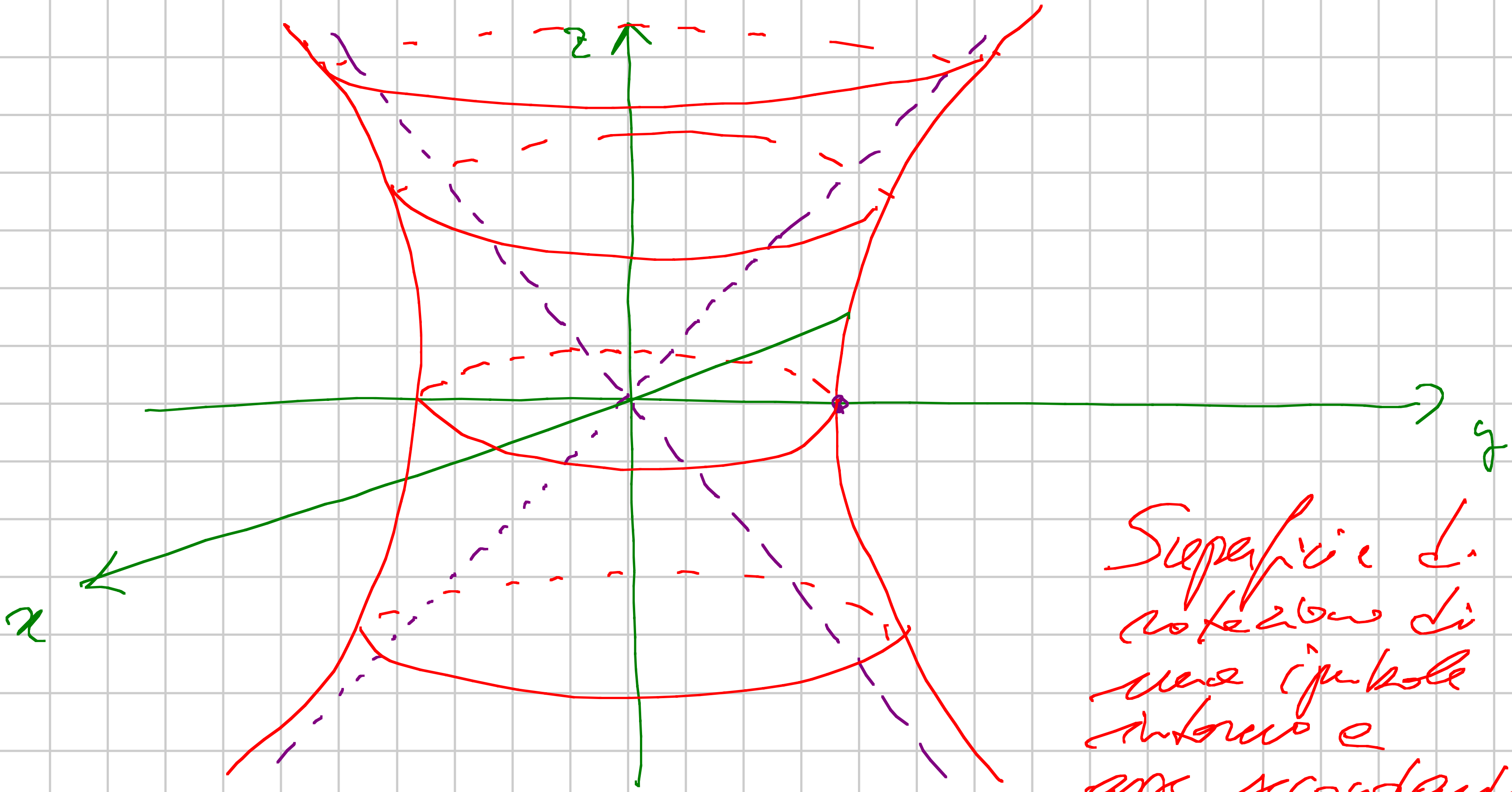
Geometria 3/5/18

Ellissoide $x^2+y^2+z^2=1$

Iperb. ellittico (2 falde) $z^2=1+x^2+y^2$

Parab. ellittico $z=x^2+y^2$

Iperboloide iperbolico (1 falda) $\underbrace{x^2+y^2}_{\text{dist}\left(\begin{pmatrix} x\\ y\\ z\end{pmatrix}, \text{asse } z\right)} = 1 + \underbrace{z^2}_{\text{coord } z}$.

