Algebra lineare 7/11/18

$A \in M_{n \times n}$ $\qquad A$ iniettiva $\iff$ A surgettiva

$\Updownarrow$ $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \Updownarrow$

$\operatorname{Ker}(A) = \{0\}$ $\qquad\qquad\qquad\qquad$ le col. di A generano $\mathbb{R}^n$

$\Updownarrow$ $\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ NOTO $\Updownarrow$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ le col. di A sono lin. indip.

l'unico $x \in \mathbb{R}^n$ t.c. $A \cdot x = 0$ è $x = 0$

$\Updownarrow$ $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \Updownarrow$

l'unico $x \in \mathbb{R}^n$ t.c. $x_1 \cdot \begin{pmatrix} a_{11} \\ \vdots \\ a_{m1} \end{pmatrix} + \dots + x_n \begin{pmatrix} a_{m1} \\ \vdots \\ a_{mn} \end{pmatrix} = 0$ è $x = 0$

Def: se $\mathcal{B}, \mathcal{B}'$ sono basi di V,

$\mathcal{B} = (v_1, \dots, v_n)$, $\mathcal{B}' = (v_1', \dots, v_n')$

chiamo <u>matrice di cambio di base</u> la

$A \in M_{n \times n}$ t.c.

$$v_i' = \sum_{j=1}^{n} a_{ji} v_j \quad \text{cioè} \quad [id_V]_{\mathcal{B}'}^{\mathcal{B}}.$$

Prop: tale A è invertibile A^{-1} è la matrice di cambio di base da $\mathcal{B}'$ a $\mathcal{B}$.

Dim: sia M la matrice di cambio di base da $\mathcal{B}'$ a $\mathcal{B}$.

$$M \cdot A = [id_V]^{\mathcal{B}'}_{\mathcal{B}} \cdot [id_V]^{\mathcal{B}}_{\mathcal{B}'} = [id_V \circ id_V]^{\mathcal{B}'}_{\mathcal{B}'}$$

$$= [id_V]^{\mathcal{B}'}_{\mathcal{B}'} = I_n .$$

Analogamente $A \cdot M = I_n$. $\square$

Teo: se ho basi $\mathcal{B}, \mathcal{B}'$ di V con matrice di cambio A da $\mathcal{B}$ a $\mathcal{B}'$ allora $[v]_{\mathcal{B}'} = A^{-1} \cdot [v]_{\mathcal{B}}$

Dim: chiamo $x = [v]_{\mathcal{B}}$ cioè $v = \sum_{i=1}^{n} x_i v_i$.

Chiamo $M = A^{-1}$ che è la matrice di cambio da $\mathcal{B}'$ a $\mathcal{B}$

cioè $v_i = \sum_{j=1}^{n} m_{ji} v_j'$. Dunque

$$v = \sum_{i=1}^{n} x_i \sum_{j=1}^{n} m_{ji} v_j' = \sum_{j=1}^{n} \underbrace{\left(\sum_{i=1}^{n} m_{ji} x_i \right)}_{(M \cdot x)_j} v_j'$$

$$\Rightarrow [v]_{\mathcal{B}'} = M \cdot x.$$

$\blacksquare$

Risolvo formule di cambio di base: $\mathcal{B} \xrightarrow{A} \mathcal{B}'$

$$v_i' = \sum_{j=1}^{n} a_{ji} v_j = (v_1, \dots, v_n) \cdot \begin{pmatrix} a_{1i} \\ \vdots \\ a_{ni} \end{pmatrix}$$

$$\underbrace{(v_1', \dots, v_n')}_{\mathcal{B}'} = \underbrace{(v_1, \dots, v_n)}_{\mathcal{B}} \cdot \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \dots & a_{nn} \end{pmatrix}$$

$\Longrightarrow$ formule di cambio base: $\mathcal{B} \xrightarrow{A} \mathcal{B}'$
se $\mathcal{B}' = \mathcal{B} \cdot A$.

