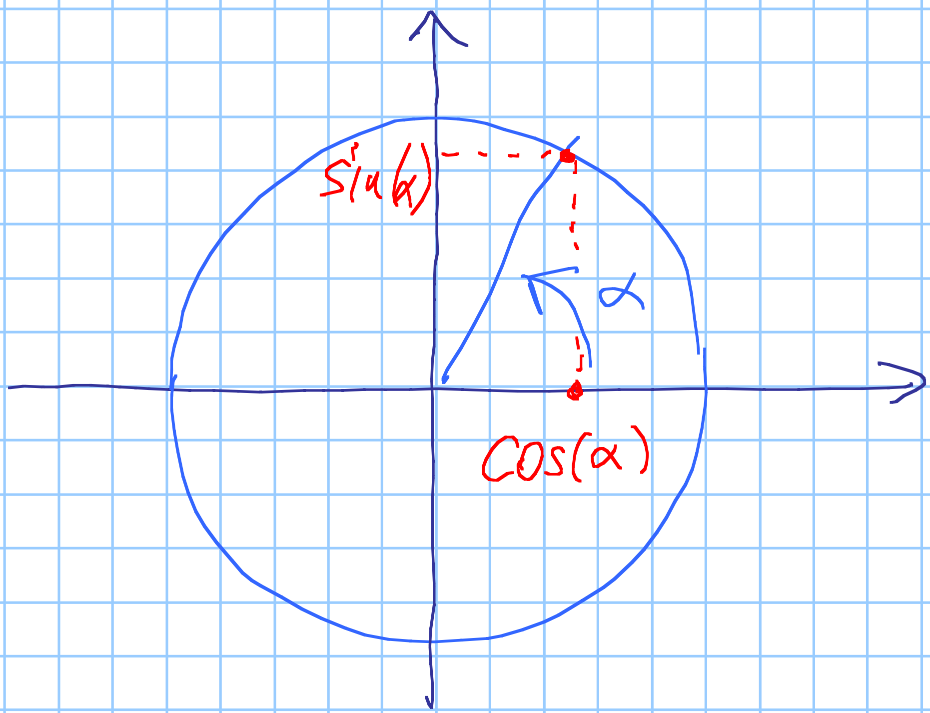
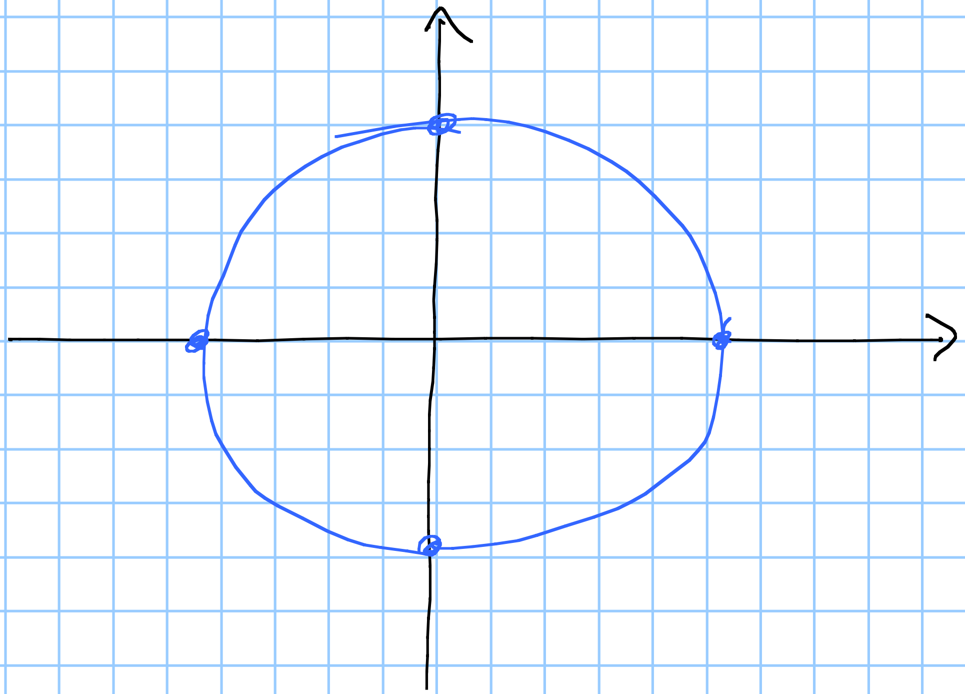
Matematica e statistica 31/10/17



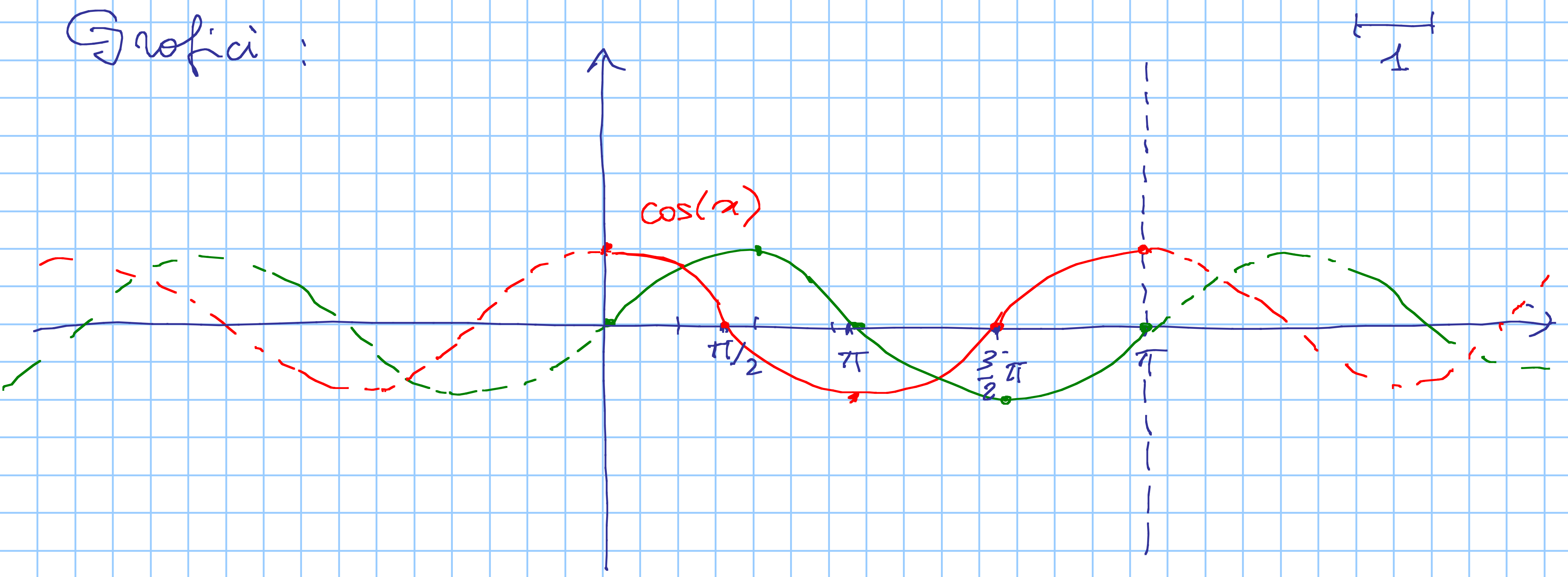
sin, cos sono periodiche di periodo 2π:

$$\cos(x+2\pi) = \cos(x) \quad \forall x$$
$$\sin(x+2\pi) = \sin(x) \quad \forall x$$



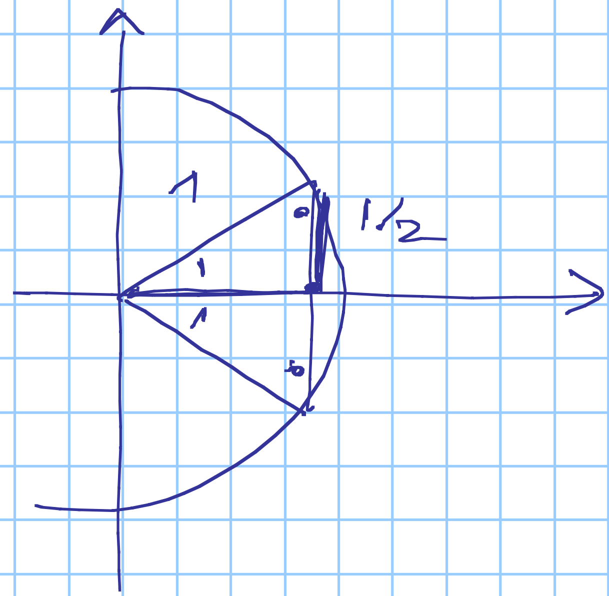
	$\cos(x)$	$\sin(x)$
$x = 0$	1	0
$x = \pi/2$	0	1
$x = \pi$	-1	0
$x = \frac{3}{2}\pi$	0	-1

