Geometria 11/5/18

Def: $\alpha : [a,b] \to \mathbb{R}^2$ regolare chiamo <u>curvatura</u> di α nel pto $\alpha(t)$ il reciproco del raggio della circonferenza avente contatto triplo con α in $\alpha(t)$.

Oss: $K = 0$ = "dritta".

Lem: se $v : [a,b] \to \mathbb{R}^n$ e $\|v(t)\| \equiv 1$ allora

$v'(t)$ è $\perp$ a $v(t)$ $\forall t$.

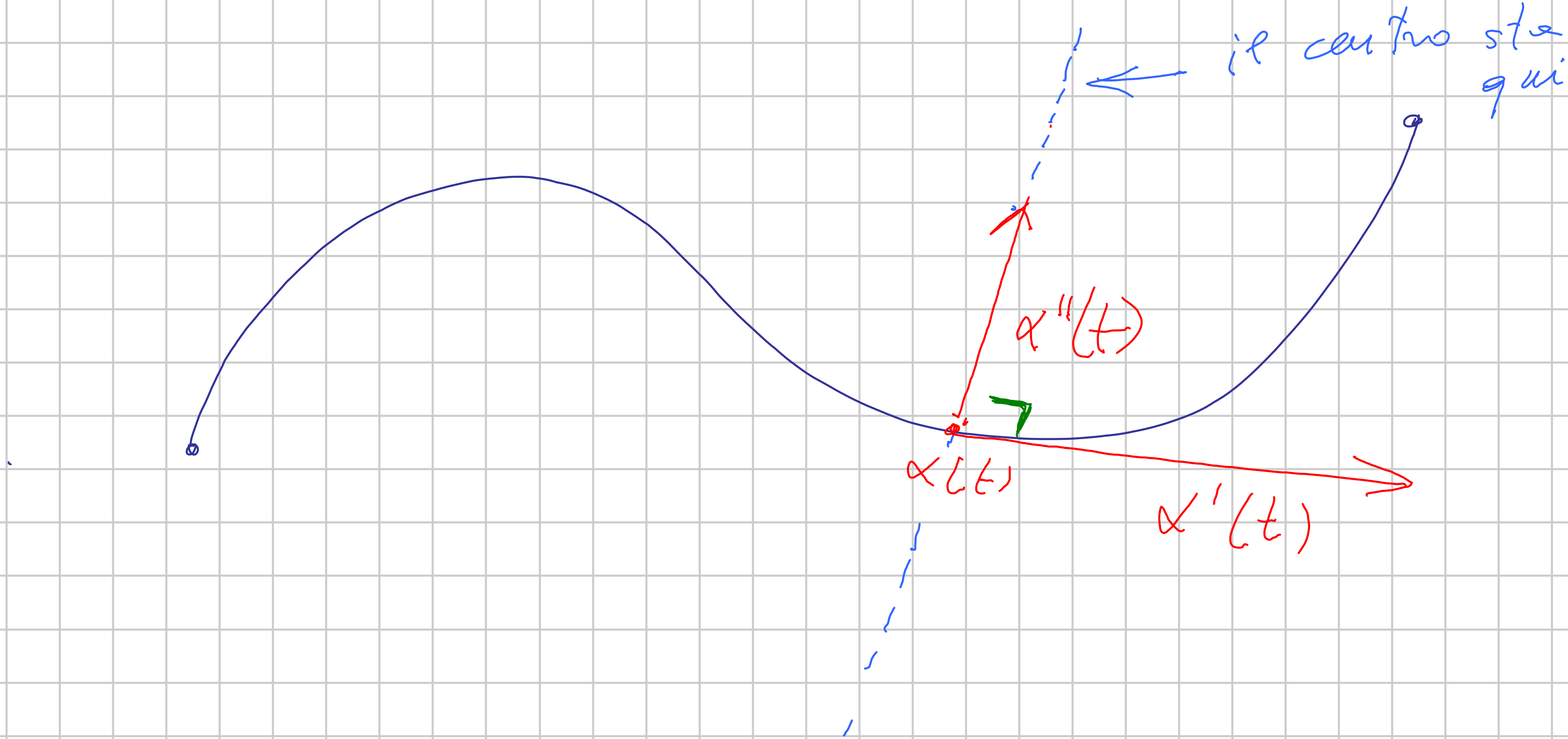
Dim: $\|v(t)\|^2 = v_1(t)^2 + \dots + v_n(t)^2 \equiv 1$

