Algebra lineare 12/12/18

$\boxed{20}$

$\boxed{1}$ Determinare $[18e_2 - 11e_1]_{\mathcal{B}}$ dove

$$\mathcal{B} = (3e_1 + 2e_2,\ 5e_2 - 2e_1)$$

$$\begin{pmatrix} -11 \\ 18 \end{pmatrix} = \alpha \begin{pmatrix} 3 \\ 2 \end{pmatrix} + \beta \begin{pmatrix} -2 \\ 5 \end{pmatrix}$$

$$19\alpha = -55 + 36$$

$\alpha = -1$

$\beta = 4$

$$\begin{pmatrix} -1 \\ 4 \end{pmatrix}$$

$\boxed{2}$ Calcolare $(A^{-1})_{32}$ con $A = \begin{pmatrix} 2 & 0 & 1 \\ -1 & 3 & 2 \\ 0 & 4 & 1 \end{pmatrix}$

$$(A^{-1})_{32} = \frac{(-1)^{3+2}}{6-4-16} \cdot 8 = \frac{4}{7}$$

$\boxed{3}$ Se $f : \mathbb{C}^8 \to \mathbb{C}^3$ è lin., $f(2ie_5 - e_7) = e_3 - ie_1$,
$\mathbb{C}^8 = W \oplus \mathrm{Ker}(f)$; $\dim W = ?$

$$1 \le \dim(\mathrm{Im}(f)) \le 3$$

$$5 \le \dim(\mathrm{Ker}(f)) \le 7$$

$$1 \le \dim(W) \le 3$$

$\boxed{4}$ $A \in M_{4\times 4}$ $\quad A = (v_1, v_2, v_3, v_4)$ $\quad \det(A) = -1/4$

$$\det(v_1 - 2v_4,\ 4v_3,\ v_2 + \sqrt{5}v_3,\ 2v_1 - v_2 + v_3 - v_4)$$

$$= \det\left((v_1, v_2, v_3, v_4) \cdot \begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & -1 \\ 0 & 4 & \sqrt{5} & 1 \\ -2 & 0 & 0 & -1 \end{pmatrix} \right)$$

$$= \det(A) \cdot (-1) \cdot 4 \cdot \det \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ -2 & 0 & -1 \end{pmatrix}$$

$$= -\frac{1}{4} \cdot (-4) \cdot \det \begin{pmatrix} 1 & 2 \\ -2 & -1 \end{pmatrix} = 3$$

$\boxed{5}$ Risolvere $\begin{cases} 2x - 3y + 4z = 5 \\ 3x + 2y - 5z = 3 \\ x - 8y + 13z = 7 \end{cases}$ ✓

$$\begin{cases} x = 8y - 13z + 7 \\ 16y - 26z + 14 - 3y + 4z = 5 \\ 24y - 39z + 21 + 2y - 5z = 3 \end{cases} \qquad \begin{cases} 13y - 22z = -9 \\ \cancel{26y - 44z = -18} \\ x = 8y - 13z + 7 \end{cases}$$

Sol. part: $y = 1$, $z = 1$ $\Rightarrow$ $x = 2$

Giacitura: $y = 22$ $z = 13$ $\Rightarrow$ $x = 176 - 169 = 7$

Solut.: $\begin{pmatrix} 2 \\ i \\ 1 \end{pmatrix} + t \cdot \begin{pmatrix} 7 \\ 2 \\ 3 \end{pmatrix}$

$\boxed{6}$ Sapendo che $2z^3 - (4+i)z^2 + 7z + 3i - 1 = 0$ ha solut. $z_1 = -\frac{i}{2}$ trovare le altre.

$$\begin{array}{c|ccc|c} & 2 & -4-i & 7 & 3i-1 \\ -i/2 & & -i & 2i-1 & 1-3i \\ \hline & 2 & -4-2i & 2i+6 & \checkmark \end{array}$$

polinomio iniz $= \left(z+\frac{i}{2}\right)\left(2z^2+(-4-2i)z+2i+6\right)$

$$= (2z+i)\left(z^2-(2+i)z+i+3\right)$$

$$\Delta = (2+i)^2-4(i+3) = 4+4i-1-4i-12 = -9$$

$$z_{2,3} = \frac{(2+i)\pm 3i}{2} = \begin{cases} 1+2i \\ 1-i \end{cases}$$

$\boxed{7}$ $W = \mathrm{Span}\left(\begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix}, \begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix} \right)$ $Z = \mathrm{Span}\begin{pmatrix} 4 \\ -1 \\ 2 \end{pmatrix}$

Trovare proiez. su W di $\begin{pmatrix} 1 \\ 4 \\ 1 \end{pmatrix}$ rispetto a $\mathbb{R}^3 = W \oplus Z$.

