \qquad 1 = \alpha_2$$

$$\alpha_1 = 5 \qquad \alpha_3 = 1$$

$$\begin{cases} 13 = \beta_2 - \beta_3 \\ 4 = \beta_1 + 2\beta_3 \\ 35 = \beta_1 + \beta_2 \end{cases}$$

$$\begin{cases} 17 = \beta_1 + \beta_2 - \beta_3 \\ 13 = \beta_2 - \beta_3 \\ 35 = \beta_1 + \beta_2 \end{cases}$$

$$18 = +\beta_3$$

$$\beta_2 = 13 + 18 = 31$$

$$\beta_1 = 35 - 31 = 4$$

$$[f]^{C}_{B} = \begin{pmatrix} 5 & 4 \\ 1 & 31 \\ 1 & 18 \end{pmatrix}$$

5.4.4 (d)

matrici antisim 3×3

$f : \mathbb{R}^2 \longrightarrow A_3$ $\quad (a_{ij} = -a_{ji})$

$$\mathcal{B} = \left(\begin{pmatrix} 3 \\ 7 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \end{pmatrix} \right)$$

$$\underset{b_1}{\overset{\parallel}{}} \qquad \underset{b_2}{\overset{\parallel}{}}$$

2 vett non nulli, non multiplo l'uno dell'altro

$$\mathcal{C} = \left(\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 2 & -1 \\ -2 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 3 \\ 1 & -3 & 0 \end{pmatrix} \right)$$

$$\underset{C_1}{\overset{\parallel}{}} \qquad \underset{C_2}{\overset{\parallel}{}} \qquad \underset{C_3}{\overset{\parallel}{}}$$

$$c_2 - 2c_1 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$\frac{1}{3}\left(-(c_2 - 2c_1) + c_3\right) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

Sono 3 vett lin indip $\Rightarrow$

$\dim \operatorname{Span}(c_1, c_2, c_3) = 3$

$\Rightarrow \mathcal{C}$ è base di A_3 che ha dim 3

$$f(b_1) = \begin{pmatrix} 0 & -3 & -1 \\ 3 & 0 & -19 \\ 1 & +19 & 0 \end{pmatrix} = \alpha_1 c_1 + \alpha_2 c_2 + \alpha_3 c_3$$

$$f(b_2) = \begin{pmatrix} 0 & -2 & 1 \\ 2 & 0 & -7 \\ -1 & 7 & 0 \end{pmatrix} = \beta_1 c_1 + \beta_2 c_2 + \beta_3 c_3$$

$$f\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 & -x_1 & 2x_1 - x_2 \\ x_1 & 0 & x_1 - 3x_2 \\ x_2 - 2x_1 & 3x_2 - x_1 & 0 \end{pmatrix}$$

$$f(b_2) = -2 \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} - 1 \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} - 7 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} =$$

$$= -2 c_1 - 1 (c_2 - 2c_1) - \frac{7}{3} (2c_1 - c_2 + c_3) =$$

$$= \frac{-6 + 6 - 14}{3} c_1 + \frac{-3 + 7}{3} c_2 - \frac{7}{3} c_3$$

$$f(b_2) = \underset{\overset{\parallel}{\beta_1}}{-\frac{14}{3}} c_1 + \underset{\overset{\parallel}{\beta_2}}{\frac{4}{3}} c_2 \underset{\overset{\parallel}{\beta_3}}{-\frac{7}{3}} c_3$$

$$f(b_1) = -3 \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} - 18 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} =$$

$$= -3c_1 + (-2c_1 + c_2) - \overset{6}{\cancel{18}} \cdot \frac{1}{\cancel{3}} (2c_1 - c_2 + c_3) =$$

$$= \underset{\overset{\parallel}{\alpha_1}}{-17} c_1 + \underset{\overset{\parallel}{\alpha_2}}{7} c_2 \underset{\overset{\parallel}{\alpha_3}}{-6} c_3$$

$$[f]_{\mathcal{B}}^{\mathcal{E}} = \begin{pmatrix} -17 & -14/3 \\ 7 & 4/3 \\ -6 & -7/3 \end{pmatrix}$$

5.4.6 X, Y, Z, W sp vett

$g : X \to Y$
$h : Z \to W$ app lin

$$A = \{ f \in \mathcal{L}(Y, Z) : h \circ f \circ g = 0 \}$$

è ssp di $\mathcal{L}(Y, Z)$

Calcolare dim A in funz di dim X, Y, Z, W,

$\dim \operatorname{Im}(h)$, $\dim \operatorname{Im}(g)$.

