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9.2.5 (k) $V = M_{2\times 2}(\mathbb{R})$

$$\langle A | B \rangle = Tr({}^tA M B) \quad , \quad M = \begin{pmatrix} 2 & -1 \\ -1 & 3 \end{pmatrix}$$

$$\langle B | A \rangle = Tr({}^tB M A)$$

$${}^tB M A = {}^tB\, {}^tM A = \quad \text{xché } M \text{ è simmetrica}$$

$$= {}^t({}^tA M B)$$

$$\Rightarrow \quad \langle A|B\rangle = \langle B|A\rangle \qquad v_1 = \begin{pmatrix} 2 & 1 \\ -1 & 3 \end{pmatrix}$$

$$v_2 = \begin{pmatrix} -1 & 5 \\ 1 & 2 \end{pmatrix}$$

$$\|v_1\|^2 = \mathrm{Tr}\left({}^t v_1 \, M \, v_1 \right)$$

$${}^t v_1 \, M \, v_2 = \begin{pmatrix} 2 & -1 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ -1 & 3 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} 5 & -1 \\ -5 & 8 \end{pmatrix}$$

$$= \begin{pmatrix} 15 & -10 \\ -10 & 23 \end{pmatrix}, \qquad \|v_1\|^2 = 15 + 23 = 38$$

$$u_1 = \frac{v_1}{\|v_1\|} = \frac{1}{\sqrt{38}} \begin{pmatrix} 2 & 1 \\ -1 & 3 \end{pmatrix}$$

$$y_2 = v_2 - \frac{\langle v_2 | v_1 \rangle}{\|v_2\|^2} v_1$$

$$\langle v_2 | v_1 \rangle = \mathrm{Tr}\left({}^t v_1 \, M \, v_2 \right)$$

$${}^t v_1 \, M \, v_2 = \begin{pmatrix} 2 & -1 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} -1 & 5 \\ 1 & 2 \end{pmatrix} =$$

$$= \begin{pmatrix} 2 & -1 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} -3 & 8 \\ 4 & 1 \end{pmatrix} = \begin{pmatrix} -10 & 15 \\ 9 & 11 \end{pmatrix}$$

$$\langle v_1 | v_2 \rangle = -10 + 11 = 1$$

$$Y_2 = v_2 - \frac{1}{38} v_1 = \begin{pmatrix} \cdot & \\ \cdot & \end{pmatrix}$$

Per concludere una volta calcolato Y_2, dobbiamo normalizzarlo:

$$\| Y_2 \|^2 = \mathrm{Tr}\left({}^t Y_2 M Y_2 \right)$$

Es. 9.2.5 (m)

$V = \mathbb{R}_{\leq 2}[t]$

$\dim V = 3$

$v_1(t) = t - 2t^2$

$v_2(t) = t$

$\langle p | q \rangle = p(-1)q(-1) + p(1)q(1) + p(2)q(2)$

t	-1	1	2
$v_1(t)$	-3	-1	-6
$v_2(t)$	-1	1	2

$\|v_1\|^2 = 9 + 1 + 36 = 46$

$\langle v_1 | v_2 \rangle = 3 - 1 - 12 = -10$

$u_1 = \frac{1}{\sqrt{46}} v_1 = \frac{1}{\sqrt{46}}(t - 2t^2)$

$$y_2 = v_2 - \frac{\langle v_1 | v_2 \rangle}{\|v_2\|^2} v_1 = t + \frac{10}{46}(t - 2t^2) =$$

$$= -\frac{10}{23} t^2 + \frac{28}{23} t$$

$$\|y_2\|^2 = \left[y_2(-1)\right]^2 + \left[y_2(1)\right]^2 + \left[y_2(2)\right]^2$$

$$u_2 = \frac{y_2}{\|y_2\|}$$

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9.3.3 $\mathbb{R}^n$, $\langle \, | \, \rangle_A$

$S =$ Span di vettori assegnati.

Trovare base ortogonale di $S^\perp$

(b) $n = 2$ $\quad A = \begin{pmatrix} 3 & -2 \\ -2 & 5 \end{pmatrix}$ $\quad v = \begin{pmatrix} 7 \\ 4 \end{pmatrix}$

$S = \text{span}(v)$ $\quad \dim S = 1$

$\dim S^\perp = 1$

Basta trovare $w \in S^\perp \setminus \{0\}$ $\quad w = \begin{pmatrix} x \\ y \end{pmatrix}$

$w \in S^\perp \iff \langle w, \begin{pmatrix} 7 \\ 4 \end{pmatrix} \rangle = 0$

$$\langle w, \begin{pmatrix} 7 \\ 4 \end{pmatrix} \rangle = (x, y) \begin{pmatrix} 3 & -2 \\ -2 & 5 \end{pmatrix} \begin{pmatrix} 7 \\ 4 \end{pmatrix} =$$

$$= (x, y) \begin{pmatrix} 13 \\ 6 \end{pmatrix} = 13x + 6y$$

Posso prendere: $w = \begin{pmatrix} 6 \\ -13 \end{pmatrix}$ $\longrightarrow$ Normalizzare x avere base $\perp$normale.

9.3.2: $\mathbb{R}^4$

$$S = \text{Span}(3e_1 + e_2 - 5e_3 + 2e_4, -2e_1 + 4e_2 + 3e_3 - e_4)$$

Base ortogonale di $S^\perp$?

$\dim S = 2$, $\quad \dim S^{\perp} = 2$ $\qquad v = \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} \in \mathbb{R}^4$

$v \in S^{\perp} \iff$

$$\begin{cases} 3x + y - 5z + 2t = 0 \\ -2x + 4y + 3z - t = 0 \end{cases}$$

Cerchiamo una base di $S^{\perp}$: risolvendo con $y = 1$ e $t = 0$

trovo: $\begin{pmatrix} 23 \\ 1 \\ 14 \\ 0 \end{pmatrix} = v_2$

e risolvendo per $y = 0$, $t = 1$ trovo: $\begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix} = v_1$

v_1 e v_2 sono una base di $S^\perp$

$\langle v_1 | v_2 \rangle = 23 + 14 = 37$

Scelgo come base: $u_1 = v_1$

$$u_2 = v_2 - \frac{\langle v_2 | v_1 \rangle}{\|v_1\|^2} v_1 = \begin{pmatrix} 23 \\ 1 \\ 14 \\ 0 \end{pmatrix} - \frac{37}{3} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix}$$