$\Rightarrow \frac{d}{dt}(\uparrow) = 0 \Rightarrow 2v_1(t)v_1'(t) + \dots + 2v_n(t)v_n'(t) = 0$

$\Rightarrow \langle v'(t) \mid v(t) \rangle = 0$. $\blacksquare$

Cor: se α è in p.d'a ($\|\alpha'(t)\| \equiv 1$)

allora $\alpha''(t) \perp \alpha'(t)$ $\forall t$.


il centro sta qui
α''(t)
α(t)
α'(t)

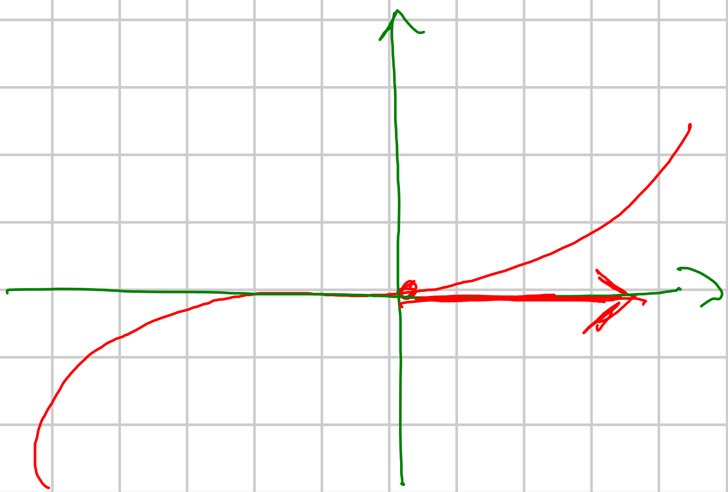
Teo: se α è in p.d'a. la circonf. con triplice contatto con α in $\alpha(t_0)$ è quella di centro $\alpha(t_0) + \dfrac{\alpha''(t_0)}{\|\alpha''(t_0)\|^2}$ e passante per $\alpha(t_0)$.

Cor: il raggio è $\left\| \alpha(t_0) + \dfrac{\alpha''(t_0)}{\|\alpha''(t_0)\|^2} - \alpha(t_0) \right\| = \dfrac{1}{\|\alpha''(t_0)\|}$

$\Longrightarrow$ curvatura $k = \dfrac{1}{\|\alpha''(t_0)\|}$.

Dimo: Tramite traslaz. e rotaz. suppongo

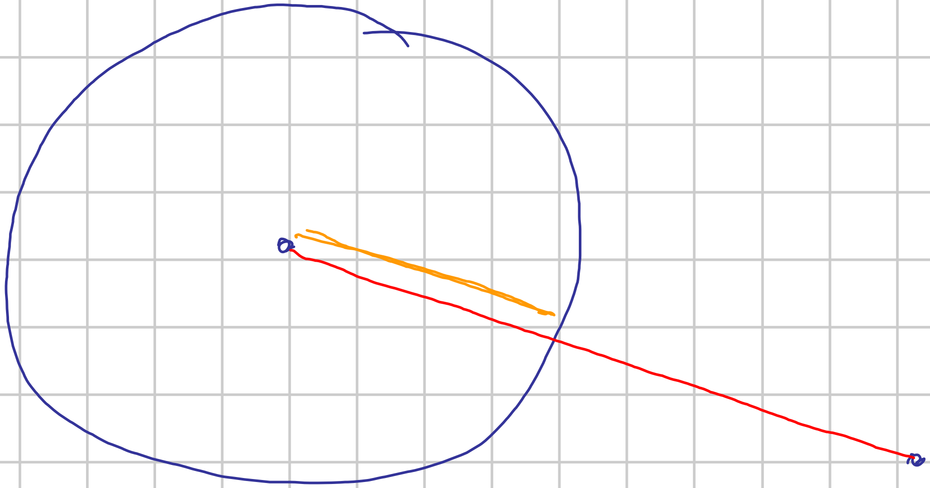
$$\alpha(t_0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \alpha'(t_0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Longrightarrow \alpha''(t_0) = \begin{pmatrix} 0 \\ c \end{pmatrix}.$$



Circonf. cercata ha centro $\begin{pmatrix} 0 \\ r \end{pmatrix}$ e raggio $|r|$.

