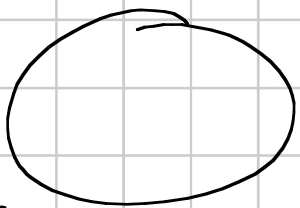
Teoria dei nodi 28/9/16

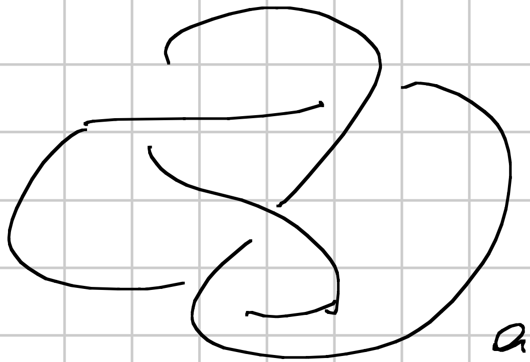
* Queste lezioni in rete (pdf + avi)
* mailto: petronio@dm.unipi.it

Vogliamo trattare

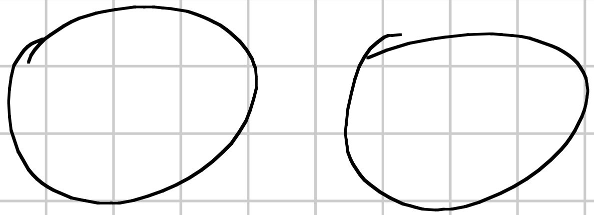


banale

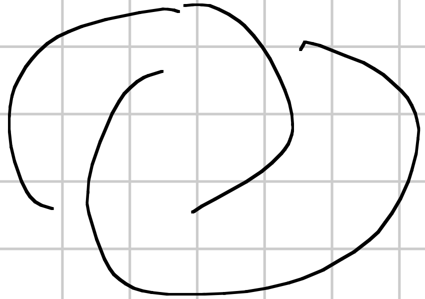
trifoglio



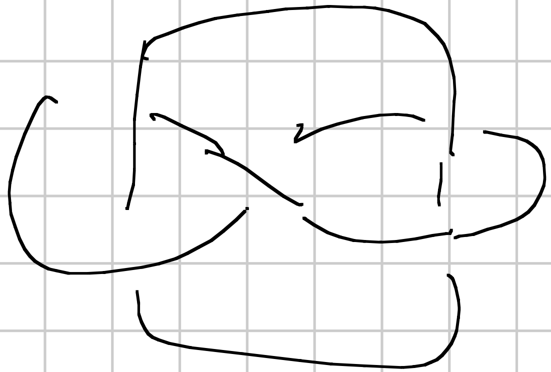
a otto



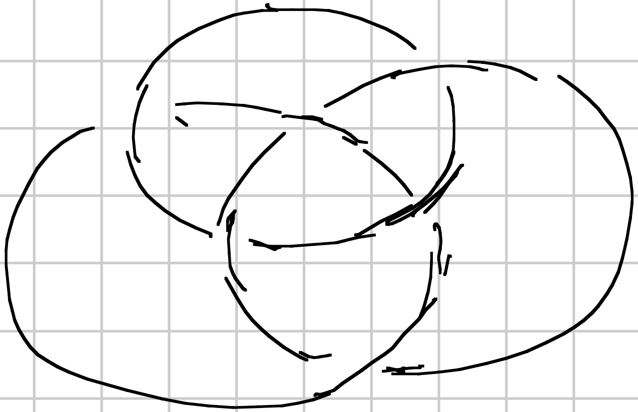
link banale



Hopf



Whitehead

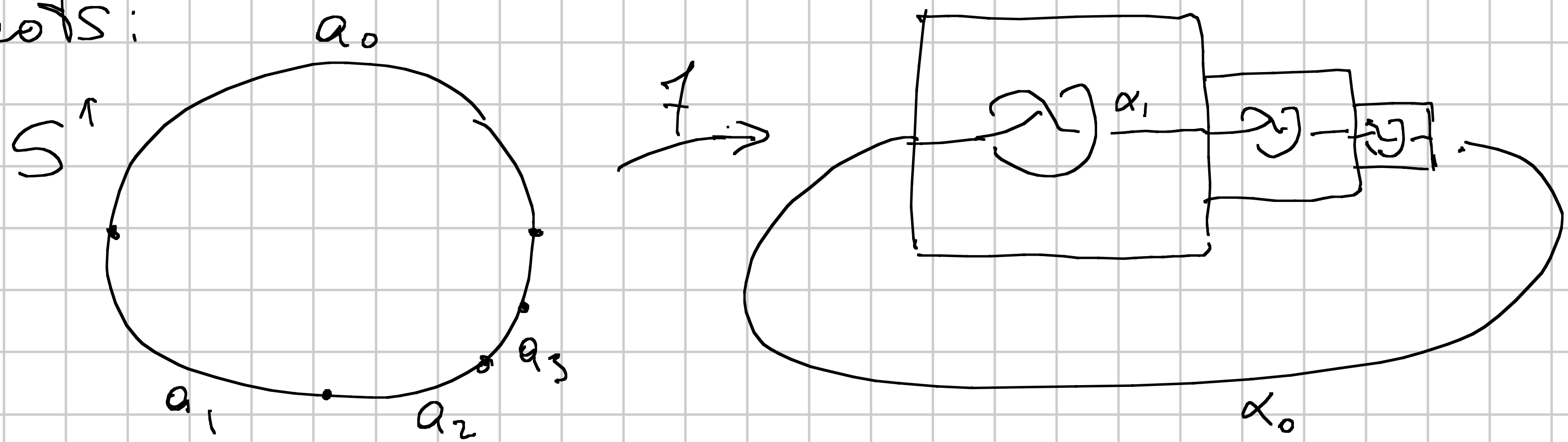


Borromeo

etc.

<u>Def</u> (sbagliata): nodo è l'immagine di un embedding di S^1 in S^3 (oppure $\mathbb{R}^3 = S^3 \setminus \{\infty\}$)

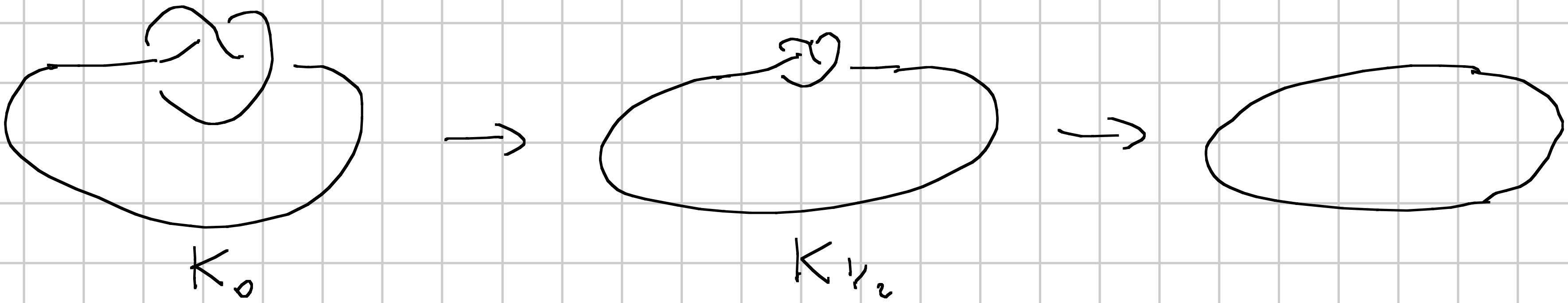
Wild knots:



f omeo sull'immagine

Def (sbagliato): dico che K_0 e K_1 sono equivalenti se esiste una famiglia continua $(K_t)_{t\in[0,1]}$, ovvero

$$S^1 \times [0,1] \longrightarrow S^3 \quad \text{t.c.} \dots$$



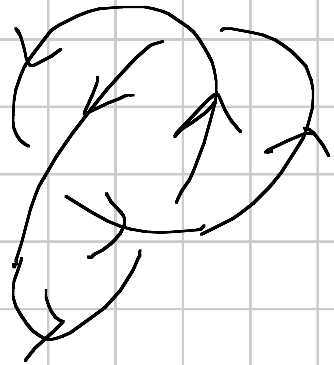
Approccio liscio: l'embedding $S^1 \hookrightarrow S^3$ e l'isotopie siano lisce con differenziale iniettivo lungo S^1

