Algebra Lineare 7/11/17

$\boxed{5.1.2}$ Trovare nucleo e immagine –

(c) $f: \mathbb{R}^3 \to \mathbb{R}^3$ $\quad f(x) = \begin{pmatrix} 2x_1 + x_2 - x_3 \\ 3x_1 + 2x_2 + 4x_3 \\ x_2 + 11x_3 \end{pmatrix}$

$f = f_A \qquad A = \begin{pmatrix} 2 & 1 & -1 \\ 3 & 2 & 4 \\ 0 & 1 & 11 \end{pmatrix}$

Ker(f) : $\begin{cases} x_2 = -11x_3 \\ 2x_1 - 12x_3 = 0 \\ 3x_1 - 18x_3 = 0 \end{cases} \qquad \begin{cases} x_2 = -11x_3 \\ x_1 = 6x_3 \end{cases}$

$$\mathrm{Ker}(f) = \mathrm{Span}\begin{pmatrix} 6 \\ -11 \\ 1 \end{pmatrix} \qquad \dim = 1$$

$$\mathrm{Im}(f) = \mathrm{Span}\left(\begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \cancel{\begin{pmatrix} -1 \\ 4 \\ 11 \end{pmatrix}} \right)$$

$$\begin{cases} 2\alpha + \beta = -1 \\ 3\alpha + 2\beta = 4 \\ \quad \beta = 11 \end{cases} \qquad \begin{cases} \beta = 11 \\ \alpha = -6 \end{cases}$$

(e) $f: \mathbb{R}^3 \to \mathbb{R}^4$ $\quad f(x) = \begin{pmatrix} 1 & 1 & -1 \\ 2 & 3 & 1 \\ 4 & 5 & -1 \\ 5 & 7 & 1 \end{pmatrix} \cdot x$

$$\operatorname{Ker}(f): \begin{cases} x_3 = 4x_1 + 5x_2 \\ -3x_1 - 4x_2 = 0 \\ 6x_1 + 8x_2 = 0 \quad \checkmark \\ 9x_1 + 12x_2 = 0 \quad \checkmark \end{cases}$$

$$\Rightarrow \operatorname{Ker}(f) = \operatorname{Span}\begin{pmatrix} 4 \\ -3 \\ 1 \end{pmatrix} \qquad \dim = 1$$

Scoperto: $\text{III col} = -4 \cdot \text{I} + 3 \cdot \text{II}$

$\dim(\operatorname{Im}(f)) = 2$

g) $f: \mathbb{R}^4 \to \mathbb{R}^3$ $\quad f(x) = \begin{pmatrix} 1 & -2 & 0 & 1 \\ 0 & 1 & 1 & -1 \\ 1 & 0 & 3 & 2 \end{pmatrix} \cdot x$

$\operatorname{Ker}(f): \begin{cases} x_4 = x_2 + x_3 \\ x_1 = x_2 - x_3 \\ x_2 - x_3 + 3x_3 + 2x_2 + 2x_3 = 0 \end{cases} \qquad \begin{cases} x_4 = x_2 + x_3 \\ x_1 = x_2 - x_3 \\ 3x_2 + 4x_3 = 0 \end{cases}$

$$\ker(f) = \operatorname{Span}\begin{pmatrix} 7 \\ 4 \\ -3 \\ 1 \end{pmatrix} \qquad \dim = 1$$

$$\operatorname{Im}(f) = \operatorname{Span}\left(\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix} \cancel{\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}} \right)$$

$\checkmark \quad \checkmark \quad \checkmark$

(i) $f : M_{n \times n} \longrightarrow M_{n \times n}$

$$f(A) = A + {}^tA$$

$\mathrm{Ker}\, f: \quad A + {}^tA = 0 \quad \Longrightarrow \mathrm{Ker}(f) = \mathcal{A}_n(\mathbb{R})$

$\dim = \dfrac{n(n-1)}{2}$

$\mathrm{Im}\, f = \mathcal{S}_n(\mathbb{R}) \qquad \dim = \dfrac{n(n+1)}{2}$

(k) $\quad f: M_{2\times 2}(\mathbb{R}) \to \mathbb{R}^3 \qquad f(A) = \begin{pmatrix} a_{11} - 3a_{12} \\ 2a_{21} + a_{12} \\ -a_{11} + 5a_{22} \end{pmatrix}$

$$\operatorname{Ker} f : \begin{cases} a_{11} = 5a_{22} \\ a_{12} = \frac{5}{3} a_{22} \\ a_{21} = -\frac{5}{6} a_{22} \end{cases}$$

$$\operatorname{Ker} f = \operatorname{Span}\left(\begin{pmatrix} 30 & 10 \\ -5 & 6 \end{pmatrix} \right) \qquad \dim \operatorname{Ker} f = 1$$

$$\operatorname{Im} f = \operatorname{Span}\left(f(E_{11}), f(E_{12}), f(E_{21}), f(E_{22}) \right)$$

$$= \operatorname{Span}\left(\cancel{\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}}, \begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 5 \end{pmatrix} \right)$$