$$d(t) = \mathrm{dist}(\text{circonf.}, \alpha(t)) =$$

$$= \sqrt{x(t)^2 + (y(t) - r)^2} - |r|$$



Voglio : $d(0) = d'(0) = d''(0) = 0$

$d(0) = d'(0) = 0$ automatico

$$d'(t) = \cancel{\frac{1}{2}} \frac{\cancel{2}XX' + \cancel{2}(Y-r)Y'}{\sqrt{X^2 + (Y-r)^2}}$$

$$d''(t) = -\underbrace{\frac{(XX' + (Y-r)Y')^2}{\sqrt{(X^2 + (Y-r)^2)^3}}}_{= \frac{0}{0} \text{ può limitato}} + \frac{X \cdot X'' + X'^2 + (Y-r)Y'' + Y'^2}{\sqrt{X^2 + (Y-r)^2}}$$

$$d''(t) = 0 \iff X \cdot X'' + X'^2 + (Y - r)\, Y'' + Y'^2 \Big|_{t = t_0} = 0$$

$$\iff 0 \cdot X'' + 1 + (-r) \cdot c + 0 = 0$$

$$\iff r = \frac{1}{c}. \qquad \blacksquare$$

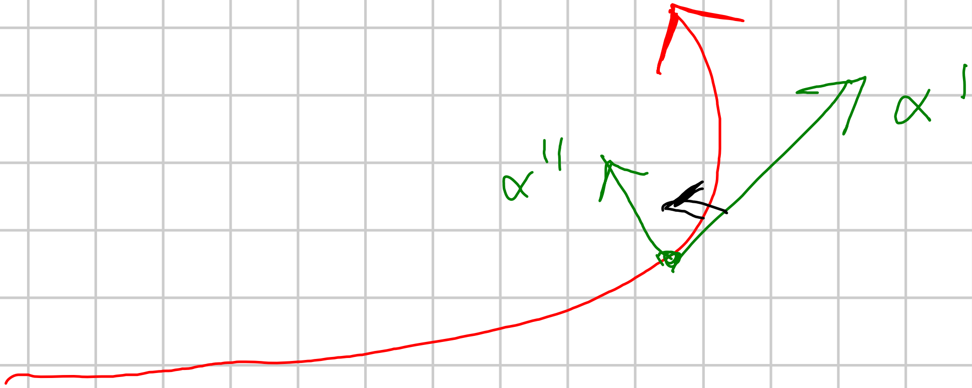
Def: la circonf. con triplice contatto è detta osculatrice.

<u>Riassunto</u>: Se α è in p.d'a. $(\|\alpha'\| \equiv 1)$

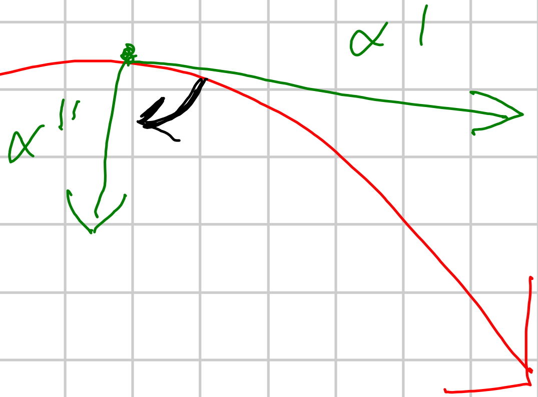
allora $\kappa = \|\alpha''\|$.

Problema: data β qualsiasi è difficile riparametrizzarla in p.d'a. —

<u>Oss</u>: se α è curva orientata posso dare un segno alla sua curvatura:



a sinistra = positive



a destra = negativa

Teo: se β è una curva qualsiasi allora

$$\kappa(t) = \frac{\det\left(\beta'(t) \;\; \beta''(t) \right)}{\|\beta'(t)\|^3}$$

(si ottiene scrivendo $\alpha = \beta \circ \tau$ in p.d'a e esprimendo $\|\alpha''\|$ in funzione di β).