Nodo: parametrizzazione liscia / isotopie lisce (1)

contiene la riparametrizzazione tramite autoomorfismi (2) di S^1 isotopi a id

immagine di una parametrizzazione

Fatto: ci sono solo due classi di autoomorfismo di S^1 a meno di isotopie: quelli che mantengono/invertono l'orientazione

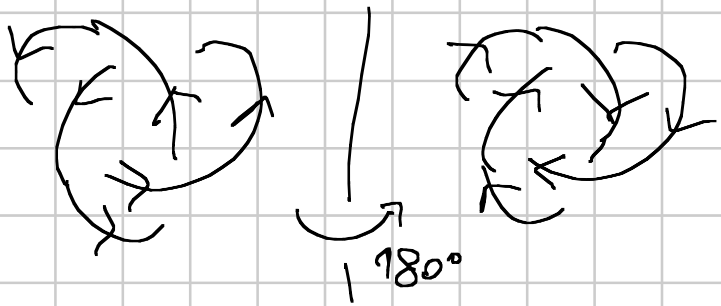


(1) $\rightsquigarrow$ nodi orientati

(2) $\rightsquigarrow$ non orientati.

Esistono nodi orientati non equiv a se stessi con l'altra orientaz?

[Sì]

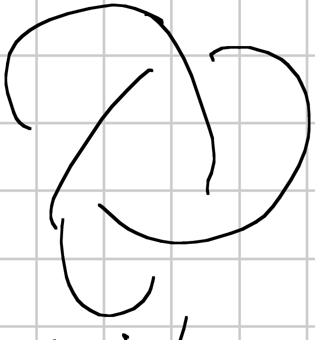


Il trif. lo è

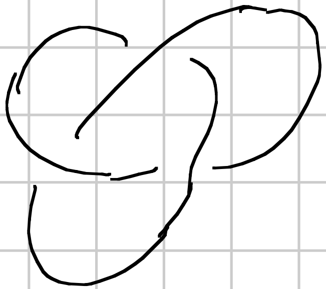
Altra variante: considerare equivalenti anche K e $\sigma(K)$ dove $\sigma : S^3 \to S^3$ inverte l'orientazione.

Esistono K t.c. K e $\sigma(K)$ non sono isotopi?

Sì:



sinistro



destro

Fatto

Non sono isotopi

<u>Normalmente</u>: si considerano equivalenti a K sia $-K$ sia $\sigma(K)$

dicendo che K è invertibile se K è isotopo a $-K$

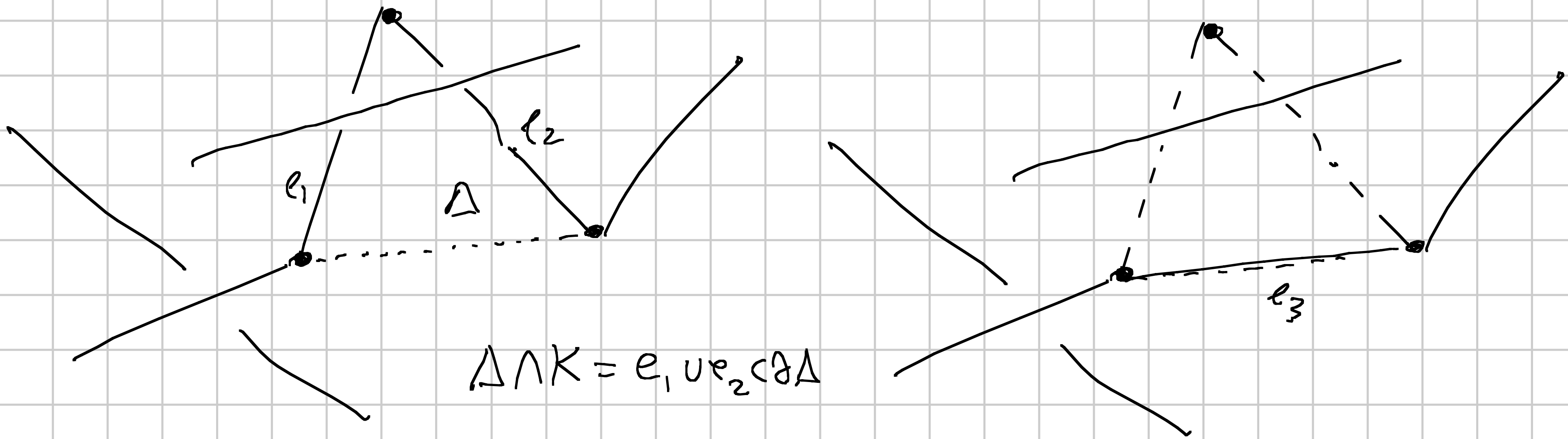
K è amphical se K è isotopo a $\sigma(K)$

(chirale se non lo è)

trifoglio chirale; nodo a otto amphichiral.

<u>APPROCCIO PL</u> Prendo $K \subset S^3$, $K \cong S^1$, K una poligonale.

Isotopie PL : generate da (suddivisione e)



$\Delta \cap K = e_1 \cup e_2 \subset \partial\Delta$

Volendo : aggiunge orientazione / dichiariamo equiv. anche K e $\sigma(K)$

I due approcci sono equivalenti

(Crowell-Fox, Cromwell).

———— o ————

Oss: sia $\mathrm{Im}(A : S^1 \times [0,1] \to S^3)$ sia un triangolo

genericamente evitano $\infty \in S^3 \Rightarrow$ teorie nodi

in $\mathbb{R}^3$ o S^3 è la stessa.

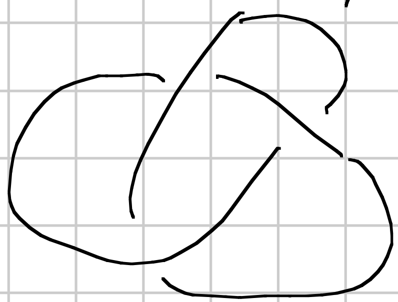
Diagrammi di nodi

Smooth $\quad \alpha : S^1 \to S^3 \quad \dot{\alpha}(z) \neq 0 \quad \forall z$

$p = \pi_{v^\perp} \circ \alpha$ a meno di piccole perturbazione di v

- $p'(z) \neq 0 \ \forall z$
- soltanto punti doppi trasversali

$\Rightarrow$ posso codificare $K = \alpha(S^1)$ tramite $p(S^1)$ con l'aggiunta del sopra/sotto agli incroci