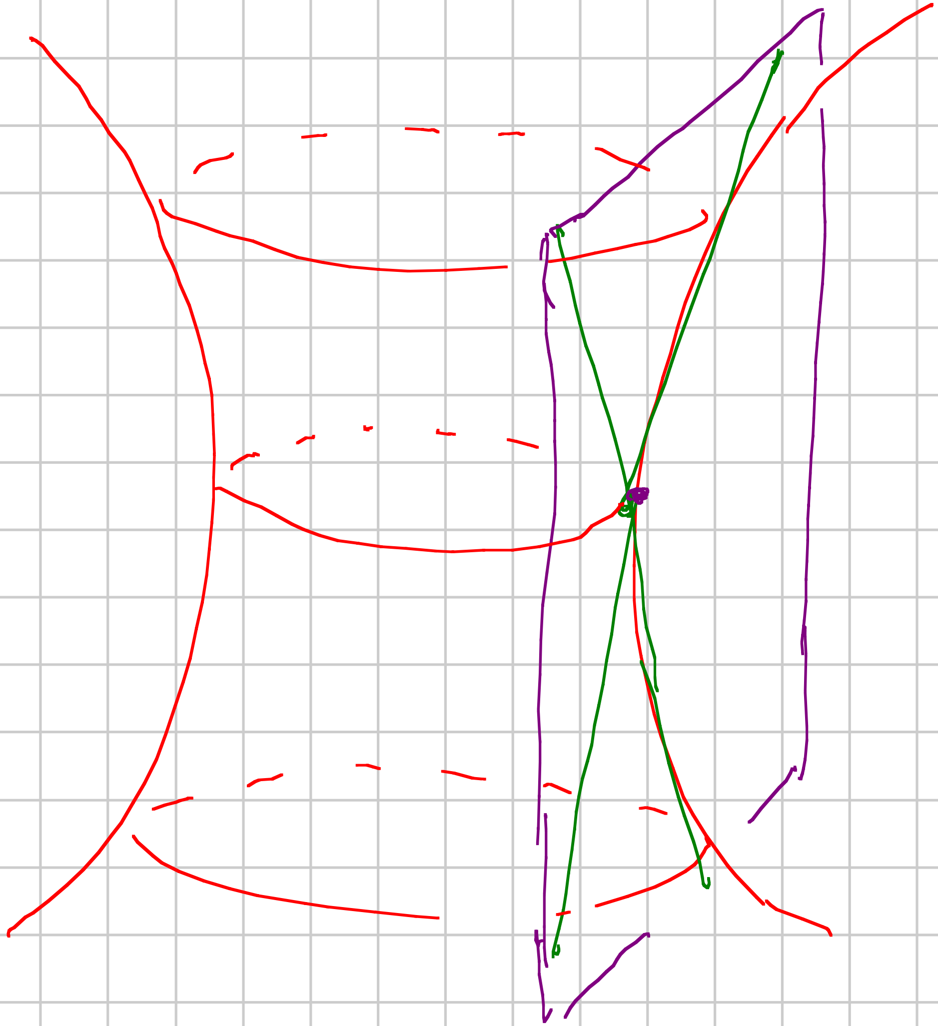


Superficie di rotazione di una iperbole intorno a asse secondario

$$\{x^2+y^2=1+z^2\} = \mathcal{G}$$

$$\mathcal{G} \cap \{y=1\}$$

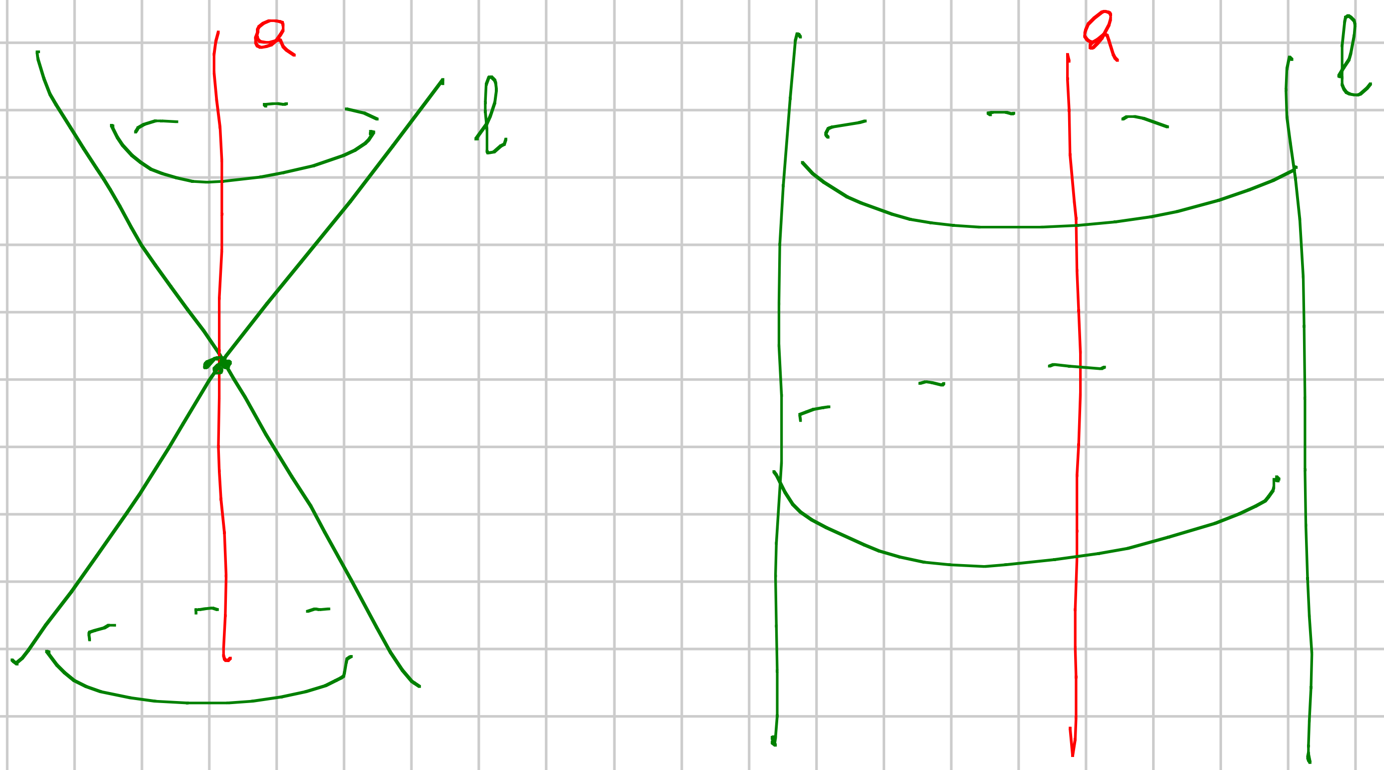
$$= \{x = \pm z\}$$



$\Longrightarrow$ per ogni punto di $\mathcal{G}$ passano due rette distinte contenute in $\mathcal{G}$.

Inoltre: $\mathcal{G}$ si ottiene per rotazione intorno all'asse di simmetria da una qualsiasi di queste rette.