Oss: se $V = \mathbb{R}^n$, $\mathcal{B}, \mathcal{B}' \in M_{n \times n}(\mathbb{R})$ e in tal caso

$$\mathcal{B}' = \mathcal{B} \cdot A$$

solito prod righe x col.

Es: $V = \mathbb{R}^2$, $\mathcal{B} = \left(\begin{pmatrix} 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 5 \\ 4 \end{pmatrix} \right)$ $\mathcal{B}' = \left(\begin{pmatrix} -1 \\ 3 \end{pmatrix}, \begin{pmatrix} 6 \\ 7 \end{pmatrix} \right)$

A = matrice di cambio da $\mathcal{B}'$ a $\mathcal{B}$:

$$\begin{pmatrix} -1 \\ 3 \end{pmatrix} = \frac{19}{11} \begin{pmatrix} 1 \\ 3 \end{pmatrix} - \frac{6}{11} \begin{pmatrix} 5 \\ 4 \end{pmatrix}$$

$$\begin{pmatrix} 6 \\ 7 \end{pmatrix} = 1 \cdot \begin{pmatrix} 1 \\ 3 \end{pmatrix} + 1 \begin{pmatrix} 5 \\ 4 \end{pmatrix}$$

$$\Longrightarrow A = \begin{pmatrix} 19/11 & 1 \\ -6/11 & 1 \end{pmatrix}$$

$$\mathcal{B}' \; ? \; \mathcal{B} \cdot A$$

$$\begin{pmatrix} 1 & 5 \\ 3 & 4 \end{pmatrix} \cdot \begin{pmatrix} 19/11 & 1 \\ -6/11 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 6 \\ 3 & 7 \end{pmatrix}$$

$$\boxed{\mathcal{B} \xrightarrow{A} \mathcal{B}' \quad \text{se} \quad \mathcal{B}' = \mathcal{B} \cdot A}$$

$\underbrace{}_{\mathcal{B}'}$ $\underline{\text{OK}}$

Riscriviamo anche la def. di $[v]_{\mathcal{B}}$:

$$[v]_{\mathcal{B}} = x \quad \text{se} \quad v = \sum_{i=1}^{n} x_i v_i \quad \text{cioè}$$

$$v = \underbrace{(v_1, \dots, v_n)}_{\mathcal{B}} \cdot \underbrace{\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}}_{[v]_{\mathcal{B}}}$$

$\Rightarrow$ la def. di $[v]_{\mathcal{B}}$: quel vettore t.c. $\boxed{v = \mathcal{B} \cdot [v]_{\mathcal{B}'}}$

<u>Teo</u> : se $\mathcal{B} \xrightarrow{A} \mathcal{B}'$ allora $[v]_{\mathcal{B}'} = A^{-1} \cdot [v]_{\mathcal{B}}$

<u>2ª dimo</u> : so che $\mathcal{B} = \mathcal{B}' A^{-1}$; inoltre

$$v = \mathcal{B} \cdot [v]_{\mathcal{B}} = (\mathcal{B}' \cdot A^{-1}) \cdot [v]_{\mathcal{B}}$$

$$= \mathcal{B}' \cdot \underbrace{\left(A^{-1} \cdot [v]_{\mathcal{B}} \right)}$$

lui è $[v]_{\mathcal{B}'}$. $\blacksquare$

<u>Teo</u>: se ho basi $\mathcal{B}, \mathcal{B}'$ di V con $\mathcal{B} \xrightarrow{N} \mathcal{B}'$

$\mathcal{C}, \mathcal{C}'$ di W con $\mathcal{C} \xrightarrow{M} \mathcal{C}'$

e $f: V \to W$ lineare, allora

$$[f]_{\mathcal{B}'}^{\mathcal{C}'} = M^{-1} \cdot [f]_{\mathcal{B}}^{\mathcal{C}} \cdot N .$$