Data $(K_t)_{t\in[0,1]}$ isotopia liscia di nodi

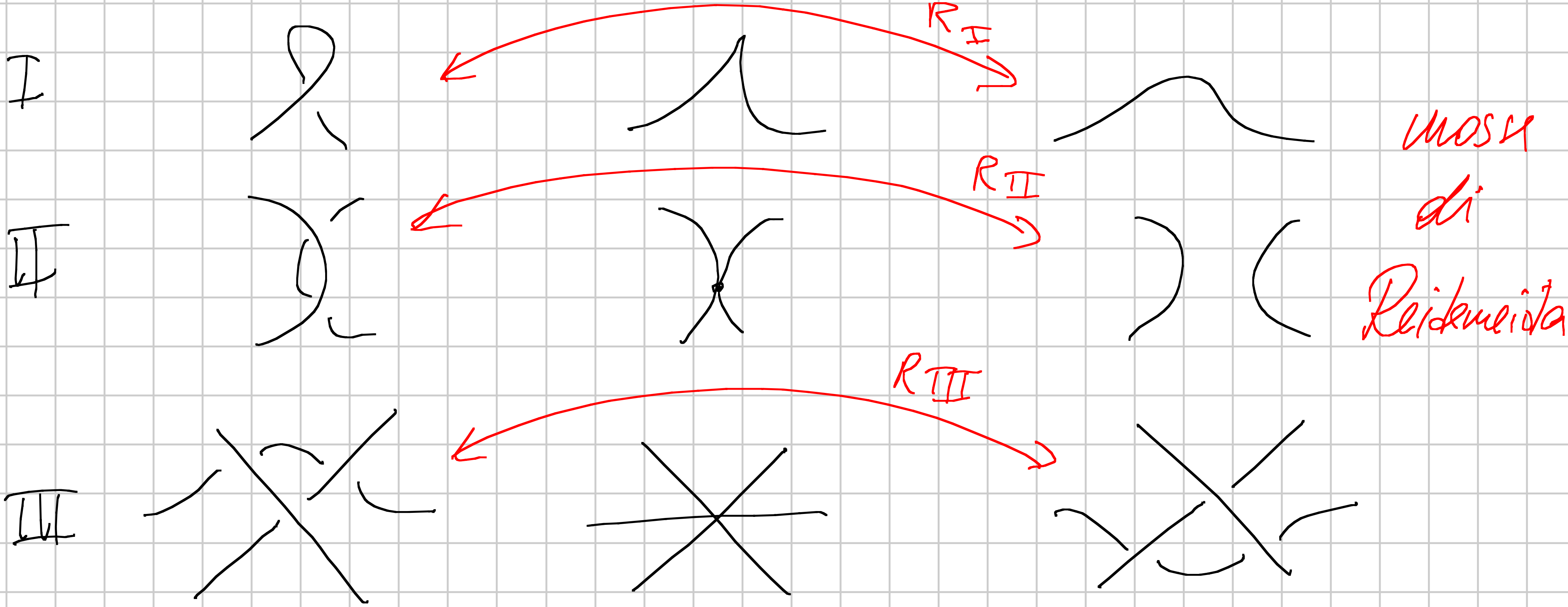
a meno perturbazione di v posso ottenere che

I • $p_t'(z)=0$ solo per z isolati, per numero finito di t
con $p_t''(z)\neq 0$

II • punti doppi non trasversali

III • punti tripli trasversali

• tutto quanto precede in numero finito e senza interazione
tra i fenomeni



escluse queste situazioni ho isotopia piana del diagramma

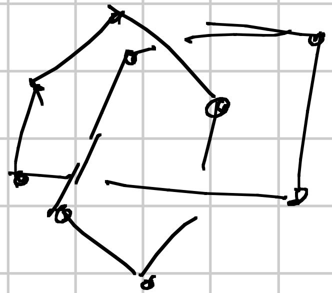
Teorema: Due diagrammi danno nodi isotopi

$\Leftrightarrow$ ottenuti uno dall'altro via isotopie piane + $R_I / R_{II} / R_{III}$

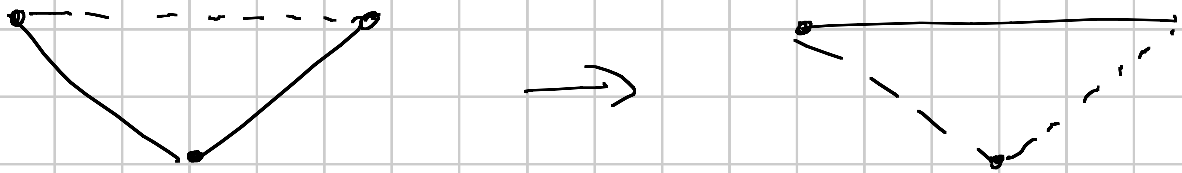
Versione PL $\quad K \subset S^3$, $K \cong S^1$ poligonale

$\pi_{v^\perp}(K)$ a meno di perturbazione di v

- i vertici vanno in punti distinti ($\Leftrightarrow$ segmenti in segmenti)
- solo pti doppi trasversali che non sono vertici

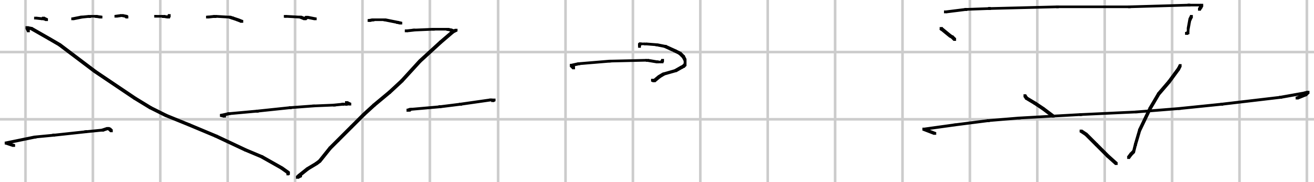


Isotopie PL: posso supporre che il triangolo lungo cui faccio le mosse si proietti e un triangolo e a meno di suddivisione che contenga ≤ 1 vertice } non
≤ 1 incroci } attraversanti