$\Longrightarrow$ $\mathcal{G}$ si ottiene ruotando una retta ℓ intorno a una retta a sghemba con ℓ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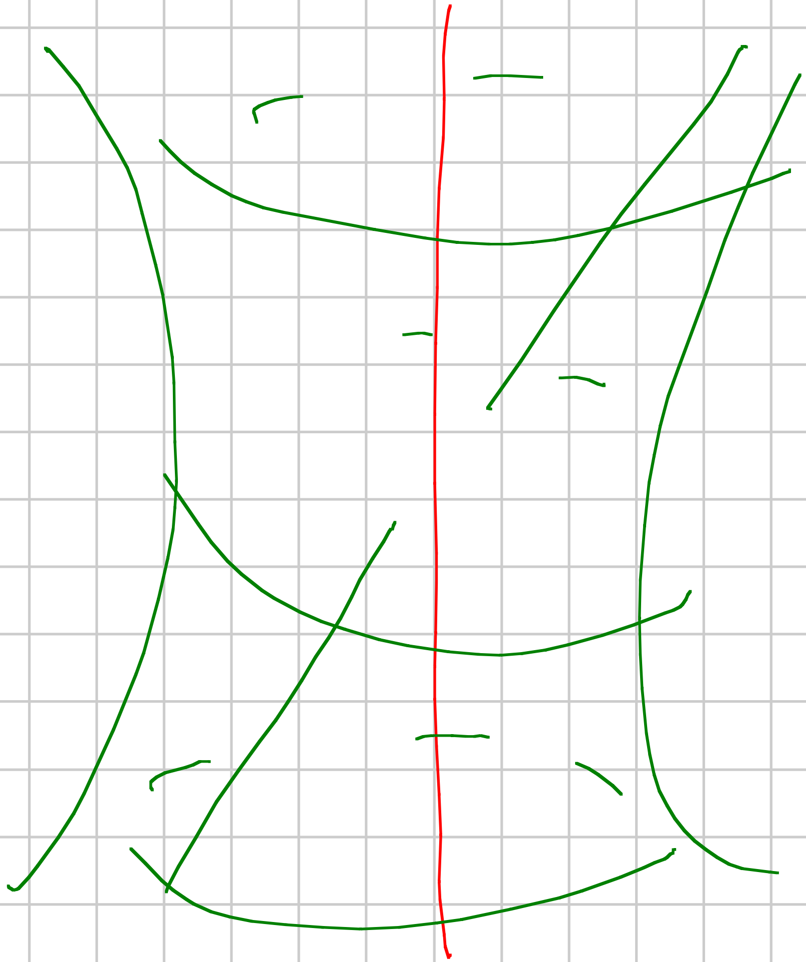
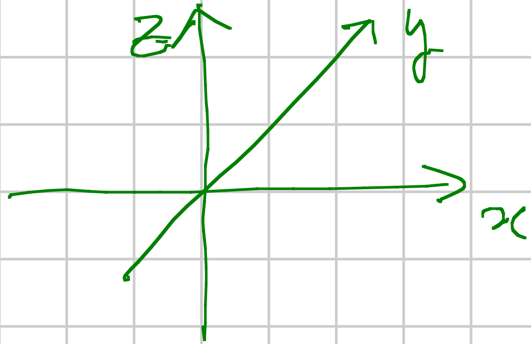
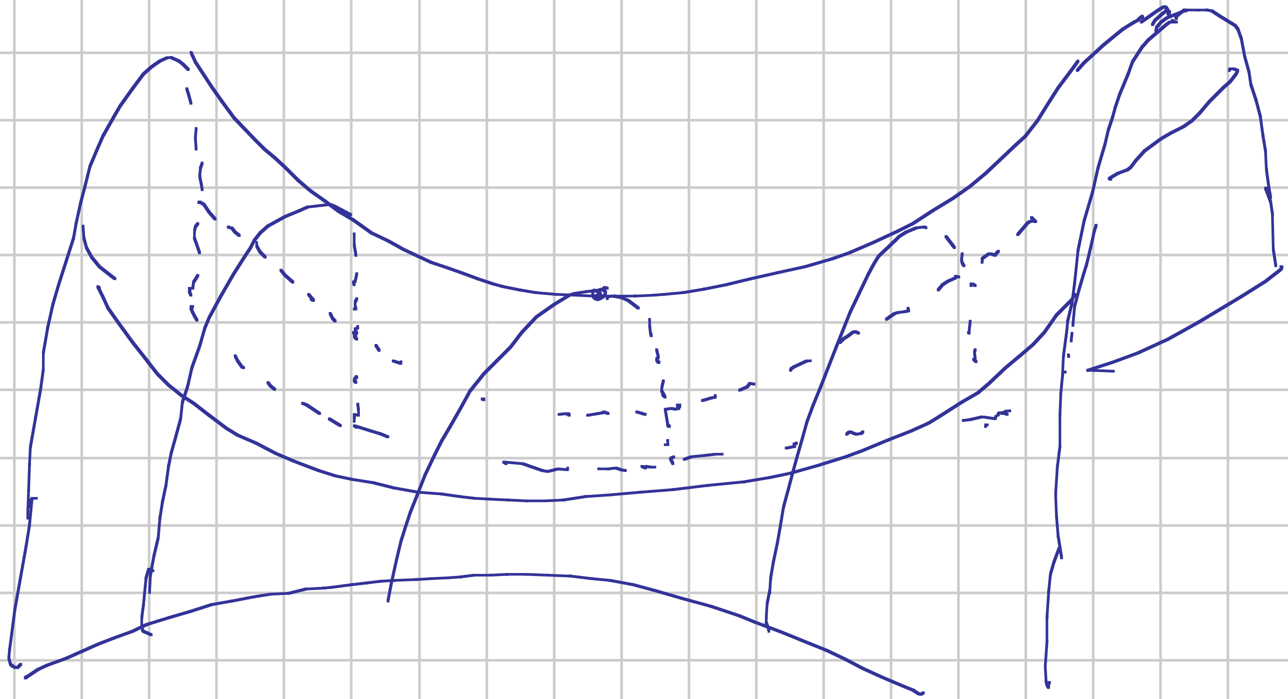


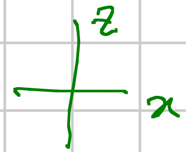
Paraboloide iperbolico $z = x^2 - y^2$.