<u>Dim</u>: chiamo $A = [f]_{\mathcal{B}}^{\mathcal{C}}$, $K = M^{-1}$; ho:

$$v_p' = \sum_{i=1}^{m} m_{ip} v_i \qquad \mathcal{B} \xrightarrow{N} \mathcal{B}'$$

$$f(v_i) = \sum_{j=1}^{m} a_{ji} w_j \qquad A = [f]_{\mathcal{B}}^{\mathcal{C}}$$

$$w_j = \sum_{q=1}^{m} k_{qj} w_q' \qquad \mathcal{C}' \xrightarrow{K} \mathcal{C}$$

$$\Rightarrow f(v_p') = f\left(\sum_{i=1}^{m} m_{ip} v_i\right)$$

$$= \sum_{i=1}^{m} m_{ip} \cdot \sum_{j=1}^{m} a_{ji} w_j$$

$$= \sum_{i=1}^{m} m_{ip} \sum_{j=1}^{m} a_{ji} \sum_{q=1}^{m} k_{qj} w_q'$$

$$= \sum_{k=1}^{m} \underbrace{\left(\sum_{j=1}^{m} k_{qj} \underbrace{\left(\sum_{i=1}^{m} a_{ji} m_{ip} \right)}_{(A \cdot N)_{jp}} \right)}_{(K \cdot A \cdot N)_{qp}} w_q'$$

$$\Longrightarrow [t]_{B'}^{e'} = K \cdot A \cdot N = M^{-1} \cdot A \cdot N.$$

Es: $f: \mathbb{R}^2 \to \mathbb{R}^2 \qquad f(x) = \begin{pmatrix} 3 & -1 \\ 4 & 7 \end{pmatrix} \cdot x$

$\mathcal{B} = \left(\begin{pmatrix} 3 \\ 1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right) \qquad \mathcal{B}' = \left(\begin{pmatrix} 4 \\ -1 \end{pmatrix} \begin{pmatrix} -3 \\ 1 \end{pmatrix} \right)$

$\mathcal{C} = \left(\begin{pmatrix} 2 \\ 7 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \end{pmatrix} \right) \qquad \mathcal{C}' = \left(\begin{pmatrix} 1 \\ 2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right)$

$\mathcal{B} \overset{N}{\rightsquigarrow} \mathcal{B}' \qquad \begin{pmatrix} 4 \\ -1 \end{pmatrix} = 6 \begin{pmatrix} 3 \\ 1 \end{pmatrix} - 7 \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$N = \begin{pmatrix} 6 & -5 \\ -7 & 6 \end{pmatrix} \qquad \begin{pmatrix} -3 \\ 1 \end{pmatrix} = -5 \begin{pmatrix} 3 \\ 1 \end{pmatrix} + 6 \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$$\mathcal{E} \xrightarrow{M} \mathcal{E}'$$

$$\begin{pmatrix} 1 \\ 2 \end{pmatrix} = -1 \begin{pmatrix} 2 \\ 7 \end{pmatrix} + 3 \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

$$M = \begin{pmatrix} -1 & -2 \\ 3 & 5 \end{pmatrix} \qquad \begin{pmatrix} 1 \\ 1 \end{pmatrix} = -2 \begin{pmatrix} 2 \\ 7 \end{pmatrix} + 5 \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

$$M^{-1}: \quad \begin{pmatrix} -1 & -2 \\ 3 & 5 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{cases} -a - 2c = 1 \\ 3a + 5c = 0 \\ -b - 2d = 0 \\ 3b + 5d = 1 \end{cases} \qquad \begin{cases} a = -2c - 1 \\ -6c - 3 + 5c = 0 \\ b = -2d \\ -6d + 5d = 1 \end{cases} \qquad \begin{cases} c = -3 \\ a = 5 \\ d = -1 \\ b = 2 \end{cases}$$