isotopie piane

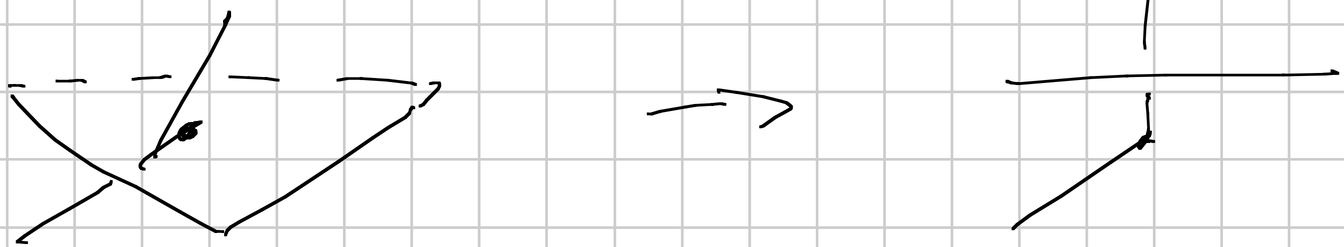
R_I



R_{II}

isotopia piana

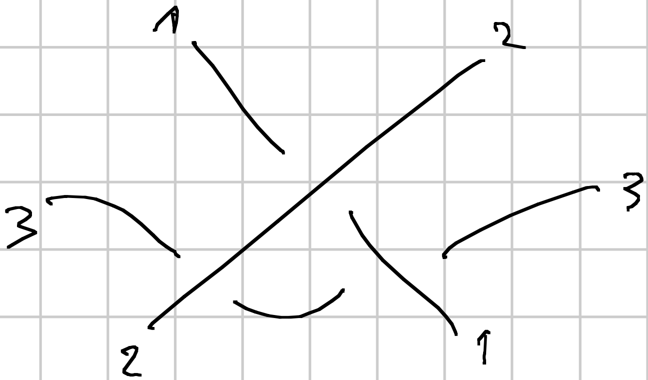
R_{VI}

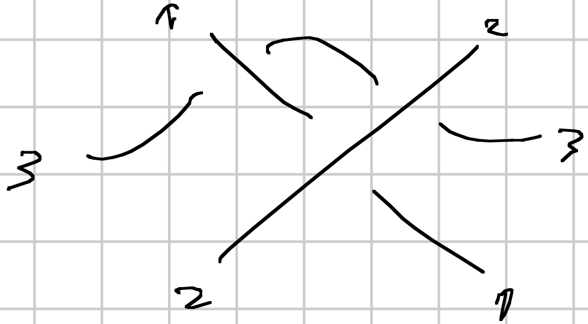


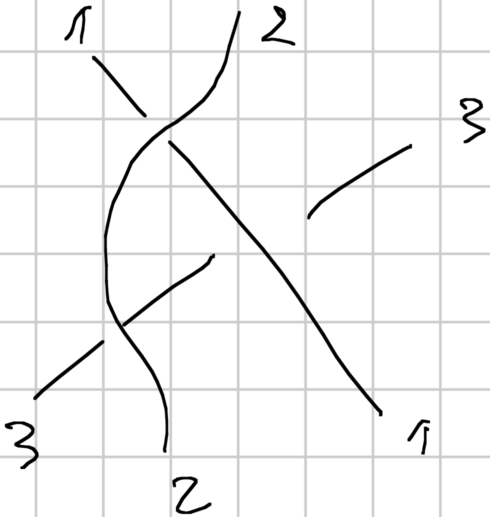
isotopie

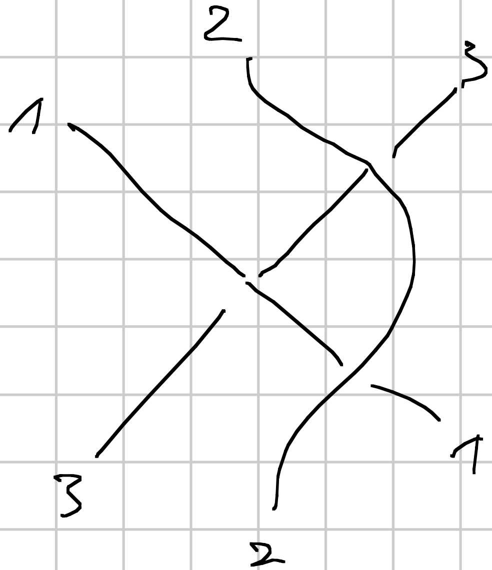
R_{III}

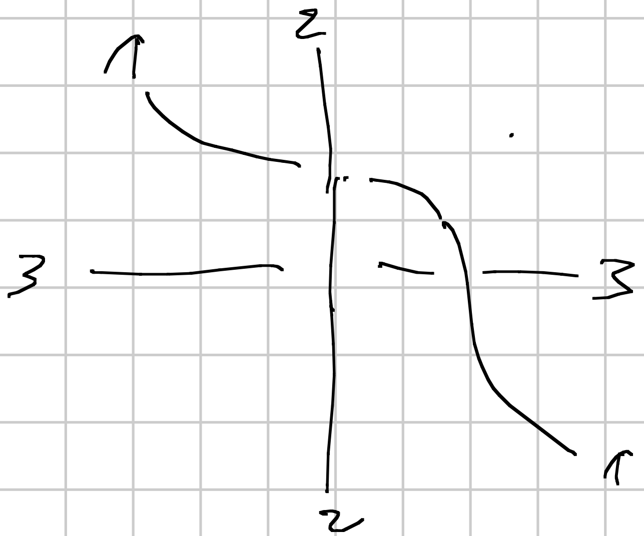
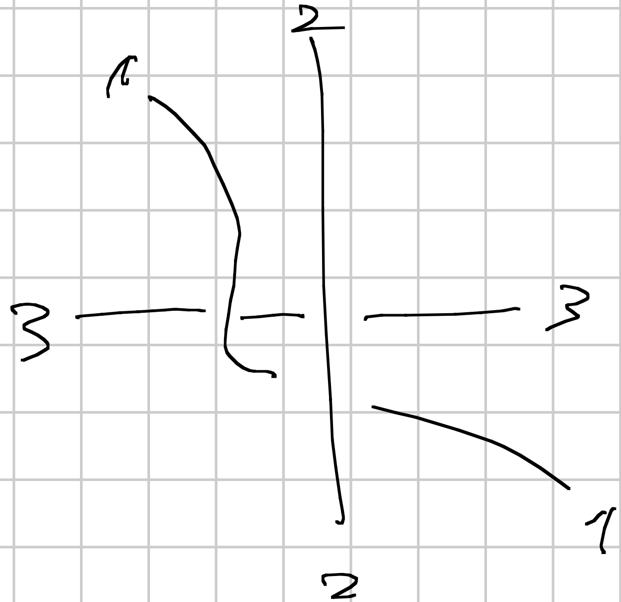
(Fatto: R_{III} è del tutto simmetrico.)