$0 \in \mathcal{L}(Y, Z)$ $\qquad 0 \in A$

$h \circ 0 \circ g = 0$

$f_1, f_2 \in A \implies h \circ f_1 \circ g = 0$, $h \circ f_2 \circ g = 0$

$h \circ (f_1 + f_2) \circ g = h \circ f_1 \circ g + h \circ f_2 \circ g =$

$= 0 + 0 = 0$

$\implies f_1 + f_2 \in A$

Analogamente, se $f \in A$ posso dire $\lambda f \in A$?
($\lambda \in \mathbb{R}$)

Vale se

$$\underbrace{h \circ (\lambda f) \circ g}_{} = 0$$

$$\| $$

$$\lambda (h \circ f \circ g) = \lambda \cdot 0 = 0 \quad \underline{ok}$$

$\Rightarrow$ A è ssp vett di $\mathcal{L}(Y, Z)$

<u>Calcoliamo dim A</u>

$$f \in A \Leftrightarrow h \circ f \circ g = 0 \Leftrightarrow f(\mathrm{Im}(g)) \subset \mathrm{Ker}(h)$$

$v \in X \qquad g(v) \in \mathrm{Im}(g)$

$f(g(v)) \in \mathrm{Ker}(h)$

$\Updownarrow$

$h(f(g(v))) = 0$

$\dim \{ f : \mathcal{L}(Y, Z) \quad A_c \quad f(\mathrm{Im}(g)) \subset \mathrm{Ker}(h) \}$

$g : X \to Y \qquad \mathrm{Im}(g) \subset Y \qquad \dim \mathrm{Im}(g) = k$

Prendo base di $\mathrm{Im}(g) = \{ v_1, \dots, v_k \} = \mathcal{B}_1$

e completo a base di Y $\qquad \dim Y = n$

$\{ v_1, \dots, v_k, v_{k+1}, \dots, v_n \} = \mathcal{B}$

$Ker(h) \subset Z$ prendo base di $Ker(h) =$

$\dim Ker(h) = l$ $\mathcal{C}_1 = \{w_1, \dots, w_l\} \subset Z$

e la completo a base di Z

$\dim Z = m$

base di $Z = \{w_1, \dots, w_l, w_{l+1}, \dots, w_m\} = \mathcal{C}$

$f \in A$ sse $f(v_1) \in Span(w_1, \dots, w_l)$

..

$f(v_k) \in Span(w_1, \dots, w_l)$

$f \in A$ sse la matrice di f risp alle basi $\mathcal{B}$, $\mathcal{C}$ è

$$[f]_{\mathcal{B}}^{\mathcal{C}} = \begin{array}{cc}
\left(\begin{array}{ccc|ccc}
* & \cdots & * & & & \\
\vdots & & \vdots & & * & \\
* & \cdots & * & & & \\ \hline
0 & \cdots & 0 & & & \\
\vdots & & \vdots & & * & \\
0 & \cdots & 0 & & &
\end{array}\right) & \begin{array}{c} \\ \\ l \\ l+1 \\ \\ m \end{array} \\
\begin{array}{cccccc} 1 & \quad & k & k+1 & \cdots & n \end{array} &
\end{array}$$

$$\dim A = m \cdot n - (k \cdot (m - l)) =$$

$$= \dim Z \cdot \dim Y - \dim \mathrm{Im}(g) \cdot \dim \mathrm{Im}(h)$$

5.4.3

$$\mathcal{B} = \left(\underset{v_1}{\begin{pmatrix} 3 \\ 8 \end{pmatrix}}, \underset{v_2}{\begin{pmatrix} 2 \\ 5 \end{pmatrix}} \right)$$

$$\mathcal{C} = \left(\underset{w_1}{\begin{pmatrix} 4 \\ 1 \\ 0 \end{pmatrix}}, \underset{w_2}{\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}}, \underset{w_3}{\begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}} \right)$$

$f : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$

$$f(x) = \begin{pmatrix} 3x_1 - 5x_2 \\ -x_1 + 2x_2 \\ 2x_1 - 3x_2 \end{pmatrix}$$

VERIFICHIAMO:

$$v = \begin{pmatrix} -3 \\ 2 \end{pmatrix} \qquad [f(v)]_{\mathcal{C}} = [f]_{\mathcal{B}}^{\mathcal{C}} \cdot [v]_{\mathcal{B}}$$