$$M^{-1} = \begin{pmatrix} 5 & 2 \\ -3 & -1 \end{pmatrix}$$

$$[f]^{C}_{B} : \begin{pmatrix} 3 & -1 \\ 4 & 7 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 8 \\ 19 \end{pmatrix} = -5 \begin{pmatrix} 2 \\ 7 \end{pmatrix} + 18 \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

$$\begin{pmatrix} -5 & 0 \\ 18 & 5 \end{pmatrix} \qquad \begin{pmatrix} 3 & -1 \\ 4 & 7 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \\ 15 \end{pmatrix} = 0 \cdot \begin{pmatrix} 2 \\ 7 \end{pmatrix} + 5 \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

$$[f]^{C'}_{B'} : \begin{pmatrix} 3 & -1 \\ 4 & 7 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ -1 \end{pmatrix} = \begin{pmatrix} 13 \\ 9 \end{pmatrix} = -4 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + 17 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} -4 & 1 \\ 17 & -7 \end{pmatrix} \qquad \begin{pmatrix} 3 & -1 \\ 4 & 7 \end{pmatrix} \cdot \begin{pmatrix} -3 \\ 1 \end{pmatrix} = \begin{pmatrix} -6 \\ -5 \end{pmatrix} = 1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} - 7 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$[f]_{\mathcal{B}'}^{\mathcal{E}'} = M^{-1} \cdot [f]_{\mathcal{B}}^{\mathcal{E}} \cdot N$$

$$[f]_{\mathcal{B}'}^{\mathcal{E}'} = \begin{pmatrix} -4 & 1 \\ 17 & -7 \end{pmatrix}$$

$$M^{-1} \cdot [f]_{\mathcal{B}}^{\mathcal{E}} \cdot N = \begin{pmatrix} 5 & 2 \\ -3 & -1 \end{pmatrix} \cdot \begin{pmatrix} -5 & 0 \\ 18 & 5 \end{pmatrix} \cdot \begin{pmatrix} 6 & -5 \\ -7 & 6 \end{pmatrix}$$

$$\begin{pmatrix} 5 & 2 \\ -3 & -1 \end{pmatrix} \begin{pmatrix} -30 & 25 \\ 73 & -60 \end{pmatrix}$$

$$\begin{pmatrix} -4 & \cdots \\ 17 & \cdots \end{pmatrix}$$

Esprimo la def di $[f]_{\mathcal{B}}^{\mathcal{C}} = A$ senza indici:

$$f(v_i) = \sum_{j=1}^{m} a_{ji} w_j = \underbrace{(w_1, \dots, w_m)}_{\mathcal{C}} \underbrace{\begin{pmatrix} a_{1i} \\ \vdots \\ a_{mi} \end{pmatrix}}_{\text{i-esima col. di } A}$$

$$\Rightarrow \big(f(v_1), \dots, f(v_n)\big) = \mathcal{C} \cdot A$$

$$\Rightarrow f \cdot (v_1, \dots, v_n) = \mathcal{C} \cdot A$$

$$\Rightarrow \boxed{\overbrace{f \cdot \mathcal{B}}^{\mathcal{B}} = \mathcal{E} \cdot [f]^{\mathcal{E}}_{\mathcal{B}}}$$

Cioè $[f]^{\mathcal{E}}_{\mathcal{B}}$ è quell'unica matrice t.c.

$$f \cdot \mathcal{B} = \mathcal{E} \cdot [f]^{\mathcal{E}}_{\mathcal{B}}$$

se $V = \mathbb{R}^n$ $W = \mathbb{R}^m$

$\mathcal{B} \in M_{n \times n}$ $\mathcal{C} \in M_{m \times m}$

$f \in M_{m \times n}$ $[f]^{\mathcal{C}}_{\mathcal{B}} \in M_{m \times n}$

e i prodotti sono i soliti riga x col.