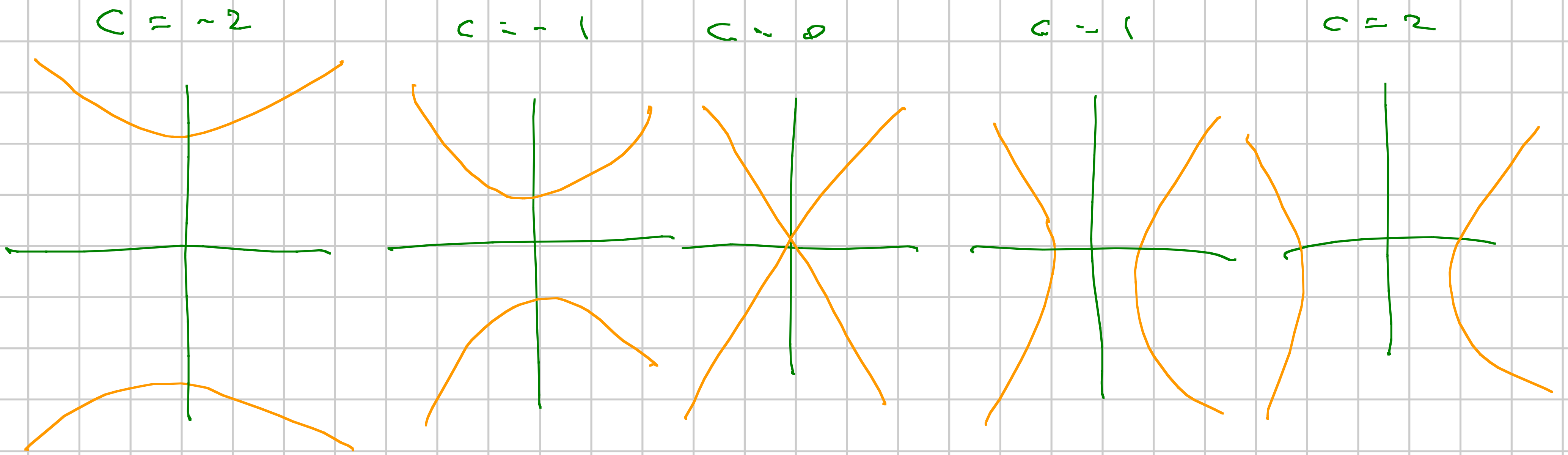




paraboloide a sella

Fette sui piani orizzontali $z = c$ $\qquad x^2 - y^2 = c$

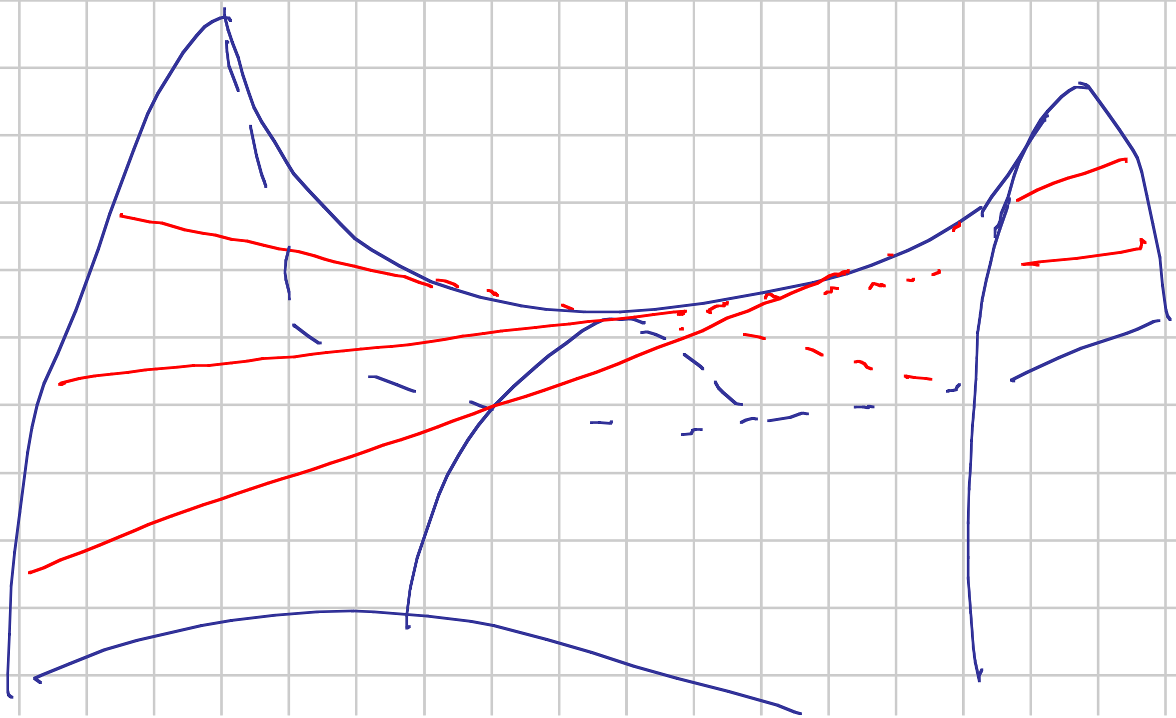




$P = \{ z = x^2 - y^2 \}$

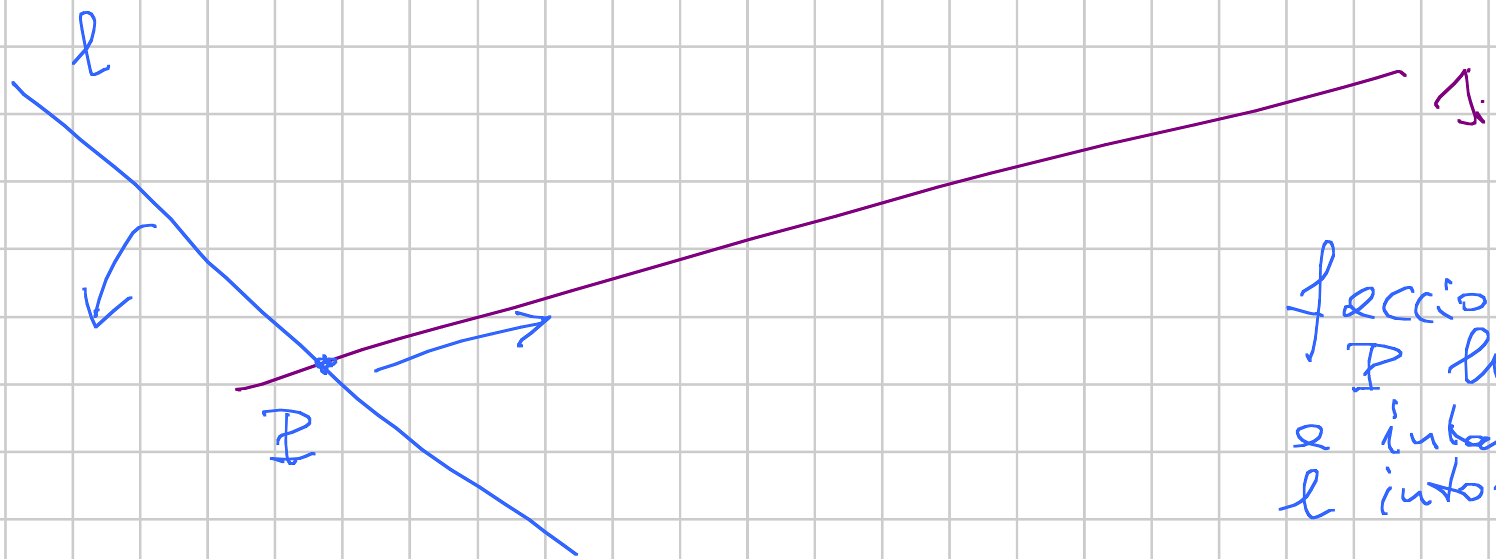
$P \cap \{ x + y = c \} = \{ z = c \cdot (x - y) \}$