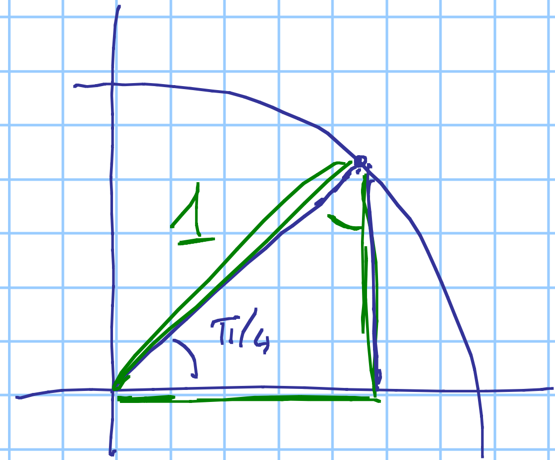
Grafici :

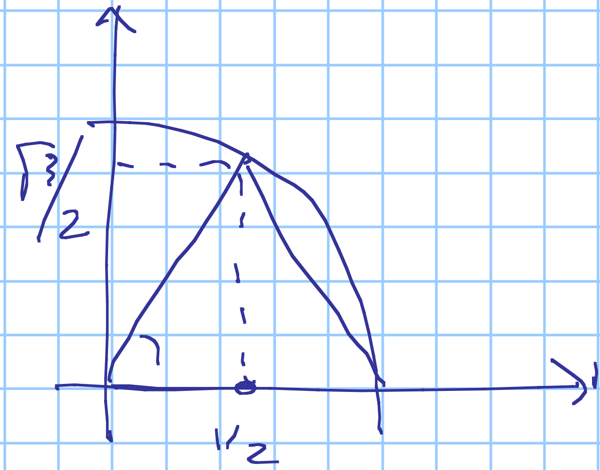


Valori su alcuni angoli :

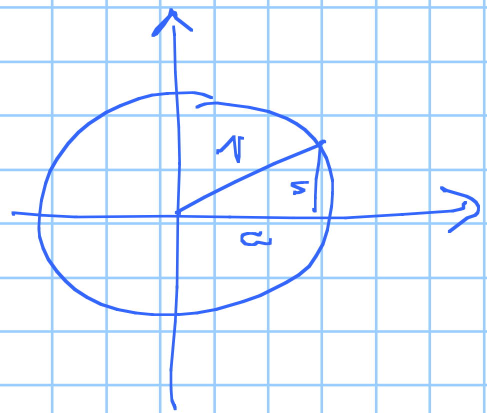
x	$\cos(x)$	$\sin(x)$
$\pi/4$	$1/\sqrt{2}$	$1/\sqrt{2}$
$\pi/6$	$\frac{1}{2}\sqrt{3}$	$1/2$
$\pi/3$	$1/2$	$\frac{1}{2}\sqrt{3}$







Fatto:



$$\cos^2(x) + \sin^2(x) = 1$$

Def:

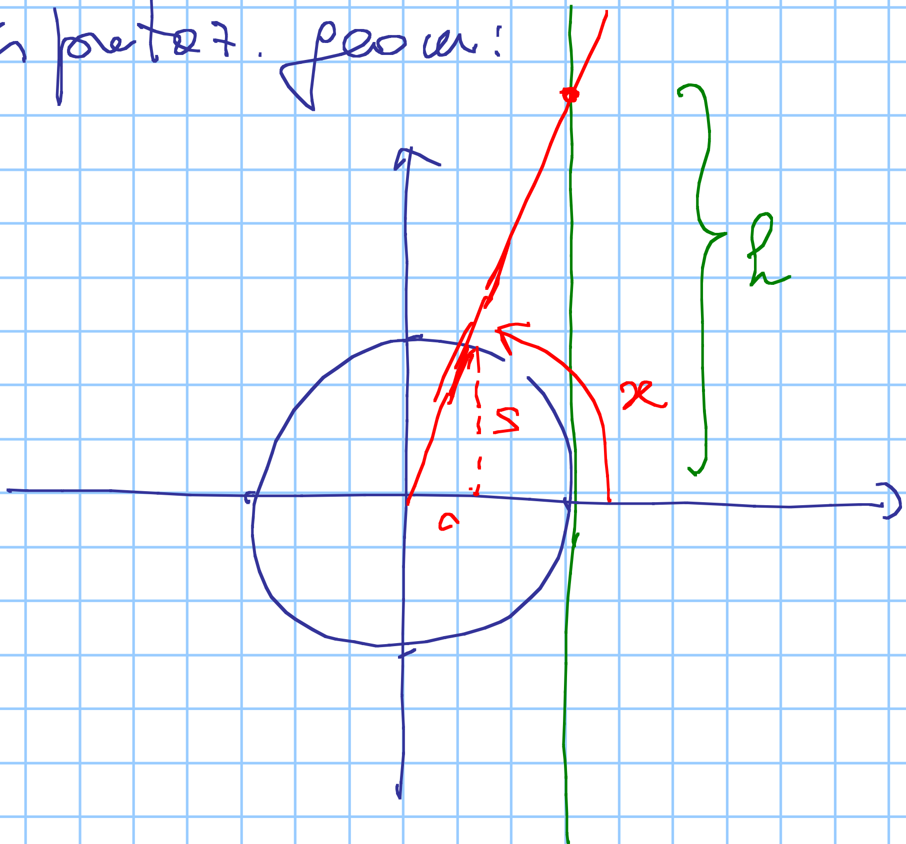
$$tg(x) = \frac{\sin(x)}{\cos(x)}$$

$$x \neq \frac{\pi}{2} + k\pi \qquad k \in \mathbb{Z}$$

$$cotg(x) = \frac{\cos(x)}{\sin(x)}$$

$$x \neq k \cdot \pi \qquad k \in \mathbb{Z}$$

Interpretaz. geom.:

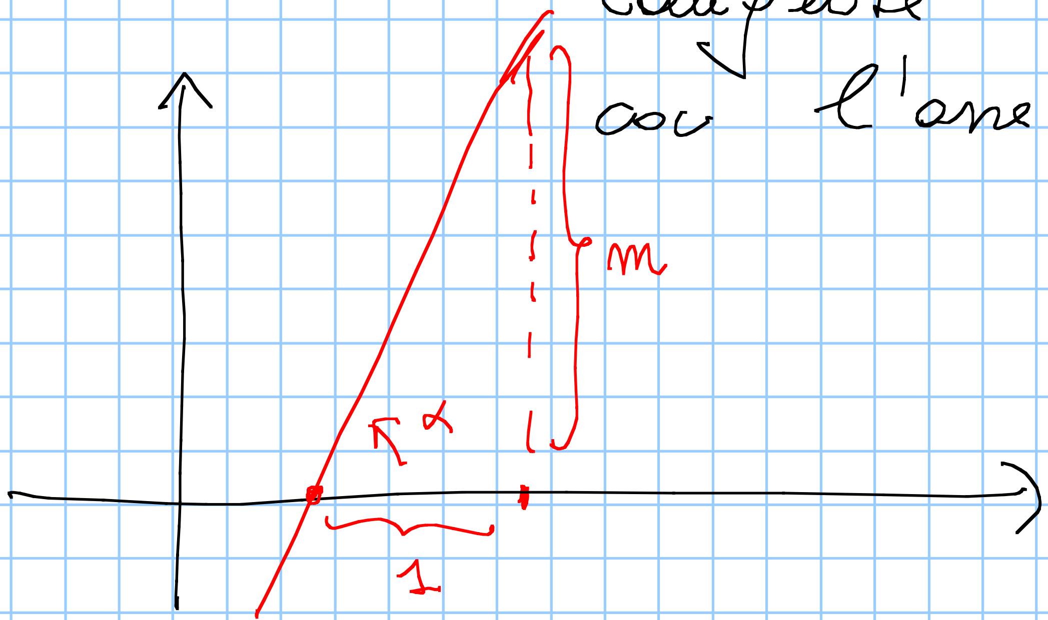


$$h : 1 = s : c$$

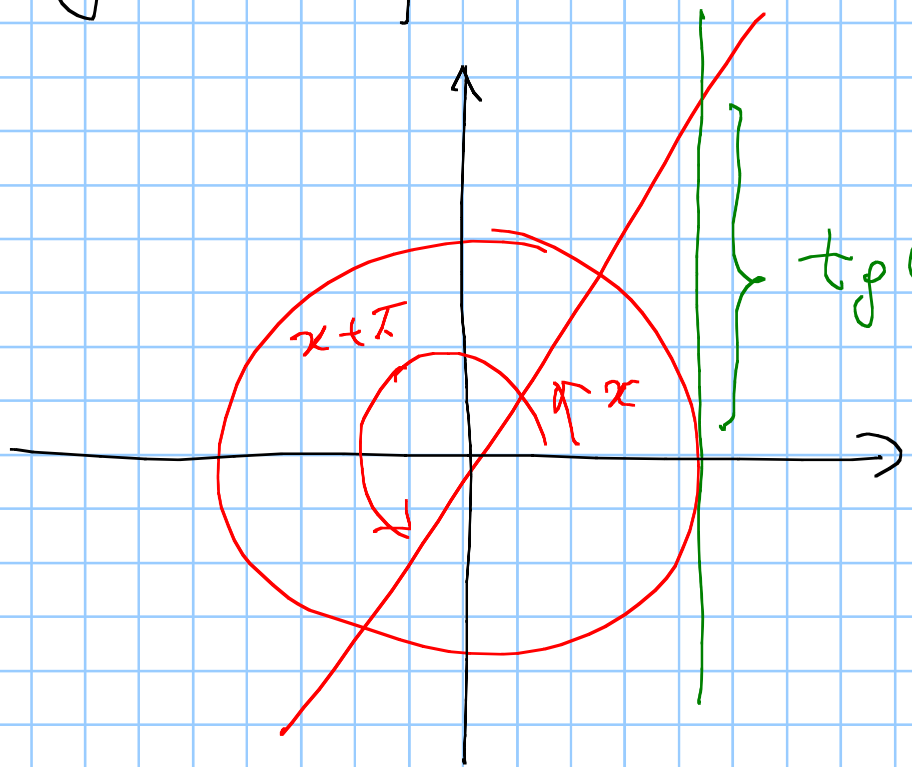
$$h = \frac{\sin(x)}{\cos(x)}$$

Conseguenze: 1) retta $y = m \cdot x + q$

m = coeff. ang. è la tangente dell'angolo formato con l'asse delle x:

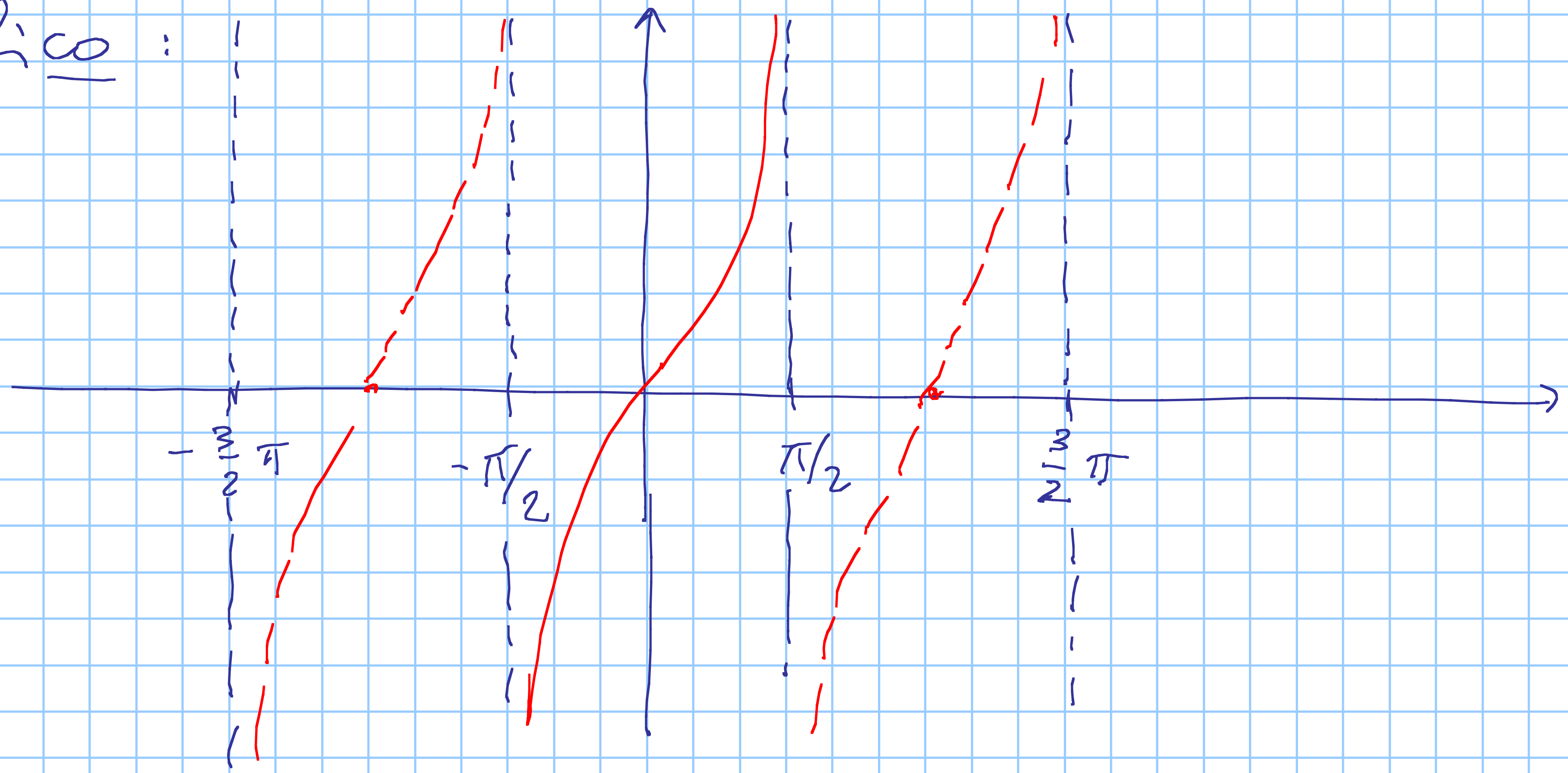


2) tg è periodica di periodo π:

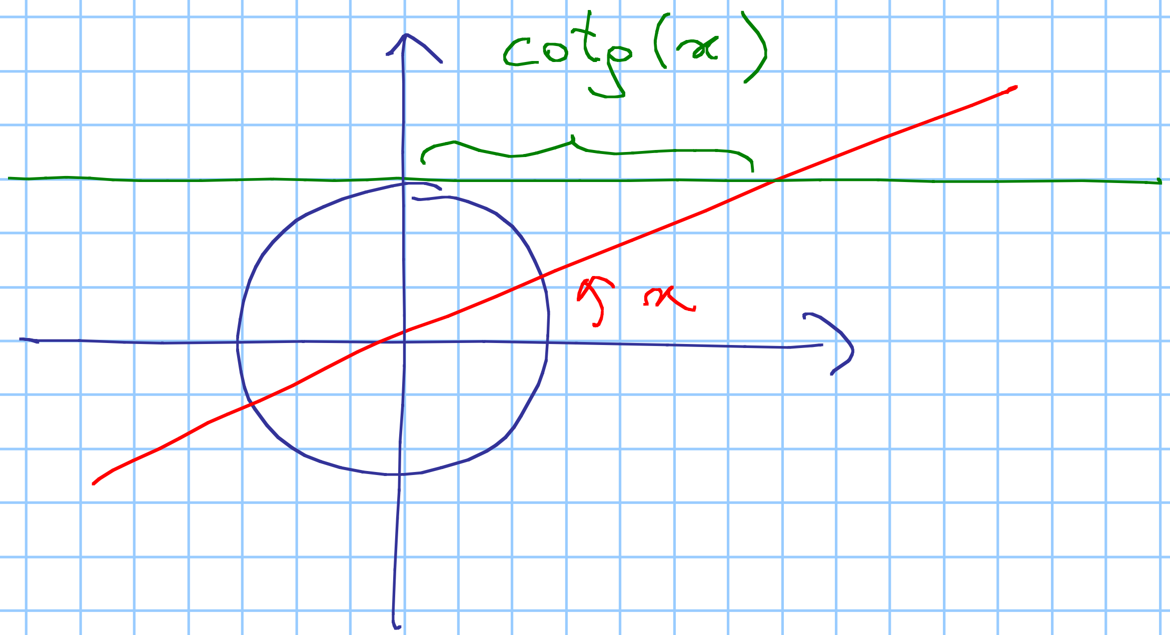


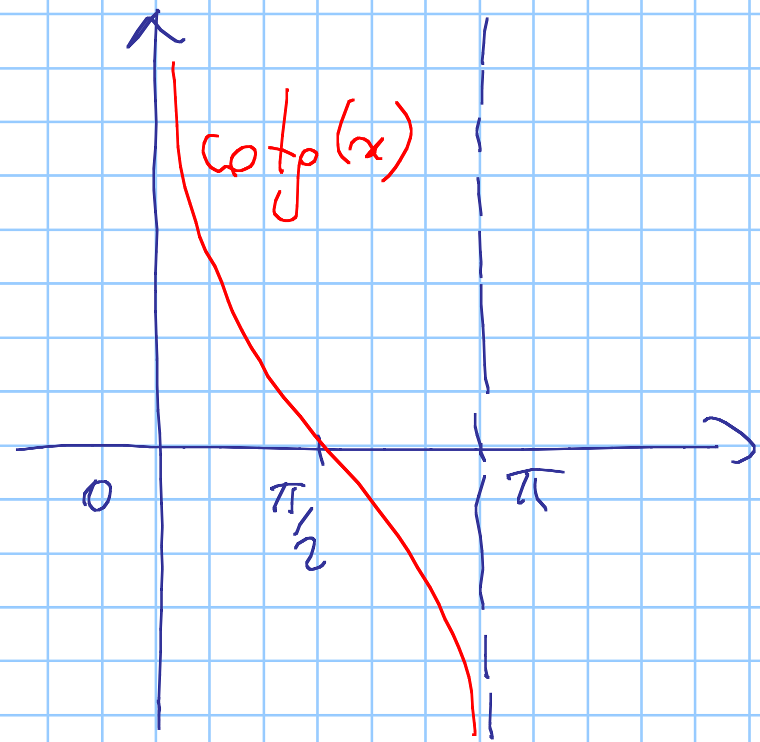
$$tg(x) = tg(x+\pi)$$

Grafico:



Cotg :

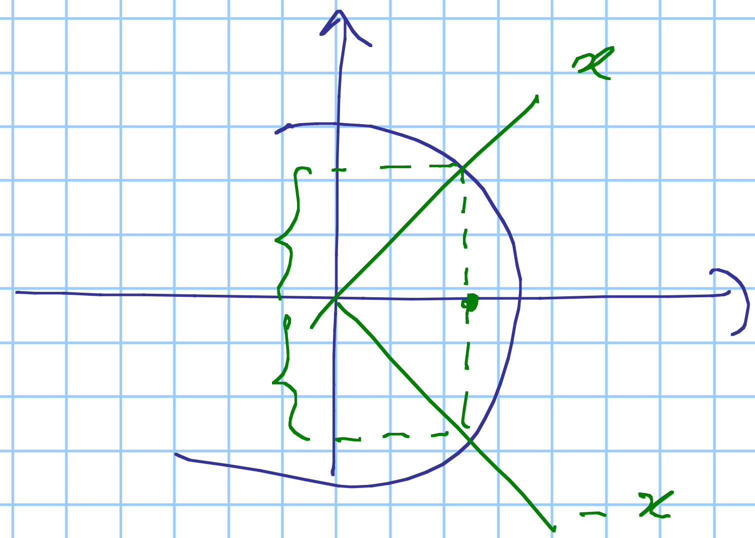




Formule per angoli con somma/differenze multiple di $\pi/2$:

1) $\cos(-x) = \cos(x)$

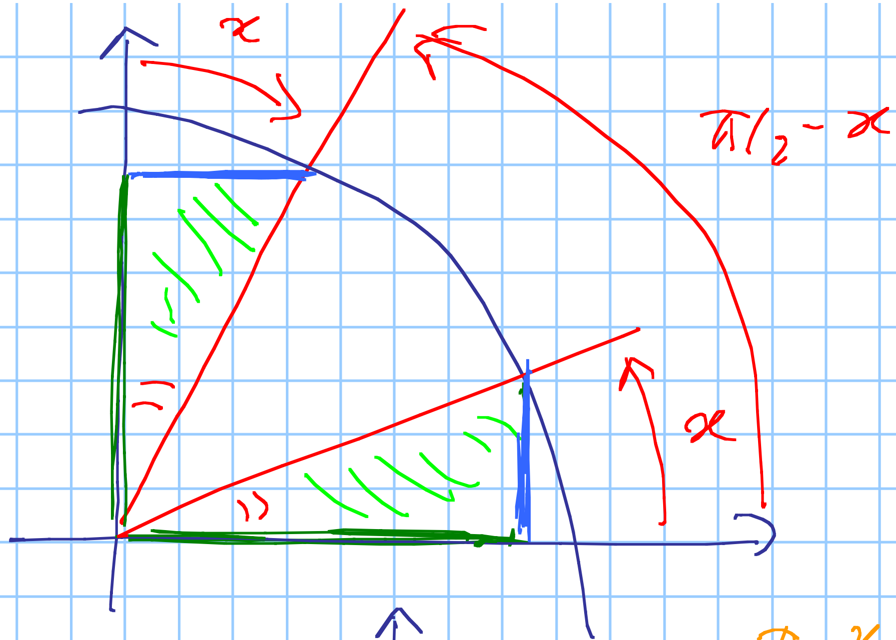
$\sin(-x) = -\sin(x)$



2)

$$\cos\left(\pi/2 - x\right) = \sin(x)$$

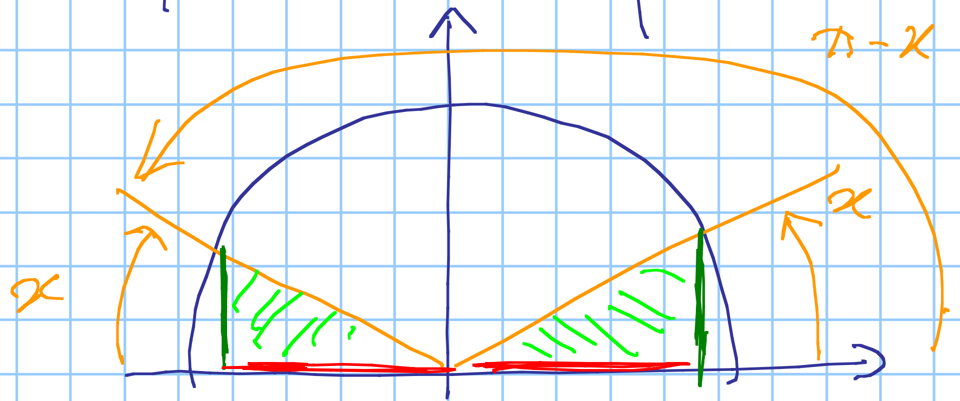
$$\sin\left(\pi/2 - x\right) = \cos(x)$$



3)

$$\cos(\pi - x) = -\cos(x)$$

$$\sin(\pi - x) = \sin x$$



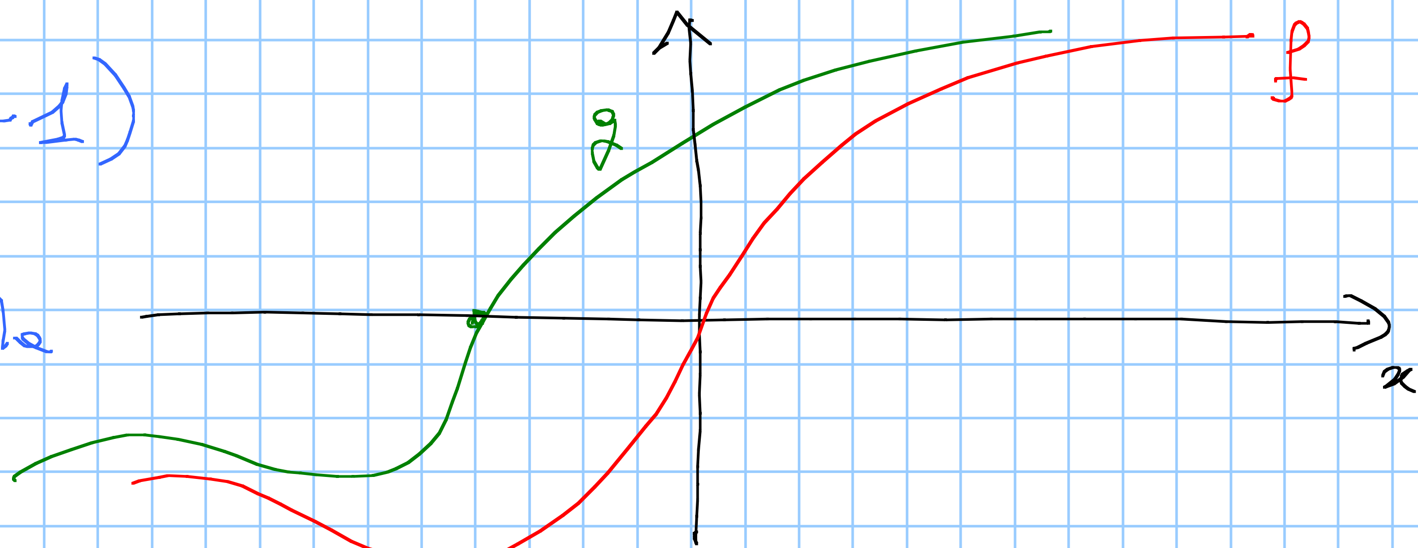
4) $\cos(\pi/2 + x) = -\sin(x)$

$\sin(\pi/2 + x) = \cos(x)$ $\Rightarrow$ il grafico del coseno è ottenuto traslando il grafico del seno di $\pi/2$ verso ... SINISTRA

In generale esaminiamo come cambia il grafico se da una funzione f passiamo a una funzione g definita...