<u>Es</u>: $\beta(t) = \begin{pmatrix} t - 2t^2 + t^3 \\ t - \log(1-t) \end{pmatrix}$ β def su $(-\infty, 1)$

$\beta'(t) = \begin{pmatrix} 1 - 4t + 3t^2 \\ 1 + \dfrac{1}{1-t} \end{pmatrix}$ $\beta''(t) = \begin{pmatrix} -4 + 6t \\ \left(\dfrac{1}{1-t}\right)^2 \end{pmatrix}$

$\beta'(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $\beta''(0) = \begin{pmatrix} -4 \\ 1 \end{pmatrix}$

$$\Rightarrow K(0) = \frac{\det \begin{pmatrix} 1 & -4 \\ 1 & 1 \end{pmatrix}}{2\sqrt{2}} = \frac{5}{2\sqrt{2}}.$$

$$\mathbb{P}^3(\mathbb{R}) \longrightarrow \mathbb{P}^1(\mathbb{C})$$

$$\left[\begin{pmatrix} x \\ y \\ r \\ s \end{pmatrix}\right] \longmapsto \left[\begin{pmatrix} x+iy \\ r+is \end{pmatrix}\right]$$

$\to$ ben. def. $\quad \begin{pmatrix} x' \\ y' \\ r' \\ s' \end{pmatrix} = \lambda \cdot \begin{pmatrix} x \\ y \\ r \\ s \end{pmatrix} \qquad \lambda \in \mathbb{R}$

$$\Rightarrow \begin{pmatrix} x'+iy' \\ r'+is' \end{pmatrix} = \lambda \cdot \begin{pmatrix} x+iy \\ r+is \end{pmatrix}, \quad \lambda \in \mathbb{R} \subset \mathbb{C}$$

→ surj.: ogni $\left[\begin{pmatrix} z \\ w \end{pmatrix}\right]$ viene da qualche $\left[\begin{pmatrix} x \\ y \\ r \\ s \end{pmatrix}\right]$

→ non iniettiva:

$$\begin{pmatrix} x'+iy' \\ r'+is' \end{pmatrix} = \mu \cdot \begin{pmatrix} x+iy \\ r+is \end{pmatrix} \qquad \mu \in \mathbb{C}$$

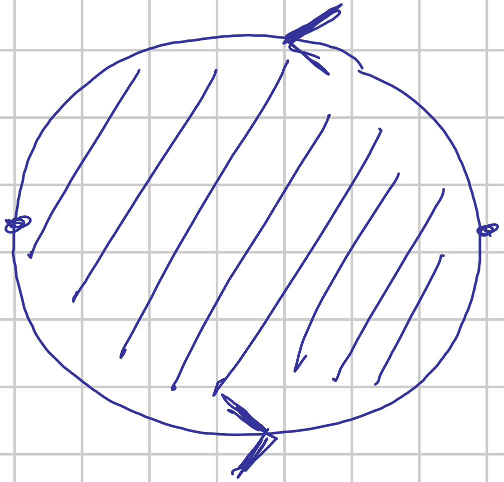
$$\not\Rightarrow \begin{pmatrix} x' \\ y' \\ r' \\ s' \end{pmatrix} = \lambda \cdot \begin{pmatrix} x \\ y \\ r \\ s \end{pmatrix}.$$

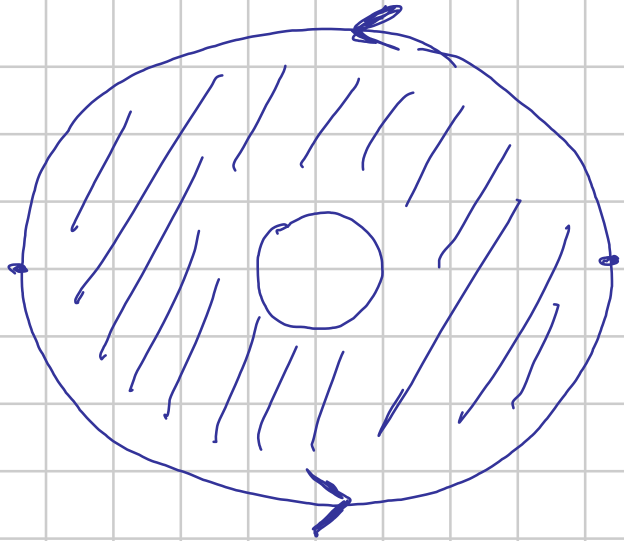
Es: $\left[\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}\right] \longmapsto \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right]$