Oss: $\dim_{\mathbb{R}} (\mathbb{R}_{\leq d}[t]) = d+1$ infatti

$$\mathbb{R}_{\leq d}[t] = \{ a_0 + a_1 t + a_2 t^2 + \dots + a_d \cdot t^d \ : \ a_0, a_1, \dots, a_d \in \mathbb{R} \}$$

Base canonica $1, t, t^2, \dots, t^d$.

$\boxed{5.1.2}$ Trovare $\mathrm{Ker}(f)$, $\mathrm{Im}(f)$

(m) $f : \mathbb{R}_{\leq d}[t] \to \mathbb{R}_{\leq d}[t]$, $f(p(t)) = p'(t)$.

$$f\left(a_0 + a_1 t + a_2 t^2 + a_3 t^3 + \dots + a_d t^d \right)$$

$= a_1 + 2a_2 t + 3a_3 t^2 + \dots + d a_d t^{d-1}$

ogni coeff del polinomio immagine è multiplo di un coeff. del polinomio di partenza.

$\mathrm{Ker}(f)$: $p(t) = a_0 + a_1 t + \dots + a_d t^d \in \mathrm{Ker}(f)$

$\Longleftrightarrow a_1 + 2a_2 t + \dots + d a_d t^{d-1} = 0$

$\Longleftrightarrow a_1 = a_2 = \dots = a_d = 0$

$\Longleftrightarrow p(t) \in \mathbb{R}$.

$$\mathrm{Im}(f) = \mathbb{R}_{\leq d-1}[t]$$

$$\underbrace{\dim(\mathrm{Ker}(f))}_{=1} + \underbrace{\dim(\mathrm{Im}(f))}_{\leq d} = \underbrace{\dim(\mathbb{R}_{\leq d}[t])}_{= d+1}$$

(a) $f : \mathbb{R}^4 \longrightarrow \mathbb{R}_{\leq 2}[t]$ (sicuro: f non iniettiva)

$$f(x) = (x_1 + 3x_2 - x_4) + (x_2 + 2x_3 - x_4)t + (x_1 - 6x_3 + 2x_4)t^2$$

Lineare: i coeff. del poli. immagine sono lineari nelle componenti del vettore argomento: <u>OK</u>.

Ker(f): $x \in \mathrm{Ker}(f)$

$$\Longleftrightarrow \begin{cases} x_1 + 3x_2 - x_4 = 0 \\ x_2 + 2x_3 - x_4 = 0 \\ x_1 - 6x_3 + 2x_4 = 0 \end{cases}$$

$$\begin{cases} x_4 = x_2 + 2x_3 \\ x_1 + 2x_2 - 2x_3 = 0 \\ \cancel{x_1 + 2x_2 - 2x_3 = 0} \end{cases}$$

$$\begin{cases} x_1 = -2x_2 + 2x_3 \\ x_4 = x_2 + 2x_3 \end{cases}$$

Dunque $Ker(f)$ ha base $\begin{pmatrix} -2 \\ 1 \\ 0 \\ 1 \end{pmatrix}$ $\begin{pmatrix} 2 \\ 0 \\ -1 \\ 2 \end{pmatrix}$; $\dim = 2$.

$$\underbrace{\dim Ker}_{2} + \underbrace{\dim Im}_{2} = \underbrace{\dim \mathbb{R}^4}_{2}$$

$Im(f)$ è generata da $f(e_1), f(e_2), f(e_3), f(e_4)$:

$1+t^2\ ,\quad 3+t\ ,\quad 2t-6t^2\ ,\quad -1-t+2t^2$

$\checkmark \qquad \checkmark \qquad \times \qquad \times$

$-6\cdot\mathrm{I}+2\cdot\mathrm{II} \qquad 2\cdot\mathrm{I}-\mathrm{II}$

$\Longrightarrow \dim = 2$

$\boxed{5.14}$ $\quad f:\{x\in\mathbb{R}^8 : x_3-2x_5+3x_8=0\}\to\mathbb{R}_{\leq 10}[t]$