- $g(x) = f(x+1)$

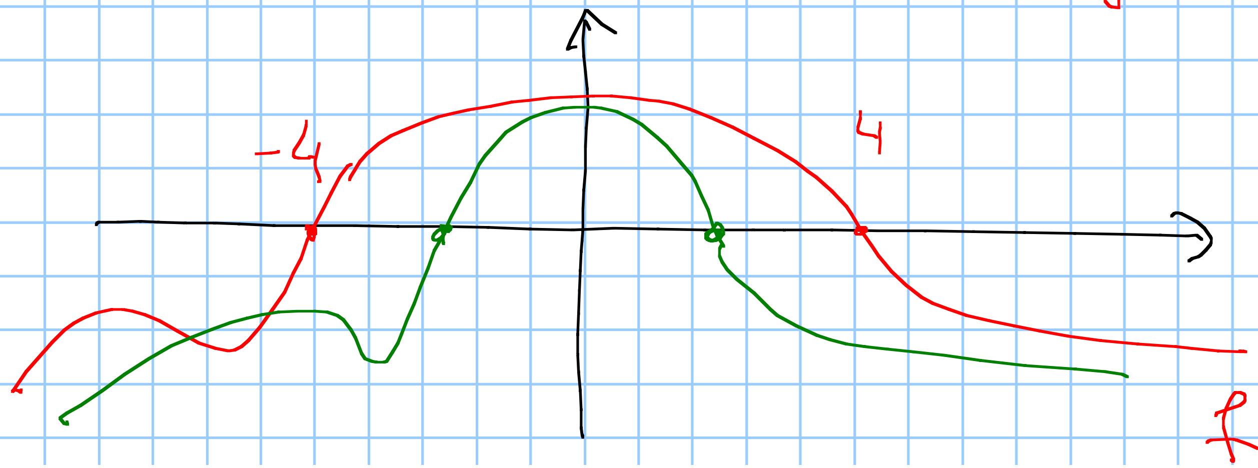
il grafico si sposta a sinistra di c (se $c > 0$)



• $g(x) = f(x) + 1$

grafico di g è quello di f traslato di 1 in alto

• $g(x) = f(2x)$



$\Rightarrow$ grafico di g ottenuto da quello di f stringendo orizzontalmente di fattore 2

- $g(x) = 2 \cdot f(x)$

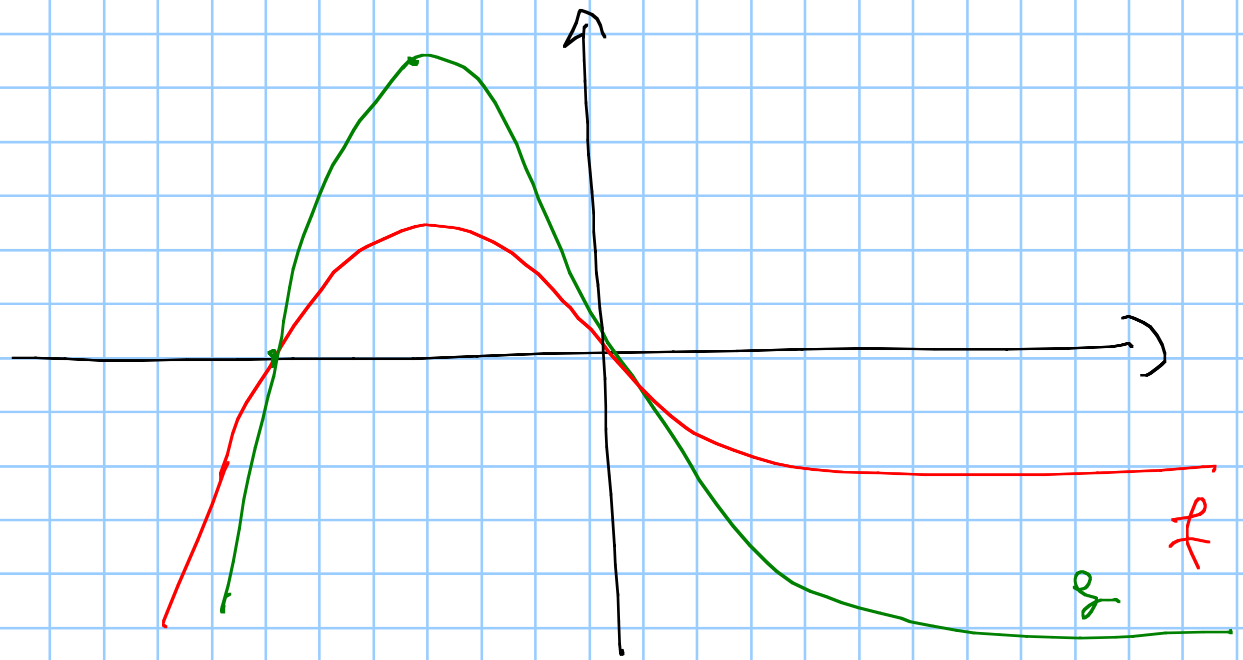
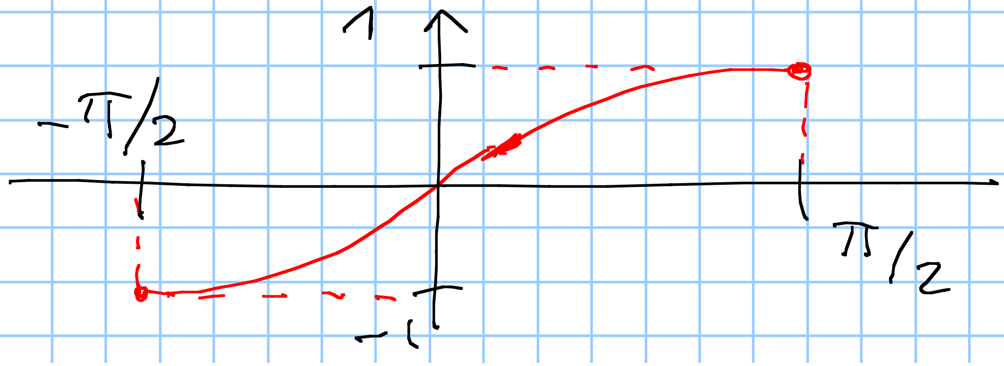


grafico di g ottenuto da quello di f dilatando in verticale di fattore 2

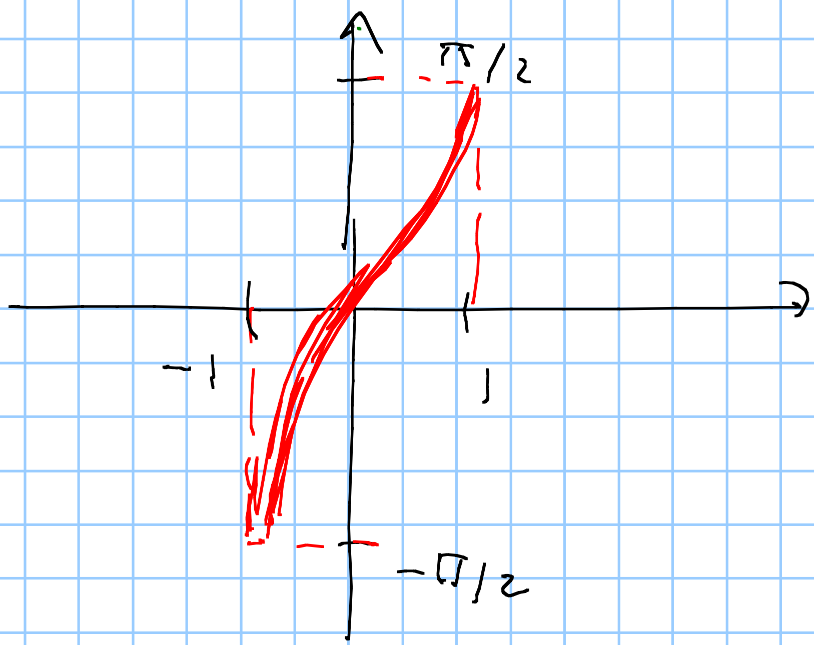
______ . ______

Oss: la funzione $\sin$ su $[-\pi/2, \pi/2]$ è biiettiva tra $[-\pi/2, \pi/2]$ e $[-1, 1]$:



Def: arcsin : $[-1,1] \longrightarrow [-\pi/2, \pi/2]$

è l'inversa di essa.