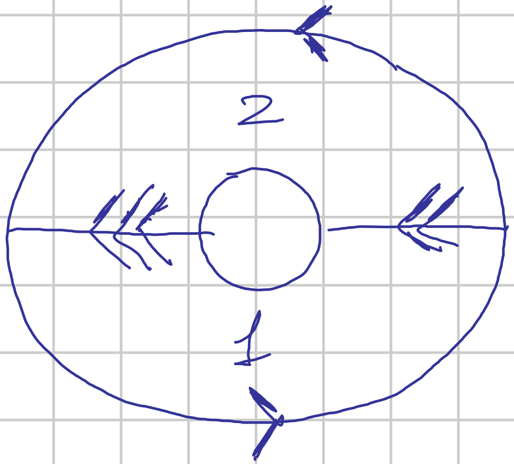
$\left[\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}\right] \longmapsto \left[\begin{pmatrix} i \\ 0 \end{pmatrix}\right]$

uguali

$\boxed{12.2.5}$ Provare che togliendo un disco a $\mathbb{P}^2(\mathbb{R})$ si ottiene Möbius.

$P^2 =$ 

 $= P^2 \setminus \text{disco}$



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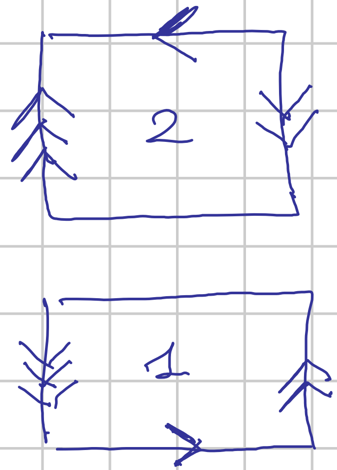

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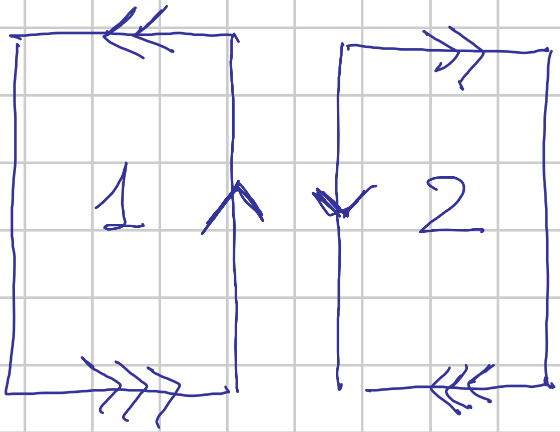

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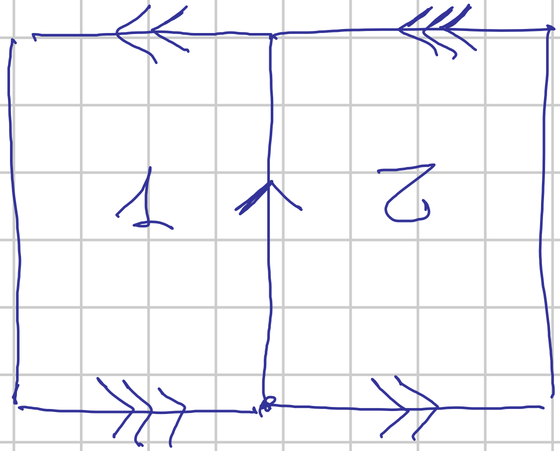