$P \cap \{ x - y = c \} = \{ z = c (x + y) \}$



$\Longrightarrow$ per ogni punto di γ passano due rette

e γ si ottiene così: $\qquad l \perp s$



faccio scorrere P lungo s
e intanto ruoto l intorno a s

ellissoide proiettivo

sul piano $\perp$ a $\vec{n}$ per P

ellissoide $\quad x^2+y^2+z^2=1 \longrightarrow x^2+y^2+z^2=w^2 \quad (3+1)$

iperb. ell $\quad z^2=1+x^2+y^2 \longrightarrow w^2+x^2+y^2=z^2 \quad (3+1)$

parab. ell $\quad z=x^2+y^2 \longrightarrow wz=x^2+y^2 \; ; \; w=u+v, \; z=u-v$

$\longrightarrow x^2+y^2+v^2=u^2 \quad (3+1)$

iperboloide proiettivo

iperb. iperb. $\quad x^2+y^2=1+z^2 \longrightarrow x^2+y^2=w^2+z^2 \quad (2+2)$

parab. iperb $\quad z=x^2-y^2 \longrightarrow \begin{cases} w=u+v \\ z=u-v \end{cases} \quad u^2+y^2=x^2+v^2 \quad (2+2)$

<u>Teo</u>: se $A = \begin{pmatrix} Q & \ell \\ {}^t\ell & c \end{pmatrix}$ è simmetrica 4×4, $\det(A) \neq 0$

(<u>non degenere</u>) allora esiste una trasf. affine di $\mathbb{R}^3$

che manda $\mathcal{L} = \left\{ x \in \mathbb{R}^3 : {}^t\begin{pmatrix} x \\ 1 \end{pmatrix} \cdot A \cdot \begin{pmatrix} x \\ 1 \end{pmatrix} = 0 \right\}$

in uno dei 6 modelli

$\emptyset$, ellissoide, iperb. ell, parab ell, iperb. iperb, parab. iperb

<u>Dim</u>: operazioni lecite sull'equazione:

- trasf affini : $A \longrightarrow A' = {}^tN \cdot A \cdot N$

$$N = \begin{pmatrix} M & v \\ 0 & 1 \end{pmatrix}$$

$$Q \longrightarrow Q' = {}^tM \cdot Q \cdot M$$

- moltiplicare A per $\lambda \neq 0$.

Come per le coniche:

I. Mi riconduco a Q diagonale (M ortog, $v = 0$)

II. Mi riconduco a Q con ± 1 o 0 sulla diago

$\downarrow$ $\left(M \text{ diago}, \ v = 0 \right)$

Oss: c'è al massimo uno 0: se due ho

$$A = \left(\begin{array}{ccc|c} 0 & 0 & 0 & * \\ 0 & 0 & 0 & * \\ 0 & 0 & * & * \\ \hline * & * & * & * \end{array}\right)$$

ma tale A ha $\det = 0$ impossibile.

III. Riordinando e/o cambiando segno mi riconduco a

$$\left(\begin{array}{ccc|c} +1 & & & \\ & +1 & & * \\ & & +1 & \\ \hline & * & & * \end{array}\right) \quad \left(\begin{array}{ccc|c} +1 & & & * \\ & +1 & & \\ & & -1 & \\ \hline & * & & * \end{array}\right)$$

$$\left(\begin{array}{ccc|c} +1 & & & \\ & +1 & & * \\ & & 0 & \\ \hline & * & & * \end{array}\right) \quad \left(\begin{array}{ccc|c} +1 & & & \\ & -1 & & * \\ & & 0 & \\ \hline & * & & * \end{array}\right)$$

$\overline{\underline{\mathrm{IV}}}$ Con traslazioni + molt. per λ + riscalare

$$\left(\begin{array}{ccc|c} +1 & & & \\ & +1 & & 0 \\ & & +1 & \\ \hline & 0 & & \pm 1 \end{array}\right) \qquad \left(\begin{array}{ccc|c} +1 & & & \\ & +1 & & 0 \\ & & -1 & \\ \hline & 0 & & \pm 1 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} +1 & & & 0 \\ & +1 & & 0 \\ & & 0 & -1/2 \\ \hline 0 & 0 & -1/2 & 0 \end{array}\right) \qquad \left(\begin{array}{ccc|c} +1 & & & 0 \\ & -1 & & 0 \\ & & 0 & -1/2 \\ \hline 0 & 0 & -1/2 & 0 \end{array}\right)$$

$x^2+y^2+z^2+1=0$ $\qquad$ $x^2+y^2-z^2+1=0$

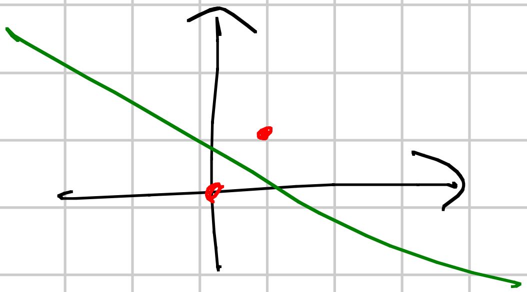
$x^2+y^2+z^2-1=0$ $\qquad$ $x^2+y^2-z^2-1=0$

$x^2+y^2=z$ $\qquad$ $x^2-y^2=z$.

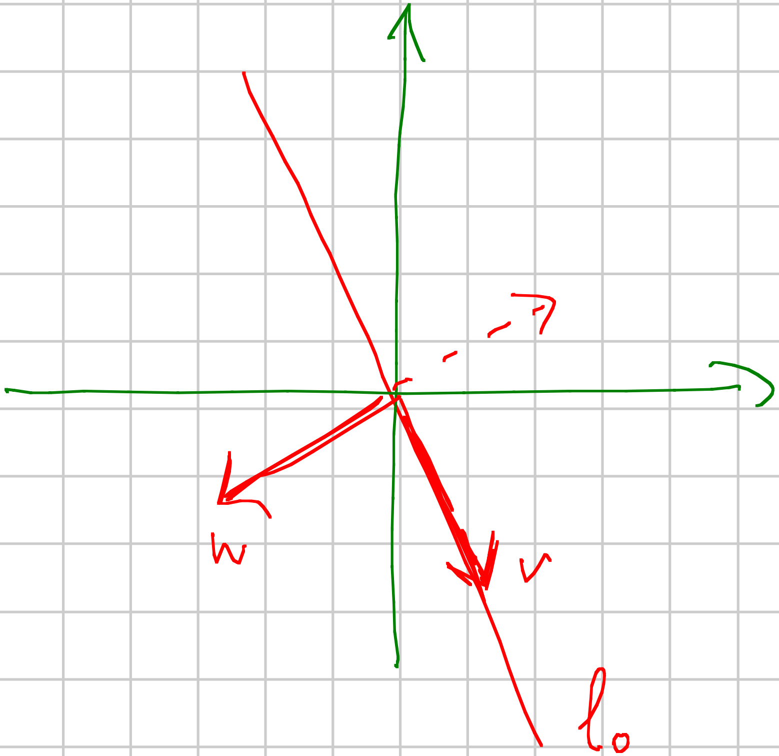
$\boxed{11.1.1}$ Esibire l'isometria φ di $\mathbb{R}^2$.