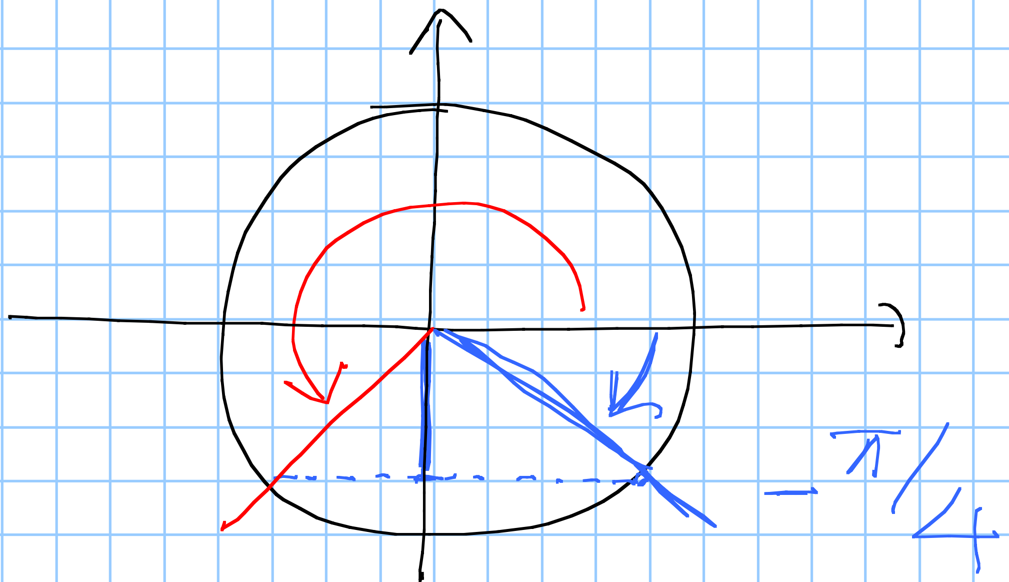
Attenzione:

$\arcsin(t)$

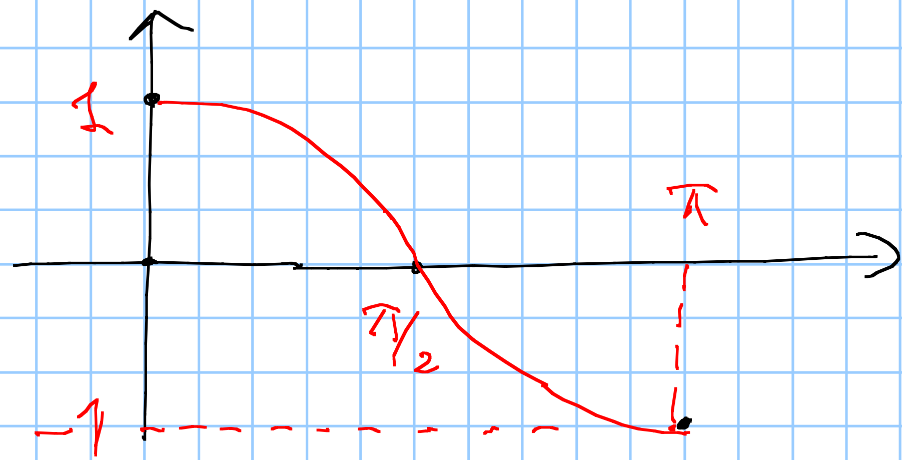
- non esiste se $|t| > 1$

• ha sempre valori in $[-\pi/2, \pi/2]$.

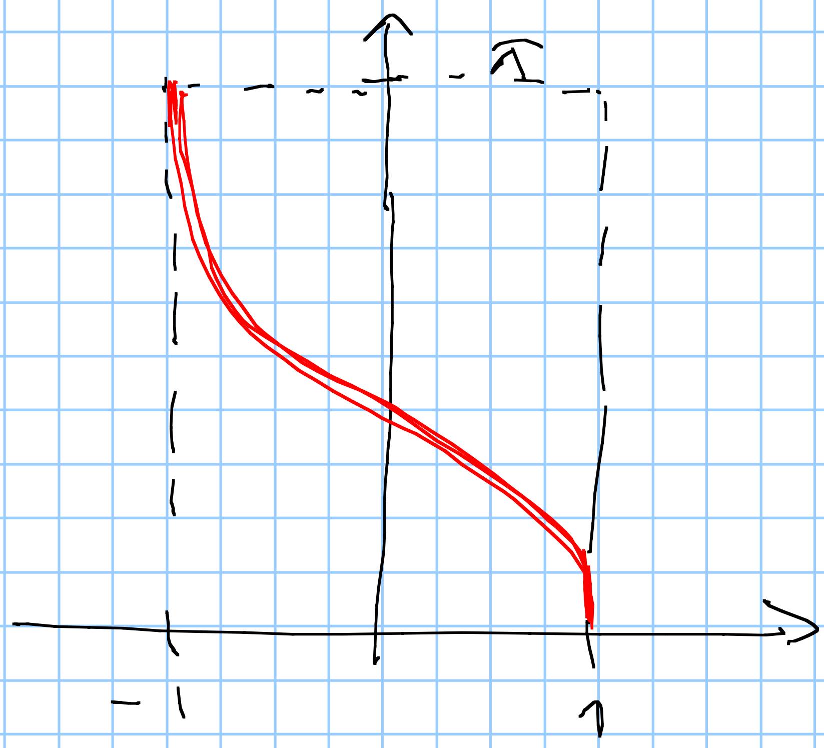
Es: $\arcsin\left(\sin\left(\frac{5}{4}\pi\right)\right) = -\frac{\pi}{4}$



Oss: cos su $[0,\pi]$ è una bijezione con $[-1,1]$ decrescente



Def: $\arccos : [-1,1] \to [0,\pi]$ è la sua inversa.


π
-1
1

Formule di addizione

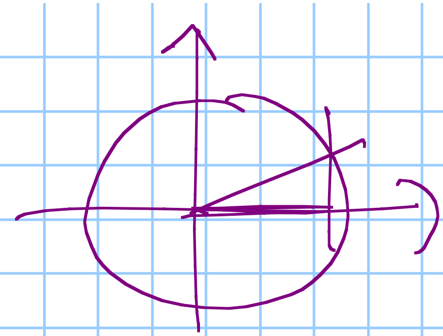
$$\begin{cases} \sin(\alpha+\beta) = \sin(\alpha)\cdot\cos(\beta) + \cos(\alpha)\cdot\sin(\beta) \\ \cos(\alpha+\beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\cdot\sin(\beta) \end{cases}$$

Es: $\sin\left(\frac{\pi}{12}\right) = \sin\left(\frac{\pi}{4} - \frac{\pi}{6}\right)$

$= \sin\left(\frac{\pi}{4}\right)\cdot\cos\left(\frac{\pi}{6}\right) + \cos\left(\frac{\pi}{4}\right)\cdot\sin\left(-\frac{\pi}{6}\right)$

$= \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}}\left(-\frac{1}{2}\right)$

$$= \frac{\sqrt{3} - 1}{2\sqrt{2}}$$



Formule di duplicazione:

$$\begin{cases} \sin(2\alpha) = 2\sin(\alpha)\cos(\alpha) \\ \cos(2\alpha) = \cos^2(\alpha) - \sin^2(\alpha) \end{cases}$$

$$= 2\cos^2(\alpha) - 1$$

$$= 1 - 2\sin^2(\alpha)$$

Conseguenza (verificare per esercizio):

Se $t = \tan\left(\frac{\alpha}{2}\right)$ (per $\alpha \neq k\pi$)

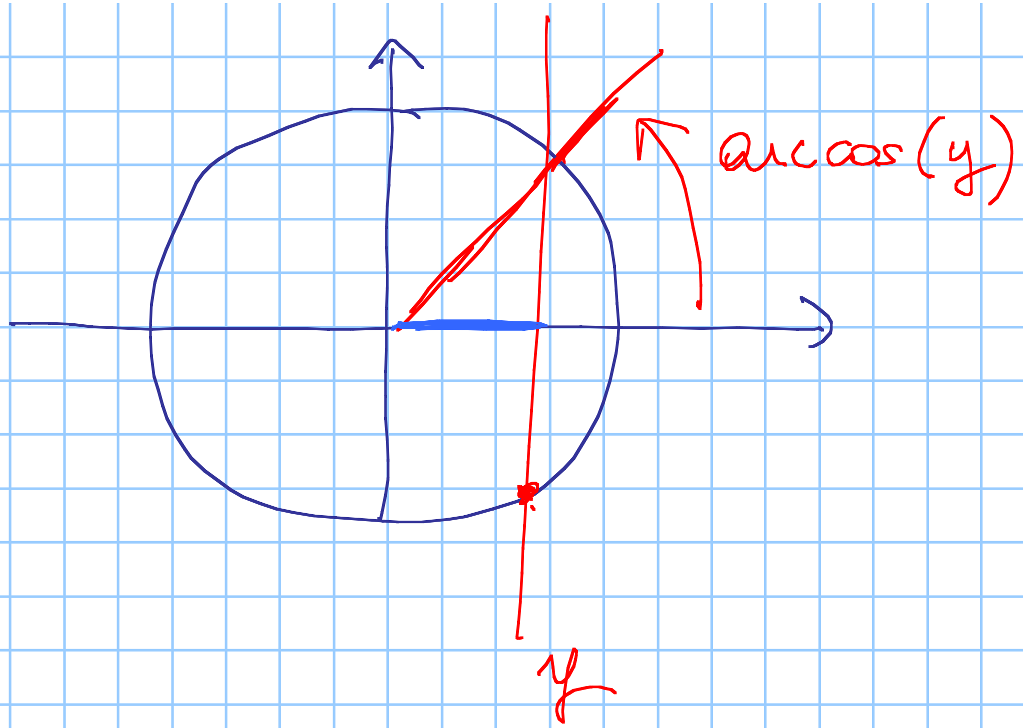
$$\cos(\alpha) = \frac{1-t^2}{1+t^2} \qquad \sin(\alpha) = \frac{2t}{1+t^2}$$