<u>I soluz</u>:

$$\begin{pmatrix} 1 \\ 4 \\ 1 \end{pmatrix} = \underbrace{\alpha \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix} + \beta \begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix}} + \gamma \begin{pmatrix} 4 \\ -1 \\ 2 \end{pmatrix}$$

trovo α, β (γ) e lui è la risposta.

II soluz: $\quad W: \; 7x + 5y + 4z = 0$. Cerco:

$$\begin{pmatrix} 1 \\ 4 \\ 1 \end{pmatrix} = P_W \begin{pmatrix} 1 \\ 4 \\ 1 \end{pmatrix} + t \cdot \begin{pmatrix} 4 \\ -1 \\ 2 \end{pmatrix}$$

ovvero t t.c. $\begin{pmatrix} 1 \\ 4 \\ 1 \end{pmatrix} - t \begin{pmatrix} 4 \\ -1 \\ 2 \end{pmatrix}$ soddisfi equaz. di W:

$$(7, 5, 4) \left(\begin{pmatrix} 1 \\ 4 \\ 1 \end{pmatrix} - t \begin{pmatrix} 4 \\ -1 \\ 2 \end{pmatrix} \right) = 0$$

$$7 + 20 + 4 - t(28 - 5 + 8) = 0$$

$t = 1 \Rightarrow P_W\begin{pmatrix}1\\4\\1\end{pmatrix} = \begin{pmatrix}-3\\5\\-1\end{pmatrix}$

$\boxed{21}$ $\boxed{1}$ $X = \{x \in \mathbb{R}^3 : 3x_1 - 2x_2 + 5x_3 = 0\}$

$v = \begin{pmatrix}1\\4\\1\end{pmatrix}$ $\mathcal{B} = \left(\begin{pmatrix}4\\11\\2\end{pmatrix}, \begin{pmatrix}3\\2\\-1\end{pmatrix}\right)$

Provare che $v \in X$, $\mathcal{B}$ base di X e $[v]_{\mathcal{B}}$.

$3 - 8 + 5 = 0$ $12 - 22 + 10 = 0$ $9 - 4 - 5 = 0$

$\begin{pmatrix}1\\4\\1\end{pmatrix} = \alpha \begin{pmatrix}4\\11\\2\end{pmatrix} + \beta \begin{pmatrix}3\\2\\-1\end{pmatrix}$

$$\begin{cases} 4\alpha + 3\beta = 1 \\ 2\alpha - \beta = 1 \end{cases} \qquad \begin{cases} \alpha = \dfrac{2}{5} \\ \beta = -\dfrac{1}{5} \end{cases}$$

$$\frac{2}{5} \cdot 11 - \frac{1}{5} \cdot 2 = +4 \quad \checkmark$$

$$\frac{1}{5}\begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

$\boxed{2}$ $A = \begin{pmatrix} -1 & 2 & 3 \\ 2 & -3 & 1 \\ 1 & -2 & 1 \end{pmatrix}$; $(A^{-1})_{23}$

$$= \frac{(-1)^{2+3}}{3+2+9 \atop -12-4-2} \cdot (-7) = -\frac{7}{4}$$

$\boxed{3}$ $\det\begin{pmatrix} 2 & 0 & 0 & -1 \\ 1 & 2 & 0 & 3 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 2 & 0 \end{pmatrix}$

$$= \det\begin{pmatrix} 2 & 0 & 0 & -1 \\ 7 & 2 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 2 & 0 \end{pmatrix} = 14-2 = 12$$