$\dim = 3$

(m) $f : \mathbb{R}_{\leq d}[t] \to \mathbb{R}_{\leq d}[t]$

$f(p(t)) = p'(t)$

$\operatorname{Ker} f : \quad p'(t) = 0$

$\ker(f) = \operatorname{Span}(1)$

$\dim = 1$

$\mathrm{Im}\, f = \mathbb{R}_{\leq d-1}[t]$. dim $= d$

(o) $f : \mathbb{R}^4 \to \mathbb{R}_{\leq 2}[t]$

$f(x) = (x_1 + 3x_2 - x_4) + (x_2 + 2x_3 - x_4)t + (x_1 - 6x_3 + 2x_4)t^2$

lineare: i coeff. del pol. $f(x)$ sono funz. lin. delle compo di x.

$$\operatorname{Ker}(f): \begin{cases} x_1 = -3x_2 + x_4 \\ x_3 = -\frac{1}{2}x_2 + \frac{1}{2}x_4 \\ -3x_2 + x_4 + 3x_2 - 3x_4 + 2x_4 = 0 \quad \checkmark \end{cases}$$

$$\operatorname{Ker}(f) = \operatorname{Span}\left(\begin{pmatrix} -6 \\ 2 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 1 \\ 2 \end{pmatrix} \right) \qquad \dim = 2$$

$$\operatorname{Im}(f) = \operatorname{Span}\Big(\underset{\checkmark}{1+t^2},\ \underset{\checkmark}{3+t},\ \underset{-3\cdot \mathrm{I} + \mathrm{II}}{\cancel{t-3t^2}},\ \underset{2\cdot \mathrm{I} - \mathrm{II}}{\cancel{-1-t+2t^2}} \Big)$$

$\dim = 2$.

$\boxed{5.1.4}$ $f : \underbrace{\{x \in \mathbb{R}^8 : x_3 - 2x_5 + 3x_8 = 0\}}_{7} \to \underbrace{\mathbb{R}_{\leq 10}[t]}_{11}$

lineare $f(e_1 + 2e_3 + e_5) = f(3e_3 - e_7 - e_8) = \underbrace{5 - 7t^9}$

Quanto può valere $\left.\begin{matrix} \\ \\ \end{matrix}\right\}$ $\dim(\operatorname{Im}(f))$?

$\Downarrow$

rett $\neq 0$ in $\operatorname{Im} f$

$$0 \neq (e_1 + 2e_3 + e_5) - (3e_3 - e_7 - e_8) \qquad \Rightarrow \dim(\operatorname{Im} f) \geq 1$$

$$\in \operatorname{Ker}(f)$$

$$\Rightarrow \dim(\operatorname{Ker} f) \geq 1$$

$$7 = \dim \operatorname{Ker}(f) + \dim(\operatorname{Im} f)$$

0	7
1	6
2	5
3	4
4	3

5
6
7

2
1
0

$1 \leq \dim(\operatorname{Im} f) \leq 6$

$\boxed{5.1.5}$

$f: \mathbb{R}^5 \longrightarrow \mathbb{R}^8 \quad \dim(\operatorname{Ker}(f)) = 2$

$Z \subset \mathbb{R}^8 \quad Z \cap \operatorname{Im}(f) = \{0\}$.

$\dim(Z) = ?$

$$\dim(\mathrm{Im}(f)) = 5 - 2 = 3$$

$$3 + \dim(Z) = \underset{\substack{\| \\ 0}}{\dim(Z \cap \mathrm{Im}(f))} + \underset{3 \leq \,\cdot\, \leq 8}{\dim(Z + \mathrm{Im}(f))}$$

$$\Longrightarrow \quad 0 \leq \dim(Z) \leq 5.$$

$\mathcal{L}(V, W) = \{ f : V \to W \ : \ f$ lineare $\}$ sp. vett.