(b) rifl. rispetto a $\ell: 4x+7y=3$

Att: 10.2.1(a)
Soluzione errata
— 2 ok, altri no



φ_0 : rifl. rispetto a $l_0 : 4x+7y=0$.



$\varphi_0(v)=v \qquad \varphi_0(w)=-w$

$$\varphi_0\begin{pmatrix}7\\-4\end{pmatrix}=\begin{pmatrix}7\\-4\end{pmatrix} \qquad \varphi_0\begin{pmatrix}4\\7\end{pmatrix}=\begin{pmatrix}-4\\-7\end{pmatrix}$$

$$\varphi_0\cdot\begin{pmatrix}7 & 4\\-4 & 7\end{pmatrix}=\begin{pmatrix}7 & -4\\-4 & -7\end{pmatrix}$$

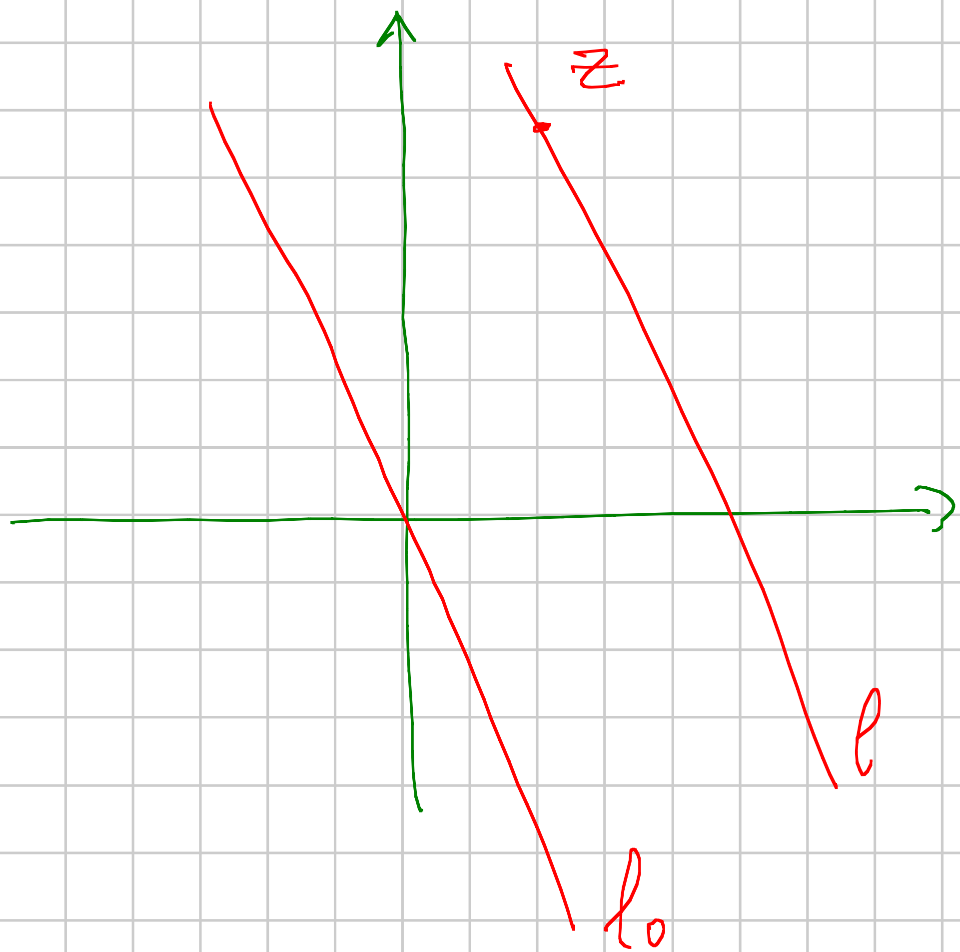
$$\Longrightarrow \varphi_0=\begin{pmatrix}7 & -4\\-4 & -7\end{pmatrix}\cdot\begin{pmatrix}7 & 4\\-4 & 7\end{pmatrix}^{-1}$$

$$= \frac{1}{65}\begin{pmatrix} 7 & -4 \\ -4 & -7 \end{pmatrix}\begin{pmatrix} 7 & -4 \\ 4 & 7 \end{pmatrix}$$

$$= \frac{1}{65}\begin{pmatrix} 33 & -56 \\ -56 & -33 \end{pmatrix}$$

(Le rifl sono $\begin{pmatrix} \cos 2\alpha & \sin 2\alpha \\ \sin 2\alpha & -\cos 2\alpha \end{pmatrix}$ sempre:

infatti $\left(\frac{33}{65}\right)^2 + \left(-\frac{56}{65}\right)^2 = 1$.)



$$\varphi = \tau_z \circ \varphi_0 \circ \tau_{-z}$$

$$\varphi\begin{pmatrix} x \\ y \end{pmatrix} =$$

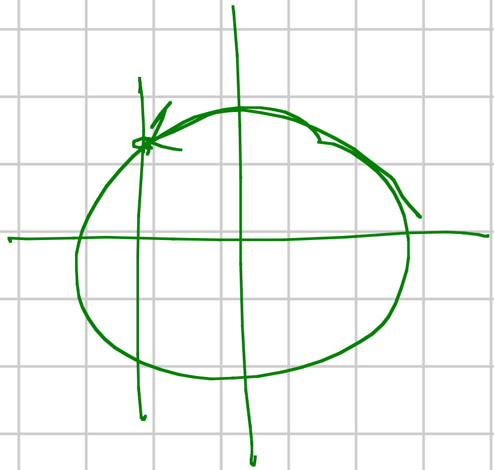
$$= \begin{pmatrix} -1 \\ 1 \end{pmatrix} \cdot \frac{1}{65}\begin{pmatrix} -33 & +56 \\ -56 & -33 \end{pmatrix} \cdot \left(\begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right)$$

$$= \frac{1}{65}\begin{pmatrix} -33 & 56 \\ -56 & -33 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \frac{1}{65}\begin{pmatrix} 24 \\ 42 \end{pmatrix}$$

$$z = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

(b) φ = rotaz. di $\vartheta = \arccos\left(-\frac{3}{5}\right)$ intorno a $\begin{pmatrix} -4 \\ 3 \end{pmatrix}$