Equazioni

- $\cos(f(x)) = y$

  $\longrightarrow$ nessuna soluz. se $|y| > 1$

  $\longrightarrow$ altrimenti esiste $\arccos(y) \in [0, \pi]$



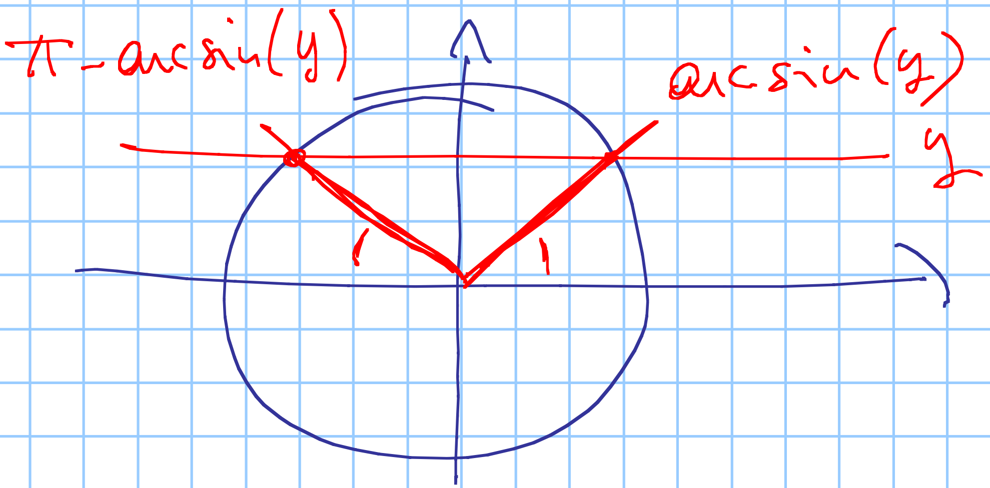
$$f(x) = \pm \arccos(y) + 2k\pi$$

$$k \in \mathbb{Z}$$

$$\sin(f(x)) = y$$

$\rightarrow$ se $|y| > 1$ nessuna soluz.

$\rightarrow$ se $|y| \leq 1$ esiste $\arcsin(y)$

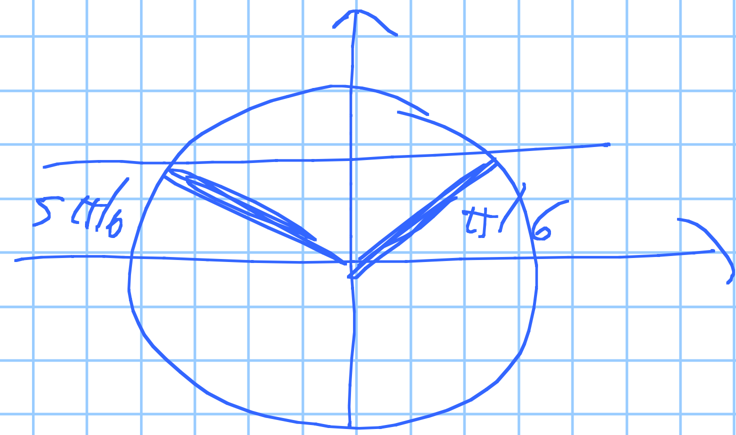


Soluz:

$$f(x) = \arcsin(y) + 2k\pi \quad \text{o}$$

$$f(x) = \pi - \arcsin(y) + 2k\pi$$

Es: $\sin(1+x^2) = \frac{1}{2}$



$1+x^2 = \frac{\pi}{6} + 2k\pi \qquad k \in \mathbb{Z}$

o

$1+x^2 = \frac{5}{6} + 2k\pi \qquad k \in \mathbb{Z}$

$x = \sqrt{\frac{\pi}{6} - 1 + 2k\pi} \qquad k \geq 1,\ k \in \mathbb{Z}$

oppure

$x = \sqrt{\frac{5}{6}\pi - 1 + 2k\pi} \qquad k \in \mathbb{N}$

Equazioni in cui posso sostituire

$t = \cos(x)$ o $t = \sin(x)$ o $t = tg(x)$ ...

risolvere rispetto a t e poi rispetto a x.

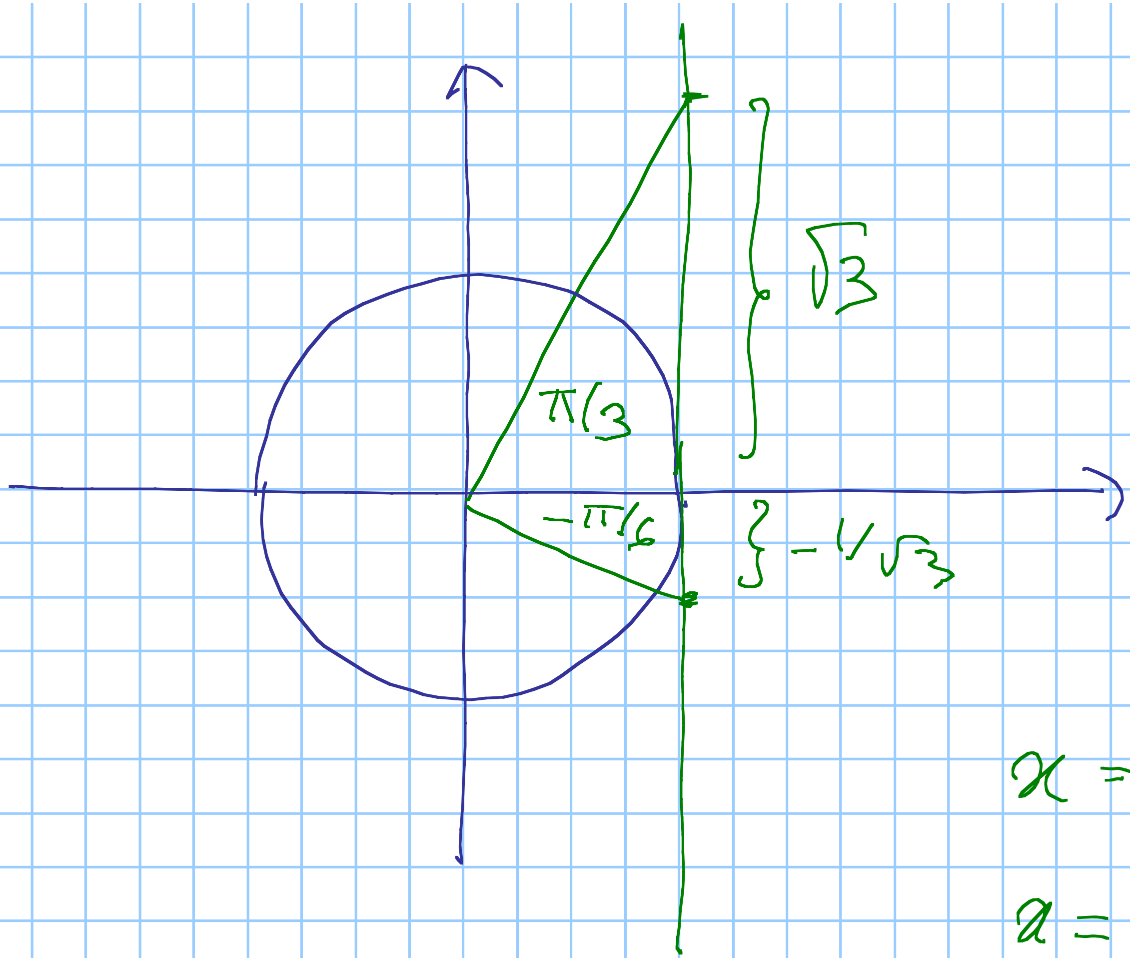
Es: $\tan^2(x) - \frac{2}{\sqrt{3}} \tan(x) - 1 = 0$