$$\mathcal{M}_{m \times n}(\mathbb{R}) \longrightarrow \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$$
$$A \longmapsto f_A \qquad \left(f_A(x) = A \cdot x \right)$$

lineare bijettiva

Inoltre: $f_A \circ f_B = f_{A \cdot B}$ –

Scriveremo A invece di f_A: indichiamo allo stesso modo una matrice e l'applicaz. ad essa associata.

$$A = \begin{pmatrix} 2 & 3 & -1 \\ 5 & -7 & 4 \end{pmatrix} \in \mathcal{M}_{2\times 3}$$

$$A : \mathbb{R}^3 \to \mathbb{R}^2 \qquad A(x) = A \cdot x = \begin{pmatrix} 2x_1 + 3x_2 - x_3 \\ 5x_1 - 7x_2 + 4x_3 \end{pmatrix}.$$

Ricordo: l'inversa $A \longmapsto f_A$ è:

$$f \longmapsto A \quad \text{t.c.} \quad f(e_i) = \sum_{j=1}^{m} a_{ji} e_j$$

Alla applicaz. f corrisponde la matrice che ha come i-esima colonna le componenti di $f(e_i)$.

$\mathcal{L}(V, W)$??

Supponiamo $\dim(V) = n \qquad \mathcal{B} = (v_1, \dots, v_n)$

$\dim(W) = m \qquad \mathcal{C} = (w_1, \dots, w_m)$

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
V \arrow[r, shift left, "f"] \arrow[r, shift right, "{[\cdot]_{\mathcal{C}}^{-1} \circ g \circ [\cdot]_{\mathcal{B}}}"'] \arrow[d, "{[\cdot]_{\mathcal{B}}}"'] & W \arrow[d, "{[\cdot]_{\mathcal{C}}}"] \\
\mathbb{R}^n \arrow[r, shift left, "g"] \arrow[r, shift right, "{[\cdot]_{\mathcal{C}} \circ f \circ [\cdot]_{\mathcal{B}}^{-1}}"'] & \mathbb{R}^m
\end{tikzcd}
$$

Prop: $\mathcal{L}(V,W) \xrightarrow{f \longmapsto [\cdot]_{\mathcal{C}} \circ f \circ [\cdot]_{\mathcal{B}}^{-1}} \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$

è lineare invertibile con inversa

$$g \longmapsto [\cdot]_{\mathcal{C}}^{-1} \circ g \circ [\cdot]_{\mathcal{B}} .$$

Cor: esiste una bijezione lineare

$$\mathcal{L}(V,W) \xrightarrow{[\cdot]_{\mathcal{B}}^{\mathcal{C}}} M_{m\times n}(\mathbb{R})$$

che dipende da $\mathcal{B}$ e $\mathcal{E}$ ottenuta componendo la precedente con $(A \longmapsto f_A)^{-1}$.

Ricordo: $(A \longmapsto f_A)^{-1}$ è la mappa che manda f in A t.c. la i-esima colonna contiene le coord. rispetto a $\mathcal{E}^{(m)}$ dell'immagine tramite f dell'i-esimo vettore di $\mathcal{E}^{(n)}$.

Poiché $[\cdot]_{\mathcal{B}}$ manda $\mathcal{B}$ in $\mathcal{E}^{(n)}$

$[\cdot]_{\mathcal{C}}$ manda $\mathcal{C}$ in $\mathcal{E}^{(m)}$

la mappa $[\cdot]^{\mathcal{C}}_{\mathcal{B}}$ è la seguente:

$[f]^{\mathcal{C}}_{\mathcal{B}}$ detta matrice associata a f rispetto alle basi $\mathcal{B}$ in partenza e $\mathcal{C}$ in arrivo

è quella matrice $A \in M_{m\times n}(\mathbb{R})$ t.c.

$$f(v_i) = \sum_{j=1}^{m} a_{ji} w_j .$$

<u>Cioè</u>: la i-esima colonna di $[f]_{\mathcal{B}}^{\mathcal{C}}$ contiene le coordinate rispetto a $\mathcal{C}$ dell'immagine tramite f dell'i-esimo elemento di $\mathcal{B}$.