$v = \lambda_1 v_1 + \lambda_2 v_2$ $\quad \lambda_1 = 19, \lambda_2 = -3$

risolvo il sistema

$$[v]_{\mathcal{B}} = \begin{pmatrix} 19 \\ -30 \end{pmatrix}$$

$[f(v)]_e$ $\qquad f(v) = \begin{pmatrix} -19 \\ 7 \\ -12 \end{pmatrix} = e_1 w_1 + e_2 w_2 + e_3 w_3$

$\downarrow$ (risolvo il sistema)

$$[f(v)]_e = \begin{pmatrix} 78 \\ -71 \\ 130 \end{pmatrix} \qquad e_1 = 78 \;,\; e_2 = -71 \;,\; e_3 = 130$$

$[f]^e_B$

$$f(v_1) = f\begin{pmatrix} 3 \\ 8 \end{pmatrix} = \begin{pmatrix} -31 \\ 13 \\ -18 \end{pmatrix} = 132 \begin{pmatrix} 4 \\ 1 \\ 0 \end{pmatrix} - 119 \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} + 220 \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$$

$$f(v_2) = f\begin{pmatrix} 2 \\ 5 \end{pmatrix} = \begin{pmatrix} -19 \\ 8 \\ -11 \end{pmatrix} = 81 \begin{pmatrix} 4 \\ 1 \\ 0 \end{pmatrix} - 73 \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} + 135 \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$$

$$[f]^{e}_{B} = \begin{pmatrix} 132 & 71 \\ -119 & -73 \\ 220 & 135 \end{pmatrix}$$

$$\begin{pmatrix} 78 \\ -71 \\ 130 \end{pmatrix} = \begin{pmatrix} 132 & 71 \\ -119 & -73 \\ 220 & 135 \end{pmatrix} \begin{pmatrix} 19 \\ -30 \end{pmatrix}$$

questo calcolo è la verifica richiesta

5.5.2 $\quad f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$$B = (\underbrace{3e_1 - 4e_2}_{v_1}, \underbrace{-2e_1 + 7e_2}_{v_2})$$

$$[f]^{B}_{B} = \begin{pmatrix} 4 & -1 \\ 5 & 6 \end{pmatrix}$$

OSS

$-7v_1 - 4v_2 = e_1$

$-2v_1 - 3v_2 = e_2$

f è invertibile perché è invertibile

la matrice $\begin{pmatrix} 4 & -1 \\ 5 & 6 \end{pmatrix}$.

$f(e_1 - 2e_2)$

$e_1 - 2e_2 = \alpha_1 v_1 + \alpha_2 v_2$

$= (-7v_1 - 11v_2) - 2(-2v_1 - 3v_2) =$

$= -3v_1 - 5v_2$ $\qquad (e_1 - 2e_2)_B = \begin{pmatrix} -3 \\ -5 \end{pmatrix}$

$(f(e_1 - 2e_2))_B = [f]_B (v)_B =$

$= \begin{pmatrix} 4 & -1 \\ 5 & 6 \end{pmatrix} \begin{pmatrix} -3 \\ -5 \end{pmatrix} = \begin{pmatrix} -7 \\ -45 \end{pmatrix}$

$$f(e_1 - 2e_2) = -7v_1 - 45v_2 =$$

$$= -7(3e_1 - 4e_2) - 45(-2e_1 + 7e_2) =$$

$$= 69e_1 - 238e_2$$

$$f^{-1}(2e_1 + e_2)$$

$$[f^{-1}]_{B}^{B} = [f]_{B}^{B^{-1}} = \begin{pmatrix} 4 & -1 \\ 5 & 6 \end{pmatrix}^{-1} = \frac{1}{29}\begin{pmatrix} 6 & 1 \\ -5 & 4 \end{pmatrix}$$

$$2e_1 + e_2 = 2(-7v_1 - 11v_2) + (-2v_1 - 3v_2) =$$

$$= -16v_1 - 25v_2$$

$$(2e_1 + e_2)_B = \begin{pmatrix} -16 \\ -25 \end{pmatrix}$$

$$\left(f^{-1}(2e_1+e_2)\right)_B = \left[f^{-1}\right]_B^B \begin{pmatrix}-16\\-25\end{pmatrix} =$$

$$= \frac{1}{29}\begin{pmatrix}6 & 1\\ -5 & 4\end{pmatrix}\begin{pmatrix}-16\\-25\end{pmatrix} = \frac{1}{29}\begin{pmatrix}-97\\-20\end{pmatrix}$$

$$f^{-1}(2e_1+e_2) = -\frac{97}{29}v_1 - \frac{20}{29}v_2 =$$

$$= -\frac{97}{29}(3e_1-4e_2) - \frac{20}{29}(-2e_1+7e_2) =$$

$$= \frac{1}{29}\left(-251\, e_1 + 927\, e_2\right)$$

$$\begin{pmatrix} 4 & -1 \\ 5 & 6 \end{pmatrix}^{-1} = \frac{1}{29}\begin{pmatrix} 6 & 1 \\ -5 & 4 \end{pmatrix}$$