φ_0 = rotaz. intorno a O di ϑ



$$\varphi_0 = \begin{pmatrix} \cos\vartheta & -\sin\vartheta \\ \sin\vartheta & \cos\vartheta \end{pmatrix}$$

$$= \frac{1}{5}\begin{pmatrix} -3 & -4 \\ 4 & -3 \end{pmatrix}$$

$$\varphi = \tau_{\begin{pmatrix}-4\\3\end{pmatrix}} \circ \varphi_0 \circ \tau_{-\begin{pmatrix}-4\\3\end{pmatrix}} = \dots$$

$\boxed{11.1.2}$ Trovare equaz. di $\mathcal{C}$:

(b) $\mathcal{C}$ ellisse di fuochi $\begin{pmatrix}-2\\3\end{pmatrix}$, $\begin{pmatrix}7\\-1\end{pmatrix}$ e parametro $2k = 18$

$$I: \quad \sqrt{(x+2)^2+(y-3)^2} + \sqrt{(x-7)^2+(y+1)^2} = 18$$

$$\sqrt{(x+2)^2+(y-3)^2} = 18 - \sqrt{}$$

$$\cancel{x^2}+4x+4+\cancel{y^2}-6y+6 = 18^2-36\sqrt{(x-7)^2+(y+1)^2}$$

$$+\cancel{x^2}-14x+49+\cancel{y^2}+2y+1$$

$$36^2\left((x-7)^2+(y+1)^2\right) = \left(18^2-18x+8y+18^2-10\right)^2$$

... equaz. di IV grado —

II. fuochi $\begin{pmatrix}-2\\3\end{pmatrix}, \begin{pmatrix}7\\-1\end{pmatrix}$ parametro $2k$.

$$2h = \sqrt{81+16} = \sqrt{97}$$

So che per fuochi $(\pm h, 0)$ l'equaz. è

$$\frac{X^2}{k^2} + \frac{Y^2}{k^2-h^2} = 1$$

Punto medio tra fuochi: $\begin{pmatrix}5/2\\1\end{pmatrix}$

Prima isometria: $T_{-\begin{pmatrix}5/2\\1\end{pmatrix}}$ che manda $\begin{pmatrix}-2\\3\end{pmatrix} \begin{pmatrix}7\\-1\end{pmatrix}$

la $\begin{pmatrix} -9/2 \\ 2 \end{pmatrix} \begin{pmatrix} 9/2 \\ -2 \end{pmatrix}$.

ruoto di: $\dfrac{1}{\sqrt{97}} \begin{pmatrix} 9 & -4 \\ 4 & 9 \end{pmatrix}$.

Equaz.: sostituire $\begin{pmatrix} x \\ y \end{pmatrix} = \dfrac{1}{\sqrt{97}} \left(\begin{pmatrix} 9 & -4 \\ 4 & 9 \end{pmatrix} \left(\begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} 5/2 \\ 1 \end{pmatrix} \right) \right)$

in $\dfrac{x^2}{81} + \dfrac{y^2}{81 - 97/4} = 1$.

(f) $\mathcal{C}$ = parabola di fuoco $\begin{pmatrix} 5 \\ -3 \end{pmatrix}$ e direttrice $4x+7y=8$

I: $\sqrt{(x-5)^2+(y+3)^2} = \dfrac{|4x+7y-8|}{\sqrt{16+49}}$

$$65\left(x^2-10x+25+y^2+6y+9\right) =$$

$$= 16x^2+49y^2+64+56xy-64x-112y$$

$$49x^2-56xy+16y^2+\dots = 0$$

$$A = \begin{pmatrix} 49 & -28 & \cdot \\ -28 & 16 & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}$$

$$d_2 = 49 \cdot 16 - 28 \cdot 28$$
$$= 7^2 \cdot 4^2 - (7 \cdot 4)^2 = 0$$

$\boxed{11.2.1}$ Classificare $\mathcal{C}$

(e) $\mathcal{C}: \ 4x^2 - 4xy + y^2 - 2y = 0$

$$A = \begin{pmatrix} 4 & -2 & 0 \\ -2 & 1 & -1 \\ 0 & -1 & 0 \end{pmatrix}$$

$d_2 = 0 \qquad d_3 = -4 \neq 0 \Longrightarrow$ parabola

(f) $x^2 - 2xy - 2x + 1 = 0$

$$A = \begin{pmatrix} 1 & -1 & -1 \\ -1 & 0 & 0 \\ -1 & 0 & 1 \end{pmatrix} \qquad d_2 = -1 < 0$$

$d_3 = -1 \neq 0 \Longrightarrow$ iperbole

(g) $9x^2+6xy+y^2-\sqrt{10}x+3\sqrt{10}y=0$

$$A=\begin{pmatrix} 9 & 3 & -\frac{1}{2}\sqrt{10} \\ 3 & 1 & \frac{3}{2}\sqrt{10} \\ -\frac{1}{2}\sqrt{10} & \frac{3}{2}\sqrt{10} & 0 \end{pmatrix} \qquad d_2=0$$

$$d_3=\frac{5}{2}\cdot\det\begin{pmatrix} 9 & 3 & -1 \\ 3 & 1 & 3 \\ -1 & 3 & 0 \end{pmatrix}$$

$$=-9-9-1-81\neq 0 \implies \text{parabola}$$

(h) $3x^2-6xy+6y^2-2x-2y-5=0$

$$A = \begin{pmatrix} 3 & -3 & -1 \\ -3 & 6 & -1 \\ -1 & -1 & -5 \end{pmatrix}$$

$d_2 = 18-9 > 0$ $\qquad$ $d_1 > 0$

$d_3 = -90-3-3-6-3$
$+45$ $< 0 \Rightarrow$ ellisse

(i) $x^2 - 3xy + 2y^2 - 2x - 5y + \sqrt{5} = 0$

$$A = \begin{pmatrix} 1 & -3/2 & -1 \\ -3/2 & 2 & -5/2 \\ -1 & -5/2 & \sqrt{5} \end{pmatrix}$$

$$d_2 = 2 - \frac{9}{4} < 0$$

$$d_3 = -\frac{8}{4}\sqrt{5} + 9 \qquad 9 \in \mathbb{Q}$$

$d_3 \neq 0$ altrimenti $\sqrt{5} = \frac{4}{3} \cdot 9 \in \mathbb{Q}$ falso.