1
2
3
3
2
1


1
2
3
3
2
1

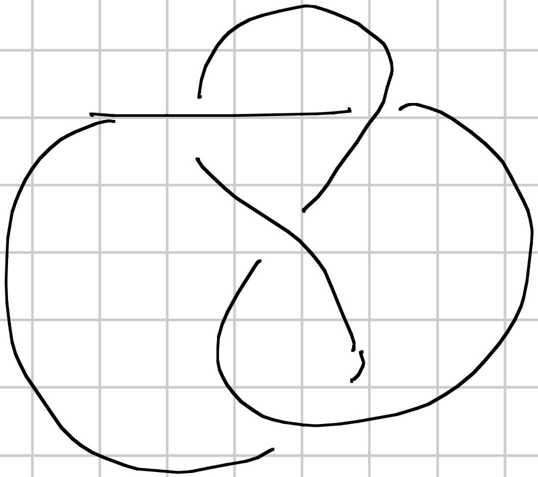

1
2
3
3
2
1

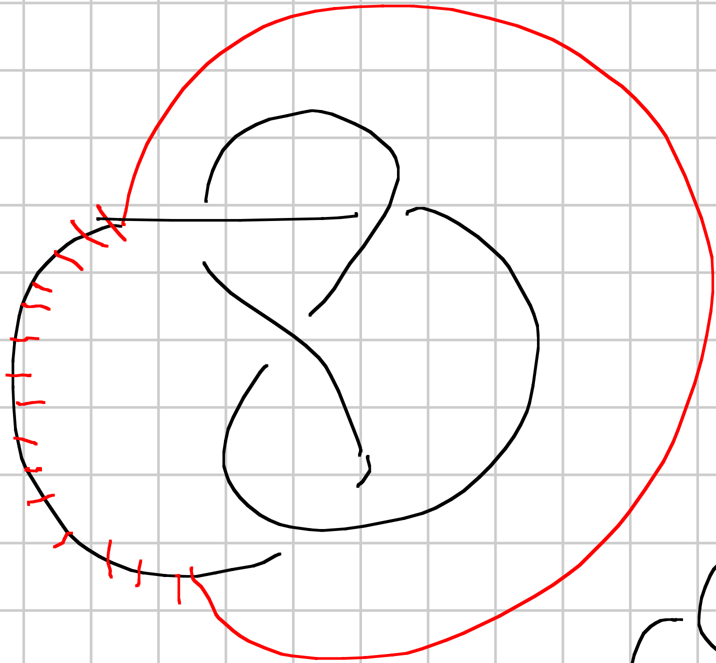

2
3
1
3
2
1

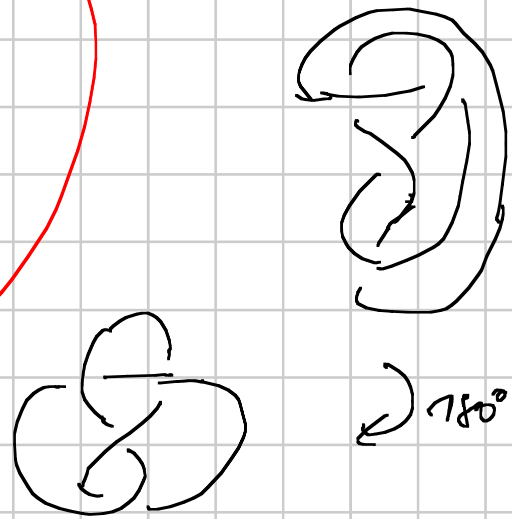

1
2
3
3
1
2

2
1
3
3
1
2

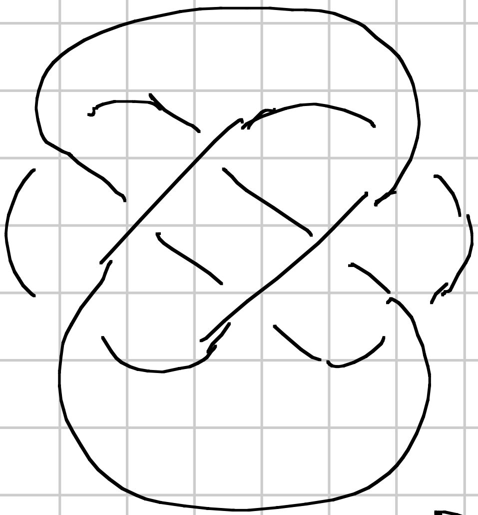
Trefoil is chiral

Fig-8 amphichiral

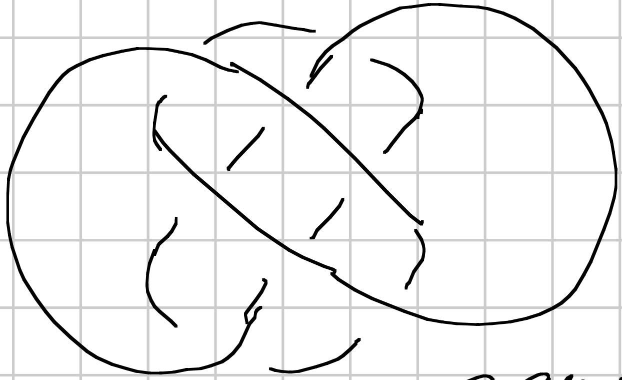






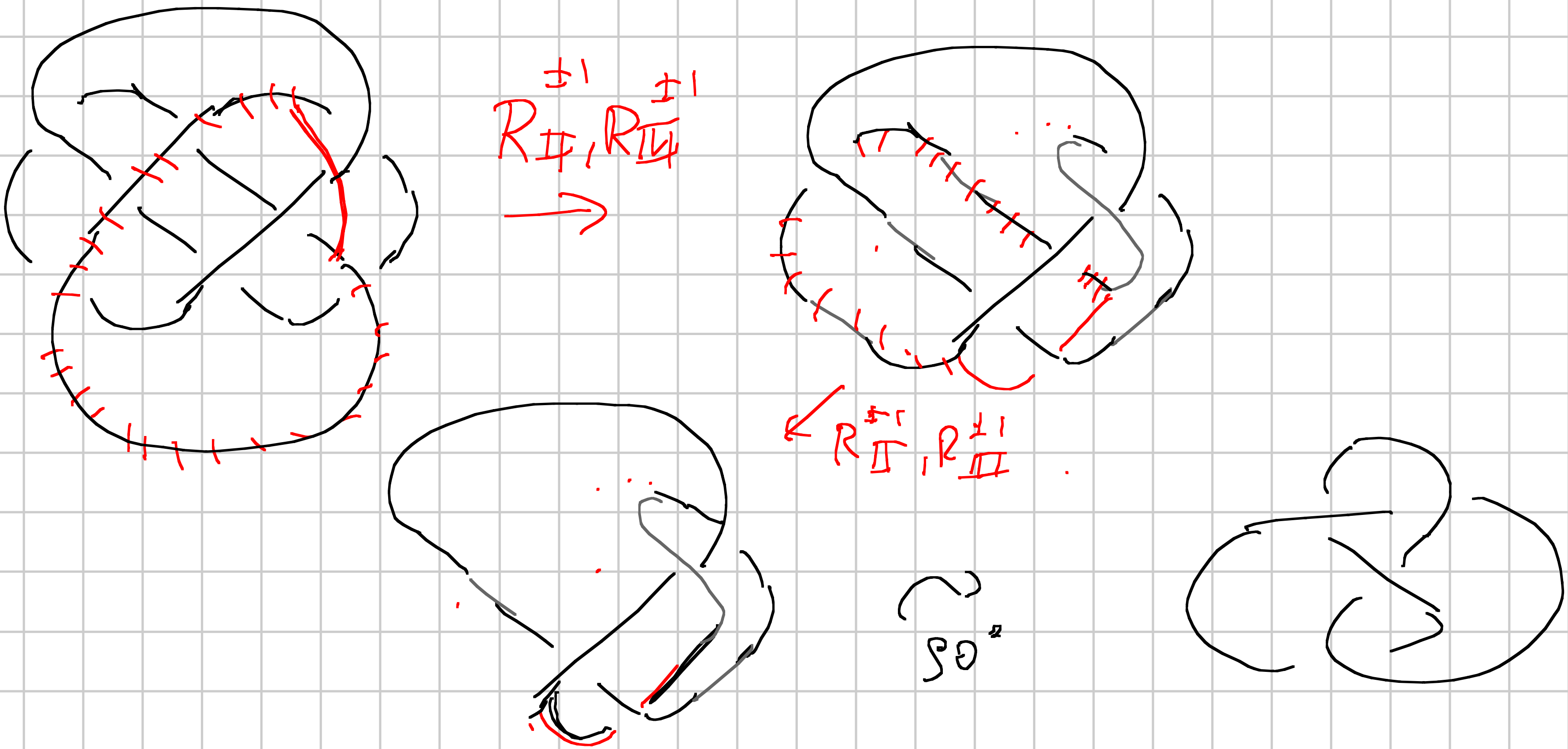


$\xrightarrow{\sigma}$



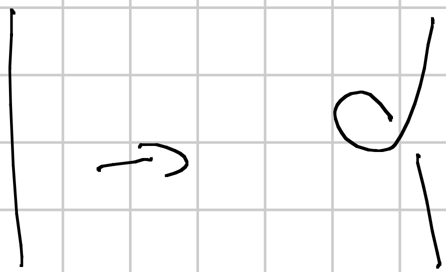
same rotated 90°

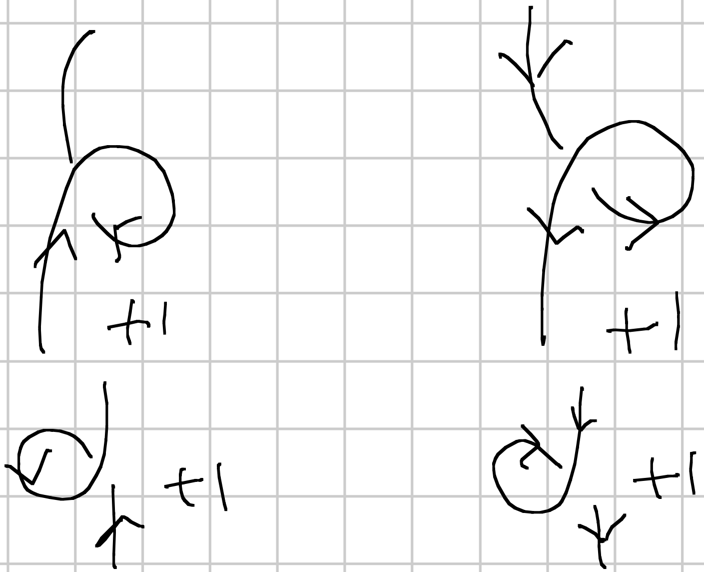