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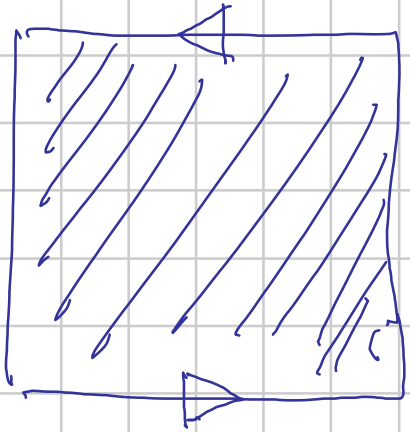
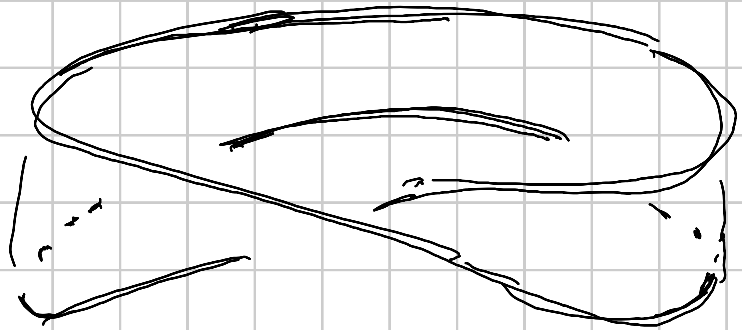
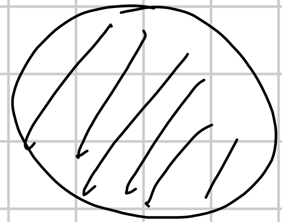
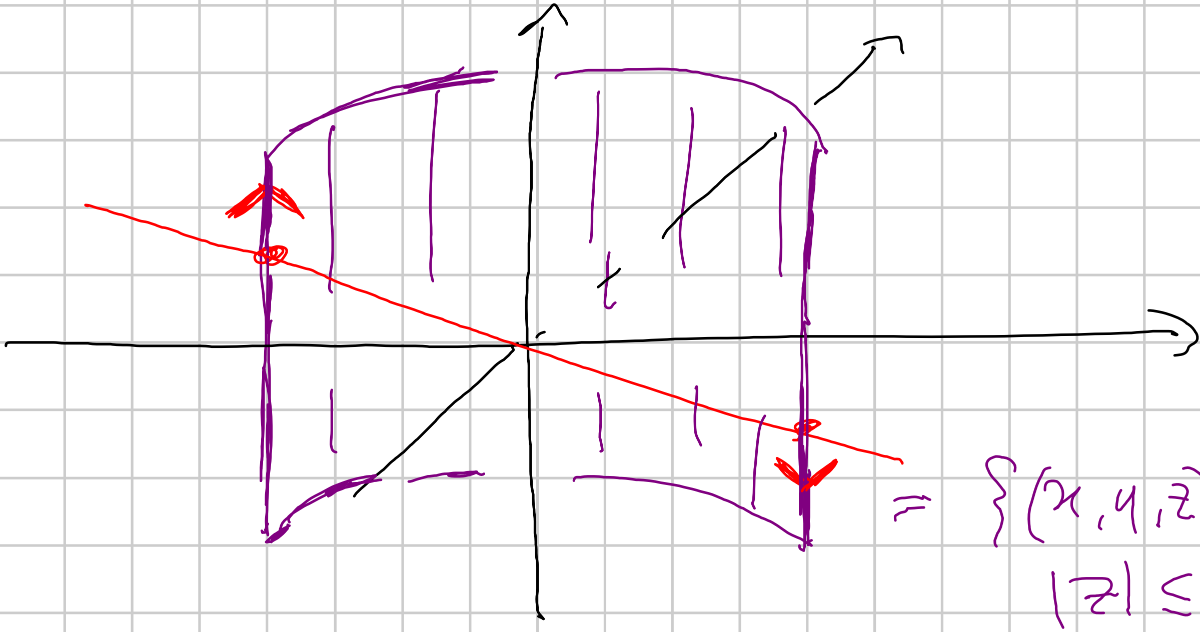