$\boxed{4}$ $f : \mathbb{C}^5 \to \mathbb{C}^7$ lin. $f(2ie_1 + e_3) = f(e_2)$

$\mathbb{C}^7 = X \oplus \mathrm{Im}(f)$; $\dim(X) = ?$

Sappiamo $f(2ie_1 - e_2 + e_3) = 0$

$$1 \le \dim(\mathrm{Ker}(f)) \le 5$$

$$0 \le \dim(\mathrm{Im}(f)) \le 4$$

$$3 \le \dim(X) \le 7$$

$\boxed{5}$ Risolvere
$$\begin{cases} 2x + 3y - 4z = 1 \\ 3x - y - 2z = 2 \\ x + 7y - 6z = 0 \end{cases}$$

$$y = 3x - 2z - 2$$

$$\begin{cases} 2x + 9x - 6z - 6 - 4z = 1 \\ \cancel{x} + 21\cancel{x} - 14z - 14 - 6z = 0 \end{cases}$$

$$\begin{cases} 11x - 10z = 7 \\ \cancel{22x - 20z = 14} \end{cases}$$

Sol part: $x = z = 7 \Rightarrow y = 5$

Gen. giac.: $x = 10 \quad z = 11 \Rightarrow y = 8$

$$\begin{pmatrix} 7 \\ 5 \\ 7 \end{pmatrix} + \operatorname{Span}\begin{pmatrix} 10 \\ 8 \\ 11 \end{pmatrix}$$

$\boxed{6}$ Trovare $z \in \mathbb{C}$ t.c. $z \cdot e_2 - 2e_3$ appartenga a $\operatorname{Span}\left((1-i)e_1 + (1+2i)e_2 + 2ie_3,\ (2+i)e_1 + (2-i)e_2 - e_3 \right)$.

non mult. f_{20} to 20

$v \in \mathbb{C}^3$; $X = \operatorname{Span}(w_1, w_2)$ $\dim_{\mathbb{C}}(X) = 2$.

$v \in X \iff v, w_1, w_2$ lin. dip.

$\iff \det(v, w_1, w_2) = 0 \iff \det \begin{pmatrix} 0 & 1-i & 2+i \\ 2 & 1+2i & 2-i \\ -2 & 2i & -1 \end{pmatrix} = 0$

$$\iff -z \cdot \det\begin{pmatrix} 1-i & 2+i \\ 2i & -i \end{pmatrix} - 2\det\begin{pmatrix} 1-i & 2+i \\ 1+2i & 2-i \end{pmatrix} = 0$$

$$\iff z = -2 \cdot \frac{2-3i-1-2-5i-2}{-1+i-4i+2}$$

$$= -2 \cdot$$ finire i conti

$\boxed{7}$ $W = \left\{ x \in \mathbb{R}^4 : \begin{array}{l} x_1 + x_2 - 2x_3 = 0 \\ 2x_1 + x_3 + x_4 = 0 \end{array} \right\}$

$Z = \left\{ x \in \mathbb{R}^4 : \begin{array}{l} x_1 - 2x_2 + x_3 = 0 \\ x_2 + 3x_3 - x_4 = 0 \end{array} \right\}$ $\qquad \mathbb{R}^4 = W \oplus Z$

$P_W \begin{pmatrix} 2 \\ 3 \\ 1 \\ 2 \end{pmatrix}$.

Generatori di Z :
(espressione parametrica)

1	-2	1	0
0	1	3	-1

+7	0
+3	+1
-1	+2
0	+7

devono essere due vett. lin. indip. che risolvono

$7 - 6 - 1 + 0$ ✓

$0 + 3 - 3 + 0$ ✓

$0 - 2 + 2 + 0$ ✓

$0 + 1 + 6 - 7$ ✓

Cerco $P_W\begin{pmatrix}2\\3\\1\\2\end{pmatrix}$ cioè

$$\begin{pmatrix}2\\3\\1\\2\end{pmatrix} = P_W\begin{pmatrix}2\\3\\1\\2\end{pmatrix} + t\begin{pmatrix}7\\3\\-1\\0\end{pmatrix} + s\begin{pmatrix}0\\1\\2\\7\end{pmatrix}$$

cioè voglio che $\begin{pmatrix} 2 \\ 3 \\ 1 \\ 2 \end{pmatrix} - t \begin{pmatrix} 7 \\ 3 \\ -1 \\ 0 \end{pmatrix} - s \cdot \begin{pmatrix} 0 \\ 1 \\ 2 \\ 7 \end{pmatrix}$

soddisfi equaz. W : trovo t, s, li sostituisco qui e ho finito.