__Fact__: this is Fig-8



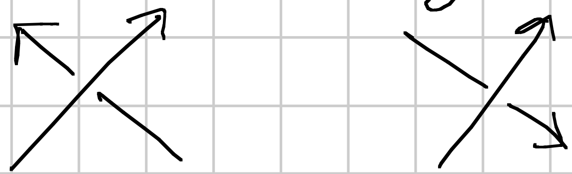
$R_{II}^{\pm 1}, R_{III}^{\pm 1}$
$R_{II}^{\pm 1}, R_{III}^{\pm 1}$
$90°$

About R_I:

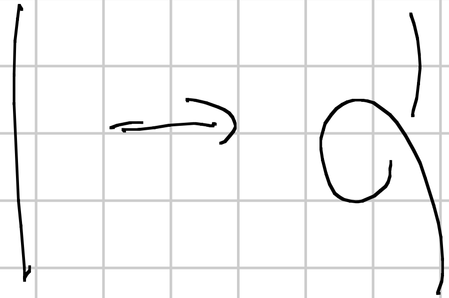
R_I^+  or 



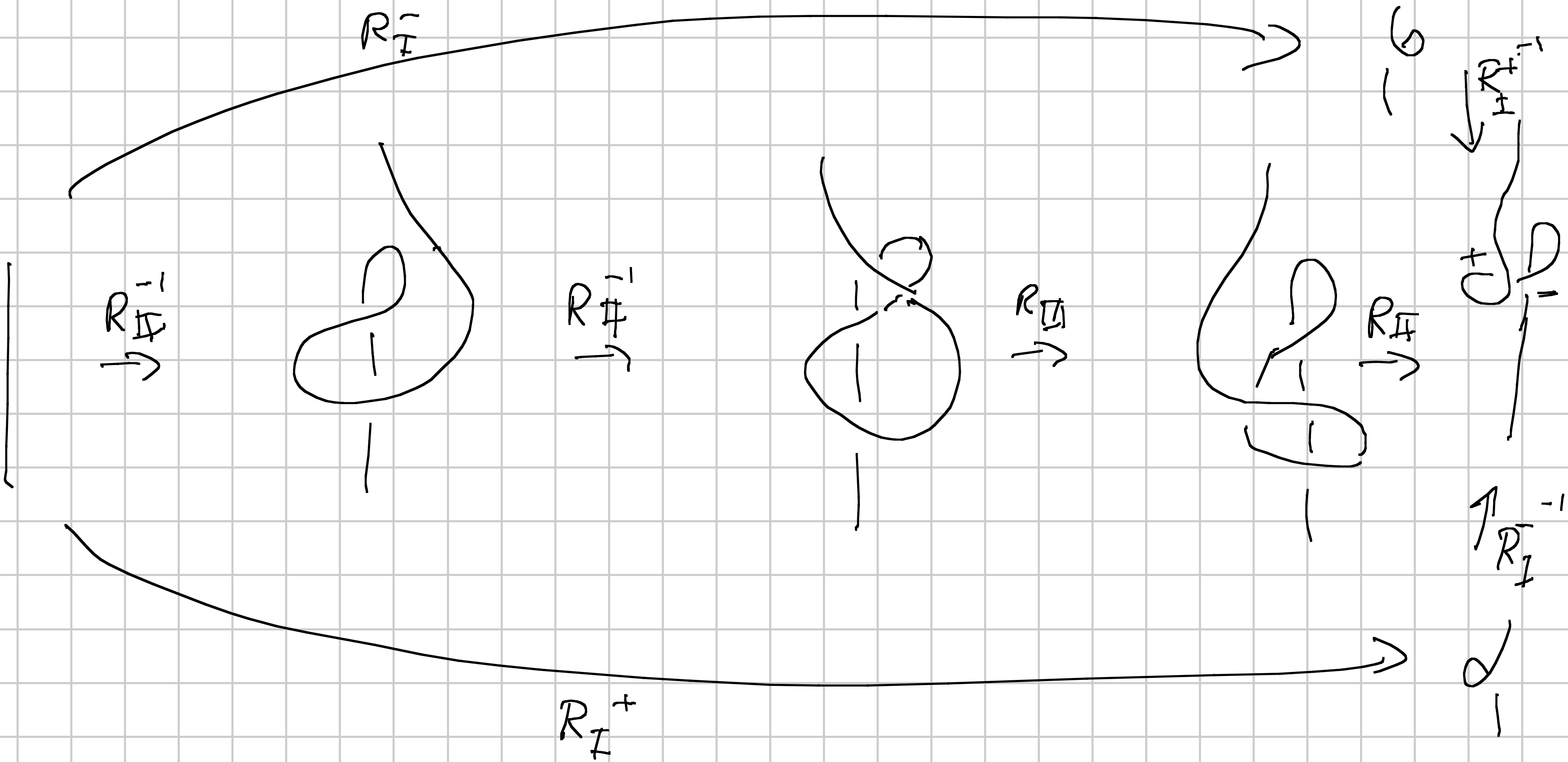
(aside sign of a crossing in oriented diagram



+1   −1)

$R_{\bar{I}}$:  on 

<u>Fact</u>: either of them is generated by R_{II}, R_{III} and the other one.



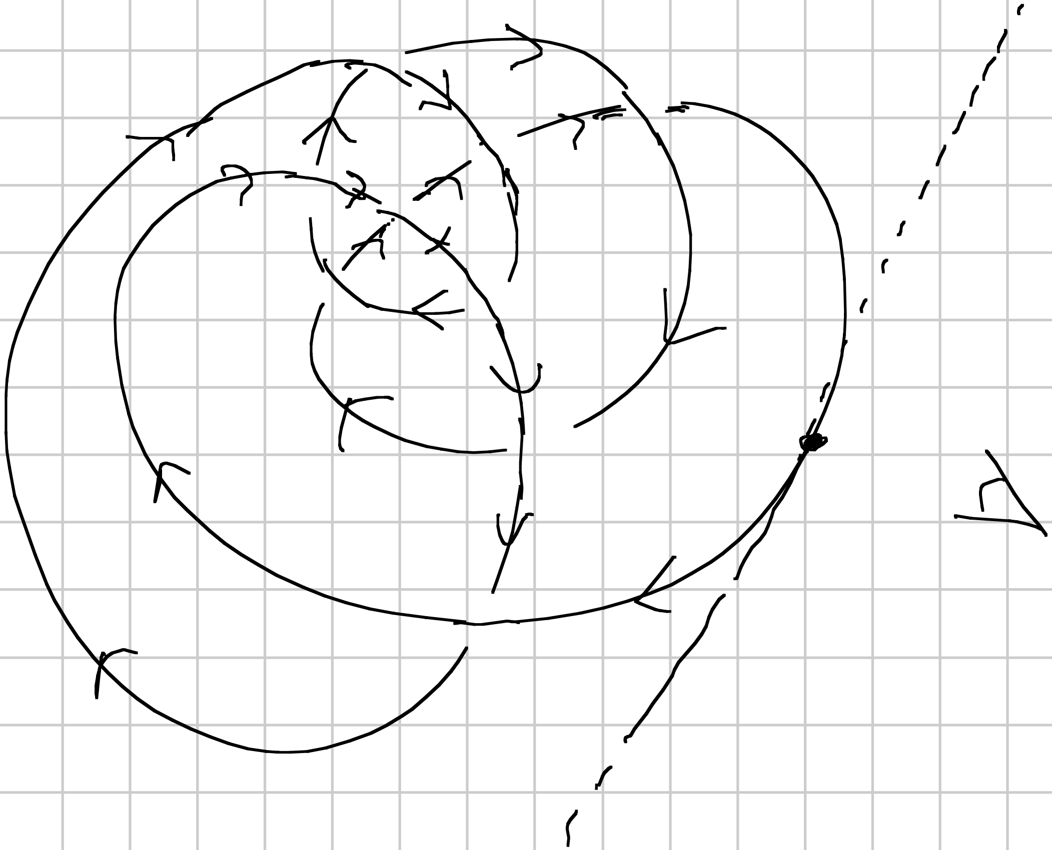
R_I^-
R_{II}^{-1}
R_{II}^{-1}
R_{II}
R_{II}
R_I^{+-1}
R_I^{--1}
R_I^+