= Möbius

Oss: $\mathbb{P}^2 =$  $\cup$ 

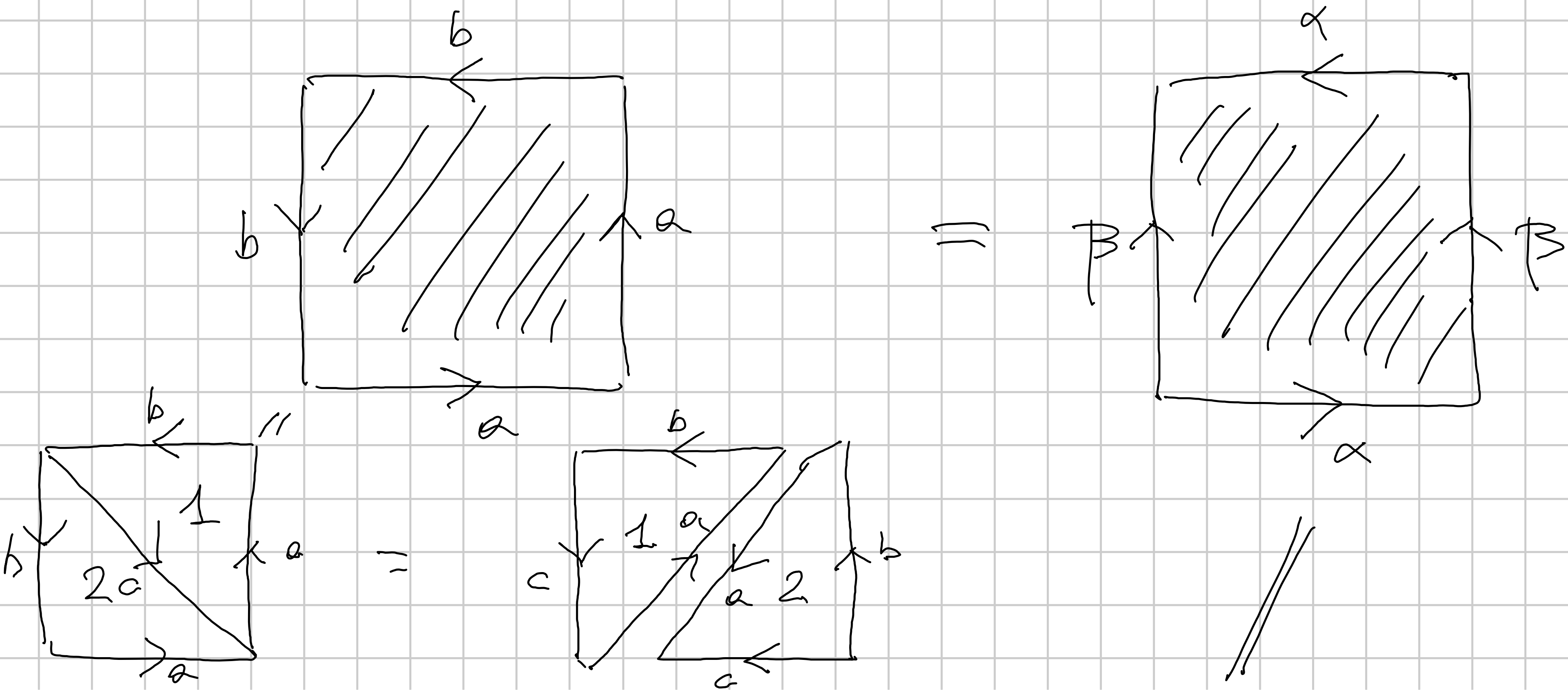
(non realizzabile in $\mathbb{R}^3$)

$\boxed{12.2.6}$ Esibire $X \subset \mathbb{R}^3 \setminus \{0\}$ t.c. la sua immagine in $\mathbb{P}^2(\mathbb{R})$ sia Möbius.



$= \{(x,y,z) : x^2+y^2 = 1$
$|z| \leq 1, \ y \geq 0\}$

$\boxed{12.2.8}$ Provare che



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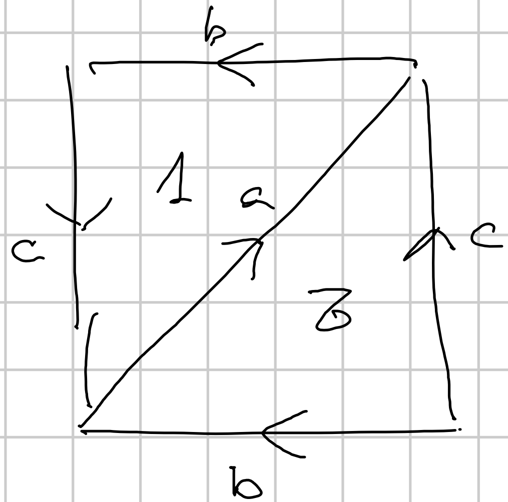
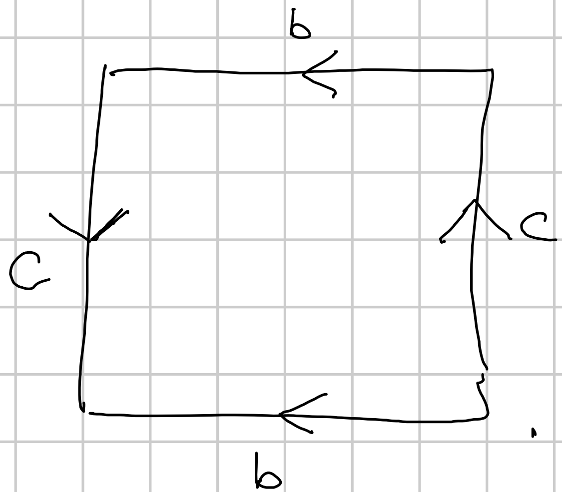


b
1
a
c
c
2
b

$=$



b
c
C
b