$f(e_1+2e_3+e_5)=f(3e_3-e_7-e_8)=\sqrt{5}-7t^9$

che dim può avere l'immagine?

$$f : \underset{7}{V} \longrightarrow \underset{11}{\mathbb{R}_{\leq 10}[t]}$$

$$\left.\underset{\substack{0 \\ \vdots \\ 7}}{\dim \operatorname{Ker}} + \underset{\substack{7 \\ \vdots \\ 0}}{\dim \operatorname{Im}} = 7 \right\} \text{ ignorando altre info.}$$

Altre info: $\dim(\operatorname{Im}(f)) \geq 1$

$\dim(\operatorname{Ker}(f)) \geq 1$

$$\dim \mathrm{Ker} + \dim \mathrm{Im} = 7$$

0	7
1	6
2	5
3	4
4	3
5	2
6	1
7	0

$$\Longrightarrow \quad 1 \le \dim(\mathrm{Im}\, f) \le 6$$

$\boxed{5.1.5}$

$f: \mathbb{R}^5 \to \mathbb{R}^8$ lin, $\dim(\mathrm{Ker}(f)) = 2$

$Z \subset \mathbb{R}^8$ stsp. $Z \cap \mathrm{Im}(f) = \{0\}$.

Che dim può avere Z?

$$\underbrace{\dim(\mathrm{Ker}\, f)}_{=\,2} + \dim(\mathrm{Im}\, f) = \underbrace{\dim(\mathbb{R}^5)}_{=\,5}$$

$$\Rightarrow \dim \mathrm{Im} f = 3$$

$$\dim Z + \underset{\substack{\| \\ 3}}{\dim \operatorname{Im}(f)} = \underset{\substack{\| \\ 0}}{\dim (Z \cap \operatorname{Im} f)} + \dim (Z + \operatorname{Im} f)$$

$$3 \leq \cdot \leq 8$$

$$0 \leq \cdot \leq 5.$$

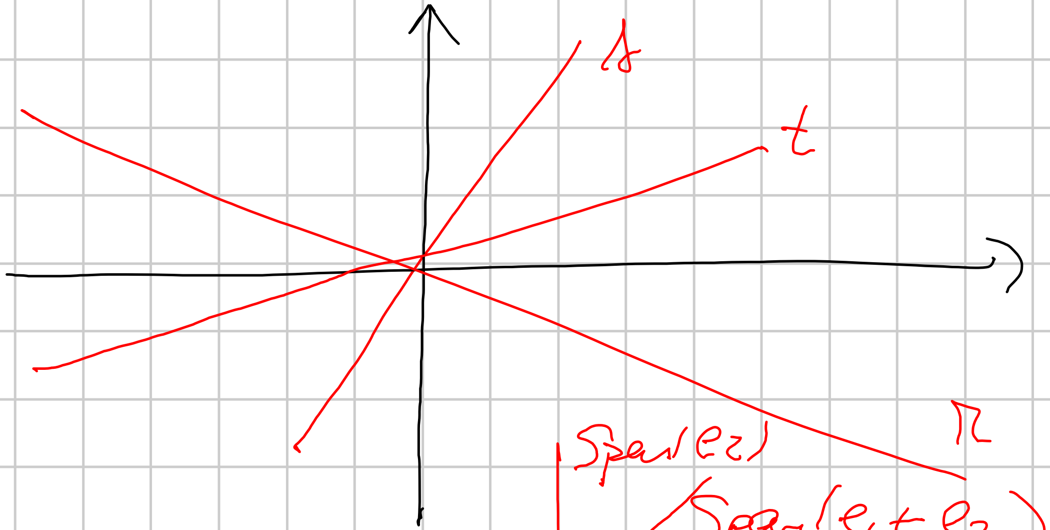
$\boxed{5.2.2}$ Esibire o provare che non esistono sttsp $X, Y, Z \subset \mathbb{R}^4$ tutti di dim $= 2$ t.c. $\mathbb{R}^4 = X \oplus Y = X \oplus Z = Y \oplus Z$.