$\Longrightarrow$ iperbole.

(j) $4x^2 - 5xy + 6y^2 - 2x - y + 22 = 0$

$$A = \begin{pmatrix} 4 & -5/2 & -1 \\ -5/2 & 6 & -1/2 \\ -1 & -1/2 & 22 \end{pmatrix}$$

$d_2 = 24 - \frac{25}{4} > 0$

$d_1 > 0$

$$d_3 = 24 \cdot 22 - \frac{5}{4} - \frac{5}{4} - 6 - \frac{25}{4} \cdot 22 - 1 > 0$$

$$\Longrightarrow \emptyset$$

(l) $2x^2 - 2xy + y^2 + 2y + 1 = 0$

$$A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

$d_2 > 0 \qquad d_1 > 0$

$d_3 = 2 - 1 - 2 < 0$ ellisse

(m) $5x^2 - 4xy + y^2 + 2y + 4\sqrt{2} = 0$

$$A = \begin{pmatrix} 5 & -2 & 0 \\ -2 & 1 & 1 \\ 0 & 1 & 4\sqrt{2} \end{pmatrix}$$

$d_2 > 0 \qquad d_1 > 0$

$d_3 = 4\sqrt{2} - 5 > 0 \;\not\Rightarrow\; \emptyset$

(9) $4x^2 - 20xy + 25y^2 - 2x + 5y - 12 = 0$

$$A = \begin{pmatrix} 4 & -10 & -1 \\ -10 & 25 & 5/2 \\ -1 & 5/2 & -12 \end{pmatrix}$$

$$d_2 = 4 \cdot 25 - 10^2 = 0$$

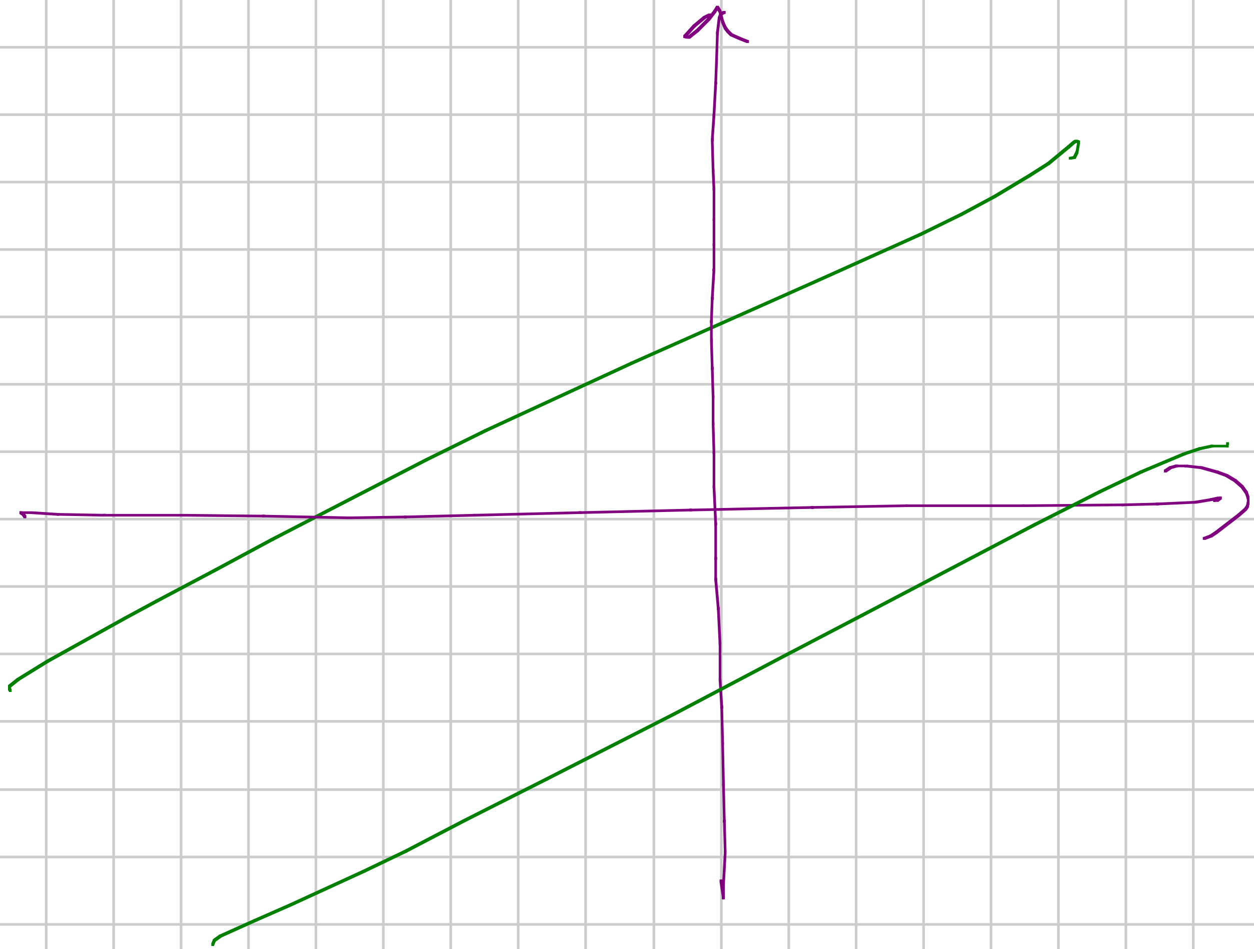
$$d_3 = -1 \cdot (-25 + 25) - \frac{5}{2}(10 - 10) - 12 \cdot 0 = 0$$

$\Rightarrow$ дерение

$$(2x - 5y)^2 - (2x - 5y) - 12 = 0$$

$$t = 2x - 5y \qquad t^2 - t - 12 = (t - 4)(t + 3)$$

$$(2x - 5y - 4)(2x - 5y + 3) = 0.$$



$\Rightarrow$ due rette
parallele.