Esempi: (1) $f : \mathbb{R}^2 \to \mathbb{R}^2$ $f(x) = \begin{pmatrix} 3x_1 - 7x_2 \\ 4x_1 + 5x_2 \end{pmatrix}$

$\mathcal{B} = \left(\begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 8 \\ -3 \end{pmatrix} \right)$ $\mathcal{C} = \left(\begin{pmatrix} 5 \\ -2 \end{pmatrix} \begin{pmatrix} 4 \\ 7 \end{pmatrix} \right)$

$[f]_{\mathcal{B}}^{\mathcal{C}} = ?$

I colonna: calcolo $f\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ e trovo le sue coordinate rispetto a $\mathcal{B}$:

$$f\begin{pmatrix}2\\1\end{pmatrix} = \begin{pmatrix}-1\\13\end{pmatrix} = -\frac{59}{43}\begin{pmatrix}5\\-2\end{pmatrix} + \frac{63}{43}\begin{pmatrix}4\\7\end{pmatrix}$$

$$\Rightarrow [f]^{\mathcal{C}}_{\mathcal{B}} = \begin{pmatrix} -59/43 & * \\ 63/43 & * \end{pmatrix}$$

II colonna: calcolo $f\begin{pmatrix}8\\-3\end{pmatrix}$ e le sue coord. rispetto a $\mathcal{C}$:

$$f\begin{pmatrix}8\\-3\end{pmatrix} = \begin{pmatrix}45\\17\end{pmatrix} = \frac{315-68}{43}\begin{pmatrix}5\\-2\end{pmatrix} + \frac{175}{43}\begin{pmatrix}4\\7\end{pmatrix}$$

$$\Rightarrow [f]^{\mathcal{C}}_{\mathcal{B}} = \frac{1}{43}\begin{pmatrix} -59 & 247 \\ 63 & 175 \end{pmatrix}$$

$$f: \mathbb{R}^3 \to \{x \in \mathbb{R}^3 : x_1 + x_2 + x_3 = 0\}$$

3

$$\mathcal{B} = \left(\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 4 \\ 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ 7 \end{pmatrix} \right) \qquad \mathcal{C} = \left(\begin{pmatrix} 5 \\ -2 \\ -3 \end{pmatrix}, \begin{pmatrix} 4 \\ -9 \\ 5 \end{pmatrix} \right)$$

2

$$f(x) = \begin{pmatrix} 3x_1 + 7x_2 - 5x_3 \\ -x_1 + x_2 - x_3 \\ -2x_1 - 8x_2 + 6x_3 \end{pmatrix}$$

$$[f]^{\mathcal{C}}_{\mathcal{B}} = \begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix} \in M_{2\times 3}$$

I col: calcolo $f\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$ e le sue coord. risp. a $\mathcal{C}$:

$$f\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 22 \\ 2 \\ -24 \end{pmatrix} = \frac{206}{37}\begin{pmatrix} 5 \\ -2 \\ -3 \end{pmatrix} - \frac{54}{37}\begin{pmatrix} 4 \\ -9 \\ 5 \end{pmatrix}$$

$$\Longrightarrow [f]^{\mathcal{C}}_{\mathcal{B}} = \begin{pmatrix} 206/37 & * & * \\ -54/37 & * & * \end{pmatrix}$$

$$f : \underset{2}{\{x \in \mathbb{R}^3 : 2x_1 - 5x_2 + 3x_3 = 0\}} \longrightarrow \underset{4}{\mathbb{R}^4}$$

$$f(x) = \begin{pmatrix} x_1 - 2x_2 + x_3 \\ 2x_1 + 5x_2 - x_3 \\ 5x_1 + x_2 - 11x_3 \\ 4x_1 + 9x_2 + 2x_3 \end{pmatrix}$$

$$\mathcal{B} = \left(\begin{pmatrix} 5 \\ 2 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right) \qquad \mathcal{C} = \left(\begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 3 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix} \right)$$

$$[f]^{\mathcal{C}}_{\mathcal{B}} = \begin{pmatrix} \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \end{pmatrix} \in M_{4\times 2}$$

I col: calcolo $f\begin{pmatrix}1\\1\\1\end{pmatrix}$ e trovo coord. risp. a $\mathcal{C}$:

$$f\begin{pmatrix}1\\1\\1\end{pmatrix} = \begin{pmatrix}0\\6\\-5\\15\end{pmatrix} = -12\begin{pmatrix}2\\1\\0\\0\end{pmatrix} + 5\begin{pmatrix}0\\1\\-1\\0\end{pmatrix} + 13\begin{pmatrix}0\\1\\0\\3\end{pmatrix} + 24\begin{pmatrix}1\\0\\0\\-1\end{pmatrix}$$

$$\delta = -2\alpha$$
$$\gamma = 1-\alpha$$

$$3-3\alpha+2\alpha = 15$$

$$\Longrightarrow [f]^{C}_{\beta_3} = \begin{pmatrix} * & -12 \\ * & 5 \\ * & 13 \\ * & 24 \end{pmatrix}$$