$t = \tan(x)$

$$t^2 - \frac{2}{\sqrt{3}}t - 1 = 0$$

$$t_{1,2} = \frac{1}{\sqrt{3}} \pm \sqrt{\frac{1}{3} + 1} = \frac{1}{\sqrt{3}} \pm \frac{2}{\sqrt{3}} \begin{cases} \sqrt{3} \\ -\frac{1}{\sqrt{3}} \end{cases}$$

$$\tan(x) = \begin{cases} \sqrt{3} \\ -\frac{1}{\sqrt{3}} \end{cases}$$



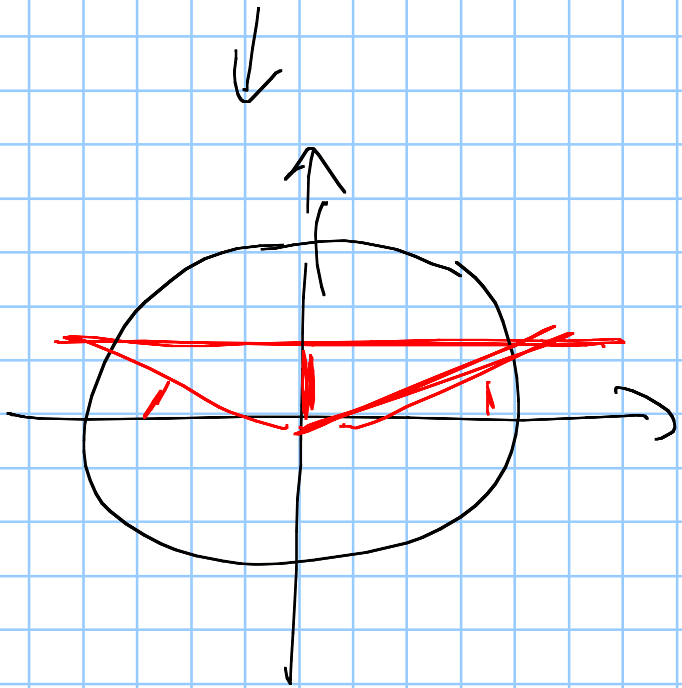
Solut:

$$x = \frac{\pi}{3} + k\pi$$

$$x = -\frac{\pi}{6} + k\pi$$

Equaz. del tipo

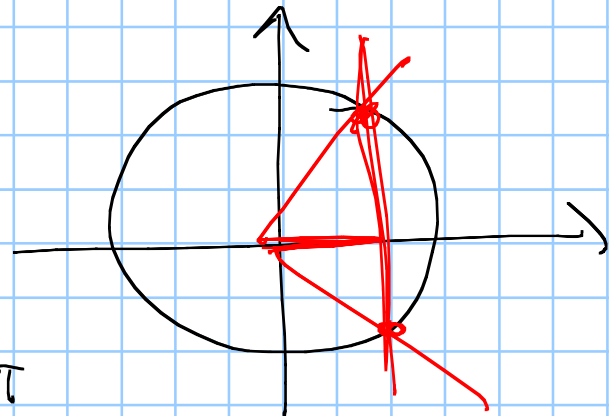
$$\cos(f(x)) = \cos(g(x))$$
$$\sin(f(x)) = \sin(g(x))$$
$$\operatorname{tg}(f(x)) = \operatorname{tg}(g(x))$$



$$f(x) = g(x) + 2k\pi$$

$$f(x) = \pi - g(x) + 2k\pi$$

$$f(x) = \pm g(x) + k\pi$$



$$f(x) = \pm g(x) + 2k\pi$$

Es: $\sin(3x) = -\sin\left(x - \frac{\pi}{3}\right)$

$$\sin(3x) = \sin\left(\frac{\pi}{3} - x\right)$$

$3x = \frac{\pi}{3} - x + 2k\pi$ oppure $3x = \pi - \left(\frac{\pi}{3} - x\right) + 2k\pi$

$x = \frac{\pi}{12} + k\frac{\pi}{2}$ oppure $x = \frac{\pi}{3} + k\pi$

———— o ————

Equazioni lineari in sin/cos:

$$a \cdot \cos(x) + b \cdot \sin(x) = c \qquad a, b, c \in \mathbb{R}.$$

Metodo di risoluzione:

- pongo $t = tg(x/2)$
- ricordo che $\cos(x) = \frac{1-t^2}{1+t^2}$ $\quad \sin(x) = \frac{2t}{1+t^2}$

- sostituisco e risolvo eq. II grado in t
- risolvo rispetto a x

ATT: così facendo perdo possibili soluz.

$$x = \pi + 2k\pi$$

perché per tali valori la sostituz. non ha senso: vanno verificati prima a mano.

Es: $\sqrt{3}\sin(x) + \cos(x) = -1$

$x = \pi$ è soluz? $\qquad \sqrt{3}\cdot 0 + (-1) \stackrel{?}{=} -1$

Sì: ho $\pi + 2k\pi$

sostituisco $\quad t = tg\left(\frac{x}{2}\right)$

$$\sqrt{3}\cdot\frac{2t}{1+t^2} + \frac{1-t^2}{1+t^2} = -1$$

$$2\sqrt{3}t + 1 \cancel{- t^2} = -1 \cancel{- t^2}$$

$$t = -\frac{1}{\sqrt{3}}$$

Ora risolvo $\tan\frac{x}{2} = -\frac{1}{\sqrt{3}}$

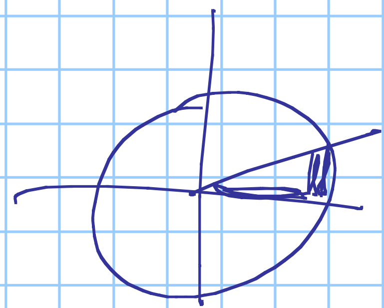
$$\frac{x}{2} = -\frac{\pi}{6} + k\pi$$

$$x = -\frac{\pi}{3} + 2k\pi$$

Metodo alternativo per

$$\sqrt{3} \cdot \sin(x) + \cos(x) = -1$$

$$\underbrace{\frac{\sqrt{3}}{2}}_{\cos\left(\frac{\pi}{6}\right)} \cdot \sin(x) + \underbrace{\frac{1}{2}}_{\sin\left(\frac{\pi}{6}\right)} \cos(x) = -\frac{1}{2}$$



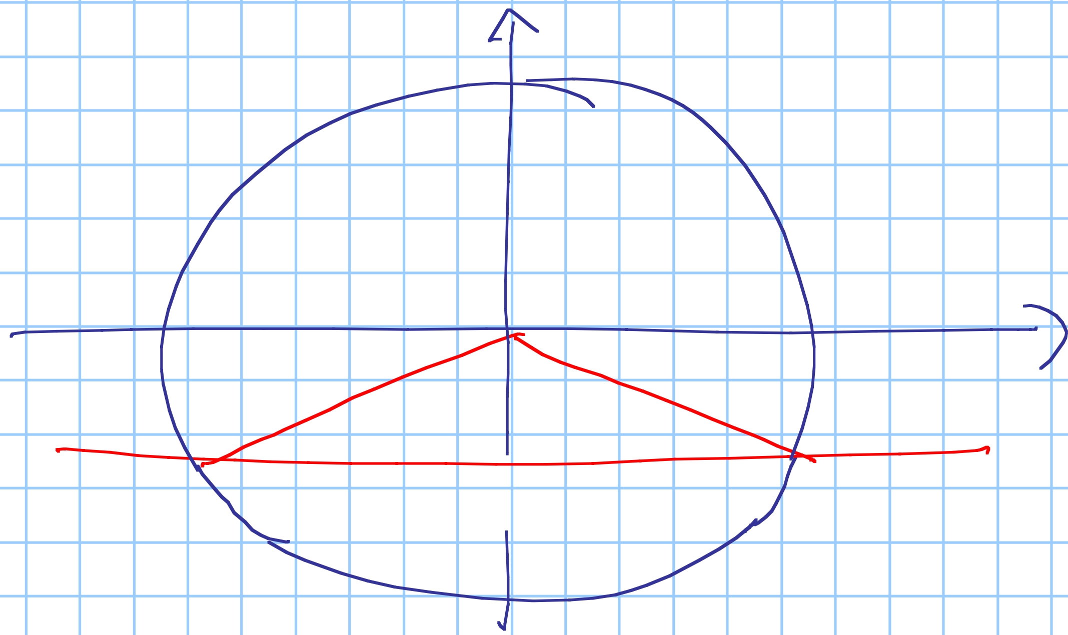
$$\sin\left(x + \frac{\pi}{6}\right) = -\frac{1}{2}$$

Soluz:

$$x + \frac{\pi}{6} = -\frac{\pi}{6} + 2k\pi$$

oppure

$$x + \frac{\pi}{6} = -\frac{5}{6}\pi + 2k\pi$$



cioè $\quad x = -\frac{\pi}{3} + 2k\pi \quad$ o $\quad x = \pi + 2k\pi$