$$\begin{pmatrix} 1 & 1 & -2 & 0 \\ 2 & 0 & 1 & -1 \end{pmatrix} \cdot \begin{pmatrix} 2-7t \\ 3-3t-s \\ 1+t-2s \\ 2-7s \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} 3 - 12t + 3s = 0 \\ 3 - 13t + 5s = 0 \end{cases} \qquad t = 2/7 \qquad s = 1/7$$

Soluz: $\begin{pmatrix} 2 \\ 3 \\ 1 \\ 2 \end{pmatrix} - \frac{2}{7} \begin{pmatrix} 7 \\ 3 \\ -1 \\ 0 \end{pmatrix} - \frac{1}{7} \begin{pmatrix} 0 \\ 1 \\ 2 \\ 7 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 1 \\ 1 \end{pmatrix}$.

$\boxed{23}$ $\boxed{1}$ Dati 3 pol. lin. indip in

$\{ p(z) \in \mathbb{C}_{\leq 7}[z] : p(i) = p'(-i) \}$

quanti vanno aggiunti per avere base?

$\dim(\mathbb{C}_{\leq 7}[z]) = 8$ $\dim(\{ \dots \}) = 8 - 1 = 7$

$\Rightarrow$ vanno aggiunti 4

$\boxed{2}$ Determinare molt. m di i come radice di

$$p(z) = i z^4 + 3 z^3 + (1-2i) z^2 + (1-2i) z - (1+i)$$

e $\quad q(z) = p(z) / (z-i)^m$.

	i	3	$1-2i$	$1-2i$	$-1-i$
i		-1	$2i$	i	$i+1$
	i	2	1	$1-i$	

$$
\begin{array}{c|cccc|c}
i & & -1 & i & & i-1 \\
\hline
 & i & 1 & 1+i & & / \\
i & & -1 & i & & \\
\hline
 & i & 0 & \times & & \\
\end{array}
$$

$m = 2$

$$q(z) = \frac{p(z)}{(z-i)^2} = iz^2 + z + (1+i)$$

$\boxed{3}$ $X = \{ x \in \mathbb{R}^6 : x_1 - 3x_5 = 2x_6 \}$

$A \in M_{3\times 6}(\mathbb{R})$

$Y = \{ x \in X : A \cdot x = 0 \}$

$Z \subset X$ t.c. $X = Y \oplus Z$. $\dim(Z) = ?$

$\dim(X) = 5$ $\quad A \Longleftrightarrow f : X^{(5)} \longrightarrow \mathbb{R}^3$

$Y = \ker(f) \Longrightarrow 2 \leq \dim(\ker(f)) \leq 5$

$0 \leq \dim(Z) \leq 3$.

$\boxed{4}$ Quante soluz. può avere sist. lin. non omogeneo di 11 equaz. in 13 incognite?

nessuna o ...

giacitura $= \mathrm{Ker}\left(\mathbb{R}^{13} \longrightarrow \mathbb{R}^{11}\right)$

$\Longrightarrow \dim \geq 2$

$\Longrightarrow$ nessuna o infinite.

$\boxed{5}$ $A = (v_1, v_2, v_3, v_4) \in M_{4\times 4}(\mathbb{C})$; $\det(A) = 1+2i$

$$\det\Big(\underset{1}{v_2} - i\underset{1}{v_4},\ \underset{2}{v_3} - 2\underset{4}{v_4},\ 2\underset{2}{v_1} + i\underset{4}{v_2},\ 3\underset{3}{v_1} - i\underset{3}{v_3}\Big)$$

$$1 \cdot (-2) \cdot 2 \cdot (-i) \cdot \det(v_2, v_4, v_1, v_3)$$

$$+ (-i) \cdot 1 \cdot i \cdot 3 \cdot \det(v_4, v_3, v_2, v_1)$$

$$= 4i \cdot (+\det(A)) + 3(+\det(A))$$

$= (3-4i)\det(A) = (3-4i)(1+2i)$

$= 11+2i$

$\boxed{6}$ $A \in M_{7\times 8}$ $B \in M_{3\times 3}$ B sottomatr. di A

$\det(B) \neq 0$; $\det(C) = 0$ $\forall C \in M_{6\times 6}$ sottomatr. di A che contiene B.

$\mathrm{rank}(A) = ?$

$$3 \leq \operatorname{rank}(A) \leq 5.$$