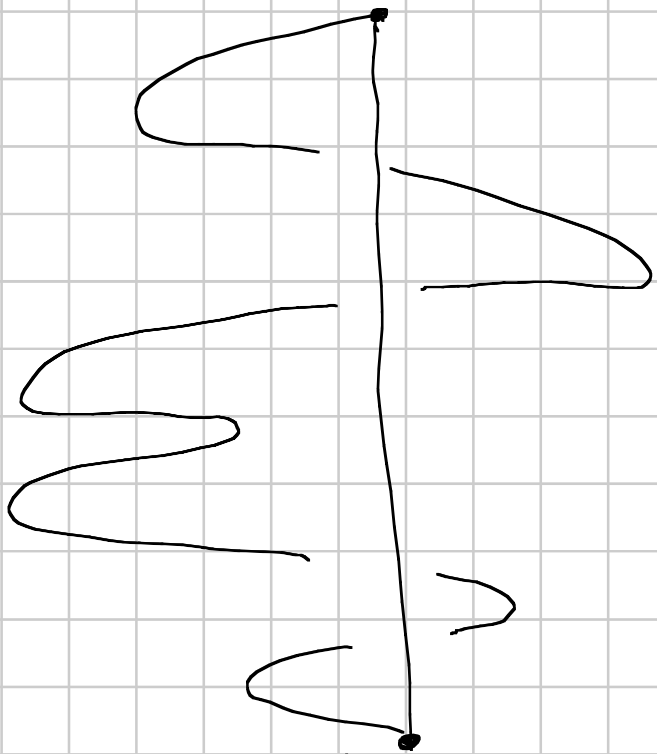
Everything so far works for links as well

Fact: up to switching crossings every diagram becomes that of an unlink:

$$\bigcirc \sqcup \dots \sqcup \bigcirc$$

For links: put components at different heights





uukuot

To show that two diagrams represent same knot

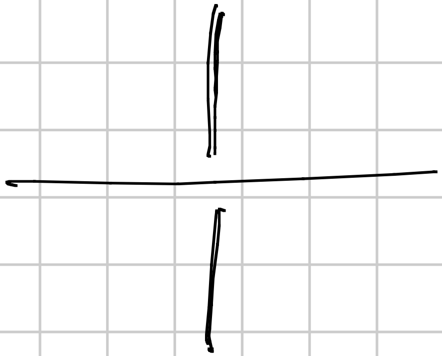
~~perhaps~~ perhaps
(oriented, adding σ) need to exhibit sequence of moves

To show they're different: need invariant
(object unchanged under moves) —

<u>3-coloring</u>

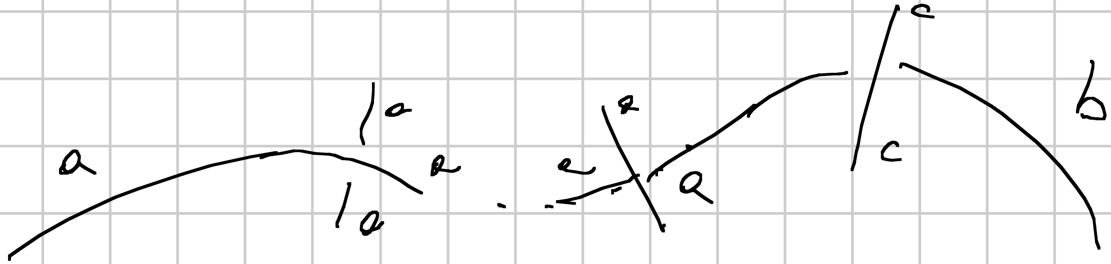
I call overarc of a diagram 

Pick $\{a,b,c\}$; a 3-coloring of a diagram is licit if at every crossing the three colors are either all the same or all different



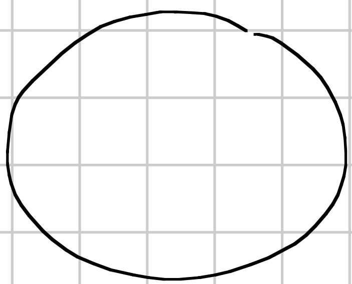
A diagram is 3-colorable if it admits a licit 3-coloring involving ≥ 2 colors.

Rem: for a knot "≥ 2" $\Leftrightarrow$ "$= 3$"

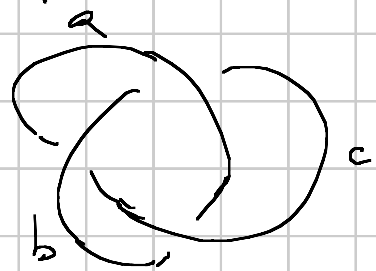


Theorem: the number of 3-colorings / $\mathfrak{S}_{\text{colors}}$ is invariant.

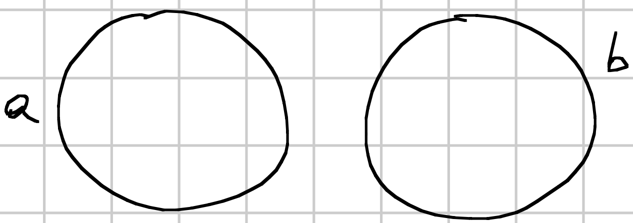
Ex:



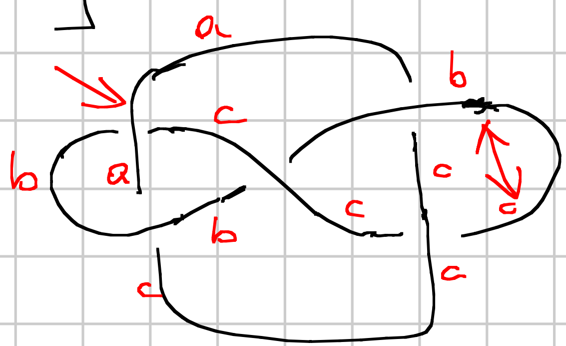
$\neq$



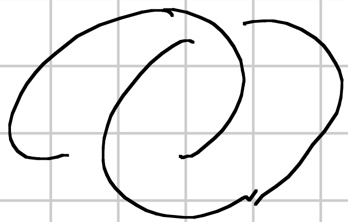
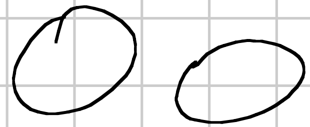
0 3-colorings   1 3-coloring