$\mathbb{R}^4 = X \oplus Y$ significa $X \cap Y = \{0\}$ e $X + Y = \mathbb{R}^4$

Però $\dim X + \dim Y = 2 + 2 = 4 = \dim \mathbb{R}^4$, dunque basta avere $X \cap Y = \{0\}$. Stesso per X, Z e Y, Z.

dim meta': $\mathbb{R}^2$, r, s, t rette t.c.

$r \cap s = r \cap t = s \cap t = \{0\}$



Scelta più facile:

In $\mathbb{R}^4$: $X = \mathrm{Span}(e_1, e_2)$

$Y = \mathrm{Span}(e_3, e_4)$

$Z = \mathrm{Span}(e_1+e_3, e_2+e_4)$

$\boxed{5.2.3}$ $X, Y \subset \mathbb{R}^9$, $Z \subset \mathbb{R}^8$ stsp.

$f : X \to \mathbb{R}^8$ iniettiva, $\mathbb{R}^9 = X \oplus Y$

$\mathbb{R}^8 = Z \oplus \mathrm{Im}(f)$, $\dim Z = 3$. $\dim(Y) = ?$

$$\left.\begin{array}{l}\dim(\operatorname{Im} f) = \dim X \\ \dim(\operatorname{Im} f) = 5\end{array}\right\} \Longrightarrow \dim X = 5$$

$$\Longrightarrow \dim Y = 4$$

Trovare le proiez. p, q associate a $V = W \oplus Z$

e verificare che $p \circ p = p$, $q \circ q = q$ $p \circ q = q \circ p = 0$.

$\boxed{5.2.5}$ $V = \mathbb{R}^2$ $W = \mathrm{Span}\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ $Z = \mathrm{Span}\begin{pmatrix} -3 \\ 2 \end{pmatrix}$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \frac{2x_1+3x_2}{7}\begin{pmatrix} 2 \\ 1 \end{pmatrix} + \frac{-x_1+2x_2}{7}\begin{pmatrix} -3 \\ 2 \end{pmatrix}$$

$$p(x) = \frac{2x_1+3x_2}{7}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \underbrace{\frac{1}{7}\begin{pmatrix} 4 & 6 \\ 2 & 3 \end{pmatrix}}_{=P} \cdot x$$

$$q(x) = \frac{-x_1+2x_2}{7} \cdot \begin{pmatrix} -3 \\ 2 \end{pmatrix} = \underbrace{\frac{1}{7}\begin{pmatrix} 3 & -6 \\ -2 & 4 \end{pmatrix}}_{=Q} \cdot x$$

Devo provare che $\underline{P} \cdot \underline{P} = \underline{P}$, $Q \cdot Q = Q$, $\underline{P} \cdot Q = Q \underline{P} = 0$.

$$\frac{1}{7} \cdot \frac{1}{7} \begin{pmatrix} 4 & 6 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} 4 & 6 \\ 2 & 3 \end{pmatrix} = \frac{1}{7} \cdot \begin{pmatrix} 4 & 6 \\ 2 & 3 \end{pmatrix} \quad \underline{\underline{ok}}$$

$$\frac{1}{7} \cdot \frac{1}{7} \begin{pmatrix} 4 & 6 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} 3 & -6 \\ -2 & 4 \end{pmatrix} = \frac{1}{49} \cdot \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad \underline{\underline{ok}}$$

$\boxed{5.2.10}$ $V = \mathbb{R}^3$ $W = \text{Span}\left(\begin{pmatrix}1\\2\\3\end{pmatrix}\begin{pmatrix}2\\-1\\1\end{pmatrix}\right)$ $Z : \begin{cases} 2x_1 - x_2 + x_3 = 0 \\ 3x_1 + x_2 - 2x_3 = 0 \end{cases}$

$\dim W = 2$ $\dim Z = 1$

Per vedere che $\mathbb{R}^3 = W \oplus Z$ basta vedere che $W \cap Z = \{0\}$.