<u>verifica</u>

$$\begin{pmatrix} 6 & 1 \\ -5 & 4 \end{pmatrix}\begin{pmatrix} 4 & -1 \\ 5 & 6 \end{pmatrix} = \begin{pmatrix} 29 & 0 \\ 0 & 29 \end{pmatrix} \checkmark$$

5.5.3 (e)

$V = \mathbb{R}^2$

$$B = \left(\overset{v_1}{\begin{pmatrix} 2 \\ -5 \end{pmatrix}}, \overset{v_2}{\begin{pmatrix} 3 \\ 7 \end{pmatrix}} \right)$$

$$B' = \left(\underset{w_1}{\begin{pmatrix} 4 \\ 1 \end{pmatrix}}, \underset{w_2}{\begin{pmatrix} 2 \\ -3 \end{pmatrix}} \right)$$

$$v_1 = a_{11} w_1 + a_{21} w_2$$

$$\begin{pmatrix} 2 \\ -5 \end{pmatrix} = -\frac{2}{7} \begin{pmatrix} 4 \\ 1 \end{pmatrix} + \frac{4}{7} \begin{pmatrix} 2 \\ -3 \end{pmatrix}$$

$$v_2 = a_{12} w_1 + a_{22} w_2$$

$$\begin{pmatrix} 3 \\ 7 \end{pmatrix} = \frac{23}{14} \begin{pmatrix} 4 \\ 1 \end{pmatrix} - \frac{25}{14} \begin{pmatrix} 2 \\ -3 \end{pmatrix}$$

$$[Id]_{B}^{B'} = \begin{pmatrix} -2/7 & 23/14 \\ 4/7 & -25/14 \end{pmatrix}$$

$$w_1 = \begin{pmatrix} 4 \\ 1 \end{pmatrix} = \frac{25}{23} \underbrace{\begin{pmatrix} 2 \\ -5 \end{pmatrix}}_{v_1} + \frac{22}{23} \underbrace{\begin{pmatrix} 3 \\ 7 \end{pmatrix}}_{v_2}$$

$$w_2 = \begin{pmatrix} 2 \\ -3 \end{pmatrix} = \frac{23}{29}\begin{pmatrix} 2 \\ -5 \end{pmatrix} + \frac{4}{29}\begin{pmatrix} 3 \\ 7 \end{pmatrix}$$

$$[Id]_{B'}^{B} = \begin{pmatrix} 25/29 & 23/29 \\ 22/29 & 4/29 \end{pmatrix}$$

si può verificare

$$\begin{pmatrix} 25/29 & 23/29 \\ 22/29 & 4/29 \end{pmatrix}\begin{pmatrix} -2/7 & 23/14 \\ 11/7 & -25/14 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

5.5.3 (d) $\quad V = \{ p(t) \in \mathbb{R}_{\leq 2}[t] : p(-1) = 0 \}$

$B = (1+t,\ 1-t^2) \qquad B' = (1+2t+t^2,\ 2-t-3t^2)$

$$p(t) = a_0 + a_1 t + a_2 t^2 \longleftrightarrow \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix}$$

$$p(-1) = 0 \qquad a_0 - a_1 + a_2 = 0$$

$$B = \left(\underset{v_1}{\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}}, \underset{v_2}{\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}} \right) \qquad B' = \left(\underset{w_1}{\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}}, \underset{w_2}{\begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}} \right)$$

$$v_1 = \frac{3}{5} w_1 + \frac{1}{5} w_2$$

$$v_2 = \frac{1}{5} w_1 + \frac{2}{5} w_2$$

$$[Id]_B^{B'} = \begin{pmatrix} 3/5 & 1/5 \\ 1/5 & 2/5 \end{pmatrix}$$

$$w_1 = 2v_1 - v_2$$

$$w_2 = -v_1 + 3v_2$$

$$[Id]_{B'}^{B} = \begin{pmatrix} 2 & -1 \\ -1 & 3 \end{pmatrix}$$

$$\begin{pmatrix} 3/5 & 1/5 \\ 1/5 & 2/5 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ -1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$