$\neq$

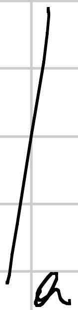


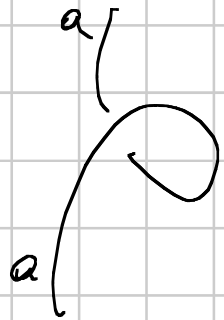
1 3-coloring   0 3-colorings

(easier to see that  is neither  nor Whitehead)

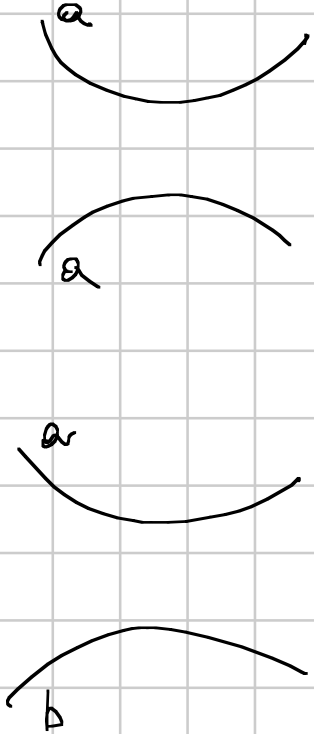
<u>Rem</u>: $\#\{3\text{-colorings}\}$ unchanged under σ (exercise)

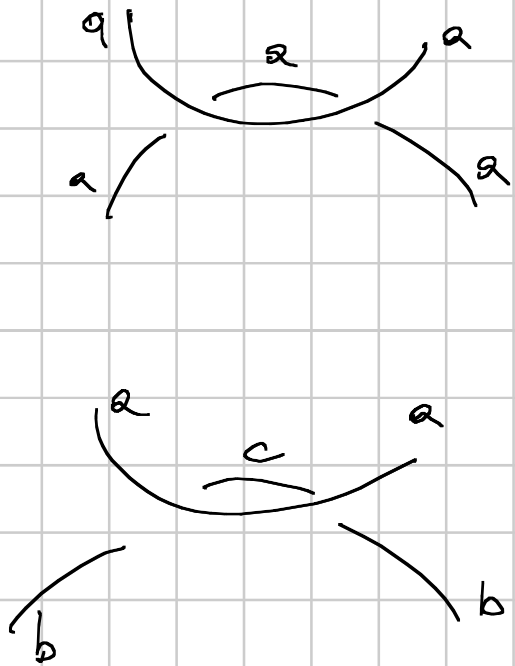
<u>Proof</u>: create a correspondence between $\{$3-colorings before a move$\} \leftrightarrow \{$other$\}$

R_I 

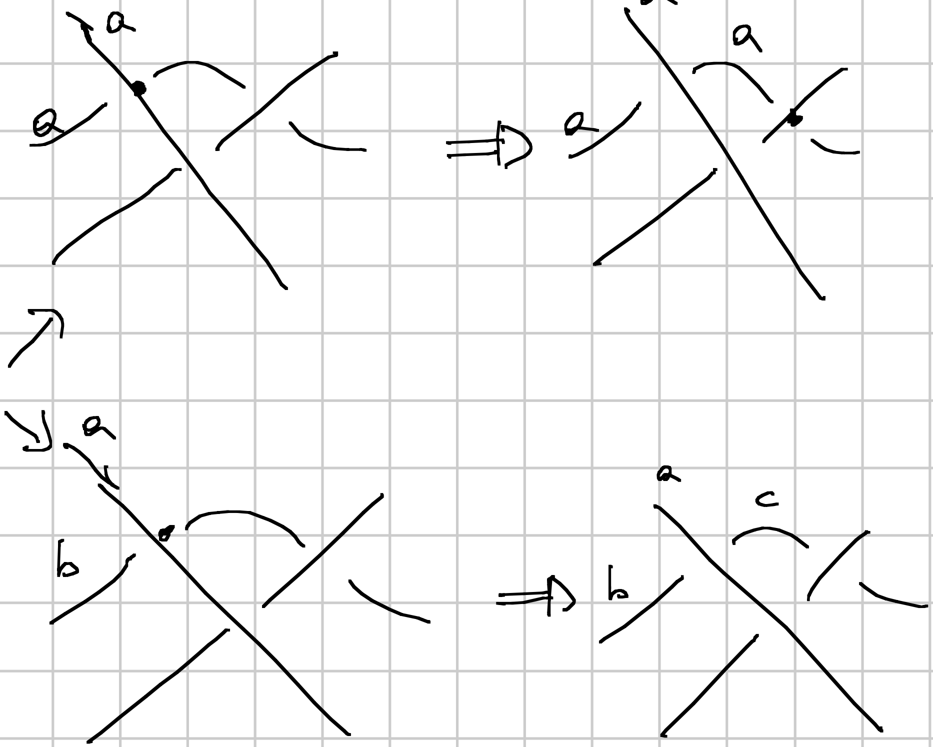
$\longleftrightarrow$ 

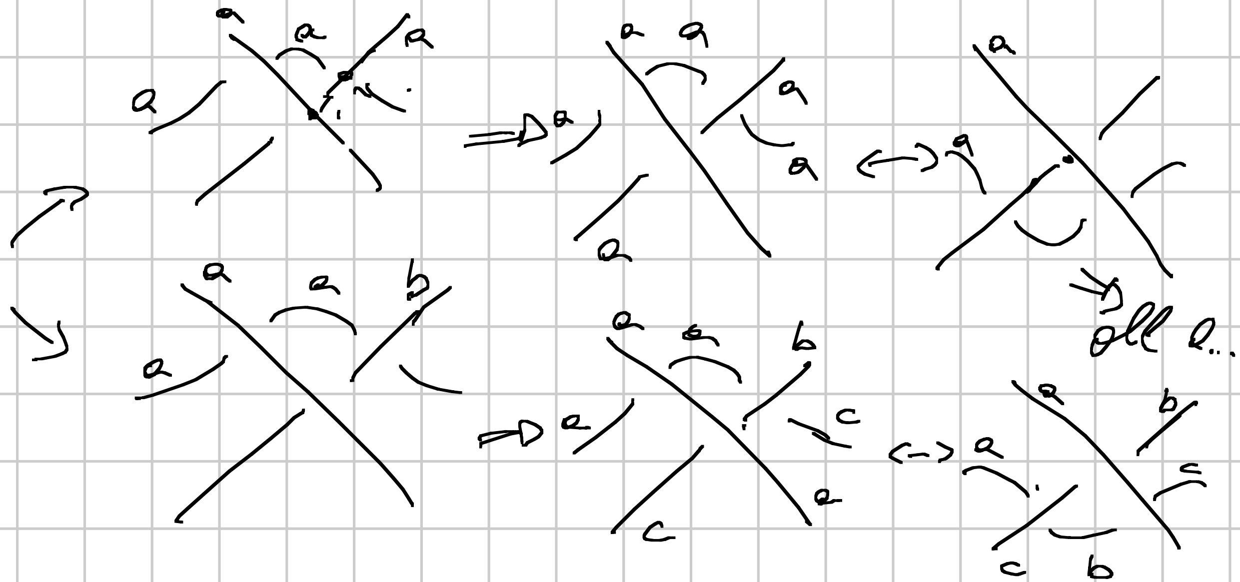
RII





R_{II}





(similar — exercise)

$\blacksquare$