In realtà trovando p e q verifico che ogni $x \in \mathbb{R}^3$ si scrive in modo unico come $x = w + z$ con $w \in W$ e $z \in Z$ $\Longrightarrow$ dimostro che $\mathbb{R}^3 = W \oplus Z$.

Devo scrivere $x = w + z$ ← deve soddisfare le equaz. di Z

$$w = \alpha\begin{pmatrix}1\\2\\3\end{pmatrix}+\beta\begin{pmatrix}2\\-1\\1\end{pmatrix}$$

dunque $x-\alpha\begin{pmatrix}1\\2\\3\end{pmatrix}-\beta\begin{pmatrix}2\\-1\\1\end{pmatrix}$ deve soddisfare le eq. di Z:

$$\begin{cases} 2(x_1-\alpha-2\beta)-(x_2-2\alpha+\beta)+(x_3-3\alpha-\beta)=0 \\ 3(x_1-\alpha-2\beta)+(x_2-2\alpha+\beta)-2(x_3-3\alpha-\beta)=0 \end{cases}$$

$$\begin{cases} -3\alpha - 6\beta = -2x_1 + x_2 - x_3 \\ \alpha - 3\beta = -3x_1 - x_2 + 2x_3 \end{cases}$$

$$\begin{cases} \alpha + 2\beta = \frac{1}{3}(2x_1 - x_2 + x_3) \\ \alpha - 3\beta = -3x_1 - x_2 + 2x_3 \end{cases}$$

$$\beta = \frac{1}{15}\left(2x_1 - x_2 + x_3 + 9x_1 + 3x_2 - 6x_3\right) = \frac{1}{15}(11x_1 + 2x_2 - 5x_3)$$

$$\alpha = \frac{1}{15}\left(10x_1 - 5x_2 + 5x_3 - 22x_1 - 4x_2 + 10x_3\right) = \frac{1}{15}(-12x_1 - 9x_2 + 15x_3)$$

$$p(x) = \frac{1}{15}\left(-12x_1 - 9x_2 + 15x_3\right)\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} +$$

$$+ \frac{1}{15}\left(11x_1 + 2x_2 - 5x_3\right)\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$$

$$= \frac{1}{15}\begin{pmatrix} -12x_1 - 9x_2 + 15x_3 + 22x_1 + 4x_2 - 10x_3 \\ -24x_1 - 18x_2 + 30x_3 - 11x_1 - 2x_2 + 5x_3 \\ -36x_1 - 27x_2 + 45x_3 + 11x_1 + 2x_2 - 5x_3 \end{pmatrix}$$

$$= \frac{1}{15}\begin{pmatrix} 10 & -5 & 5 \\ -35 & -20 & 35 \\ -25 & -25 & 40 \end{pmatrix} x = \underbrace{\frac{1}{3}\begin{pmatrix} 2 & -1 & 1 \\ -7 & -4 & 7 \\ -5 & -5 & 8 \end{pmatrix}}_{P} x$$

$$Q = I_3 - P \qquad (P + Q = I)$$

$$P \cdot P \overset{?}{=} P \qquad \frac{1}{3}\cdot\frac{1}{3}\begin{pmatrix} 2 & -1 & 1 \\ -7 & -4 & 7 \\ -5 & -5 & 8 \end{pmatrix}\begin{pmatrix} 2 & -1 & 1 \\ -7 & -4 & 7 \\ -5 & -5 & 8 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 2 & -1 & 1 \\ \vdots & \vdots & \vdots \end{pmatrix}$$

$P \cdot Q = Q \cdot P = 0$ $\quad\quad$ $Q \cdot Q = Q$ (fotelo)