Teoria dei nodi 14/12/16

<u>Quandle</u>: Γ insieme con operaz. binarie $\circ$ e $/$ t.c.

(1) $a \circ a = a$

(2) $(a \circ b) / b = a \qquad (a / b) \circ b = a$

(3) $(a \circ b) \circ c = (a \circ c) \circ (b \circ c)$

Conseguenze: (1') $a / a = a$

(4) $(a \circ b) / c = (a / c) \circ (b / c)$

(5) $(a / b) \circ c = (a \circ c) / (b \circ c)$

(6) $(a / b) / c = (a / c) / (b / c)$

<u>Es</u>: Γ gruppo $\quad a \circ b = bab^{-1} \quad a /b = b^{-1}ab$ è un quandle.

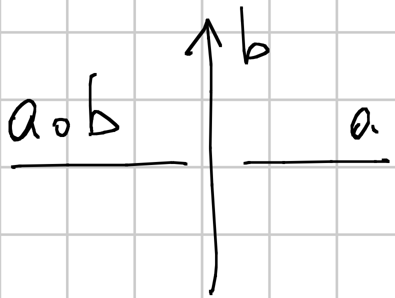
(1), (2) facili

(3) $(a \circ b) \circ c = c(bab^{-1})c^{-1}$

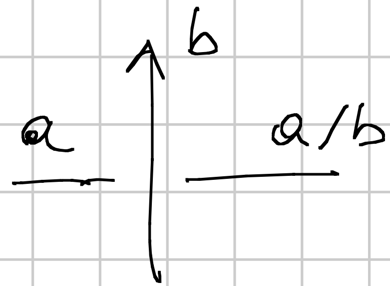
$$(a \circ c) \circ (b \circ c) = (cbc^{-1}) \cdot (cac^{-1}) \cdot (cbc^{-1})^{-1} \quad \checkmark$$

<u>Oss</u>: sufficiente dare $\circ$ con $a \circ a = a$, $(a \circ b) \circ c = (a \circ c) \circ (b \circ c)$ e chiedere che $x \circ b = a$ abbia unica soluz. $\forall a, b$.

Def: dato Γ quandle chiamo Γ-colorazione di un diagramma $\vec{D}$ una colorazione in Γ degli overarc t.c.



ovvero



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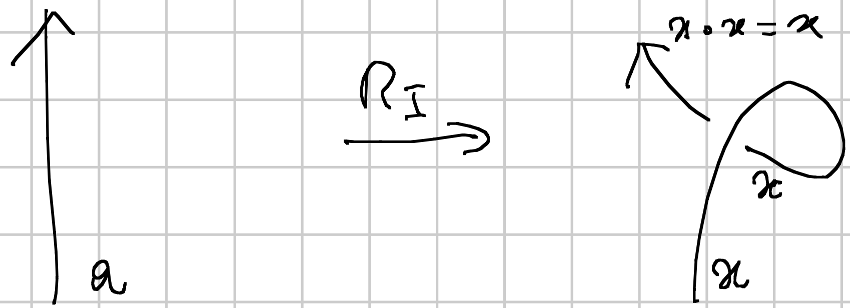
<u>Teo</u>: se D e D' sono collegate da $R_{I/II/III}$

allora $\{\Gamma\text{-coloraz. di } D\} \longleftrightarrow \{\Gamma\text{-coloraz. di } D'\}$

<u>Cor</u>: $\#\Gamma < +\infty \Rightarrow \#\{\Gamma\text{-coloraz.}\}$ è invariante

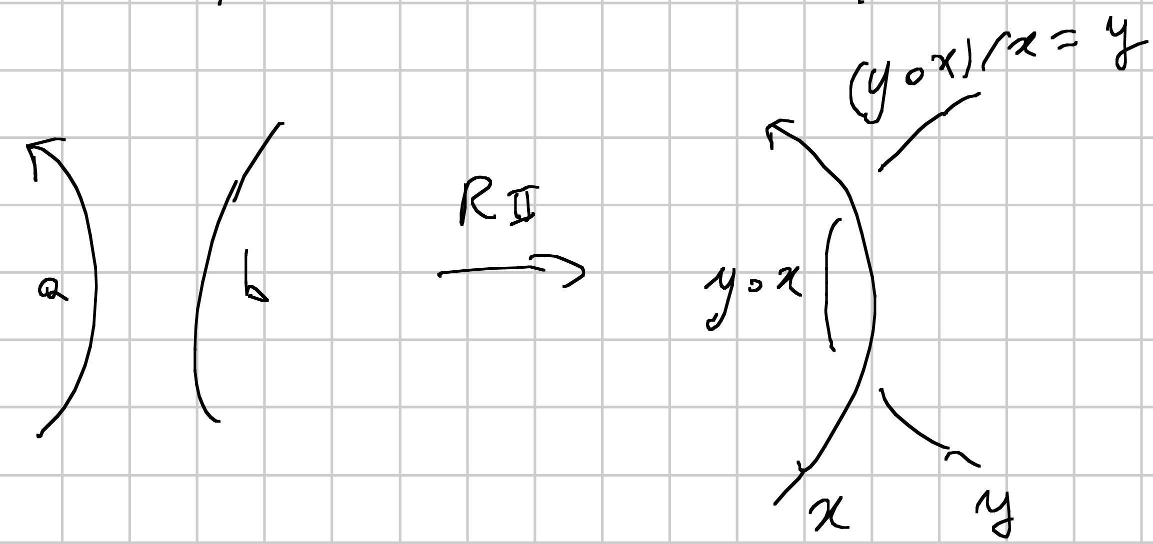
<u>Dim</u> (teo): faccio un esempio per ogni R

R_I :



$a \leftrightarrow x$

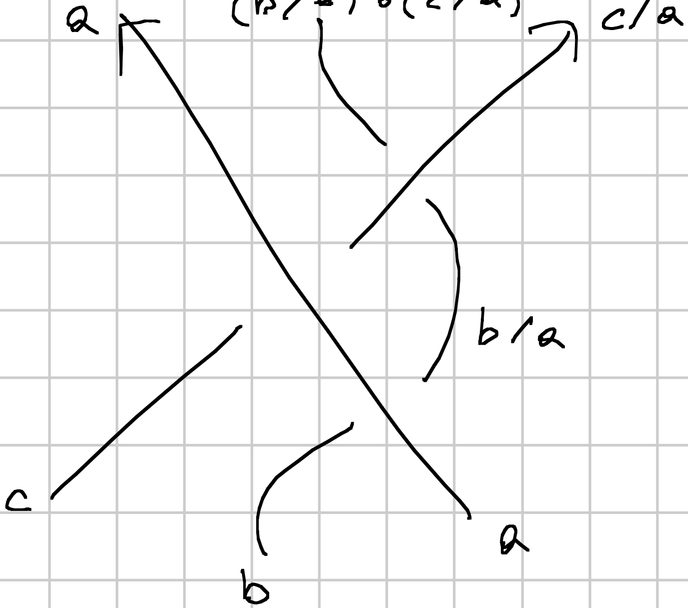
R_{II} :

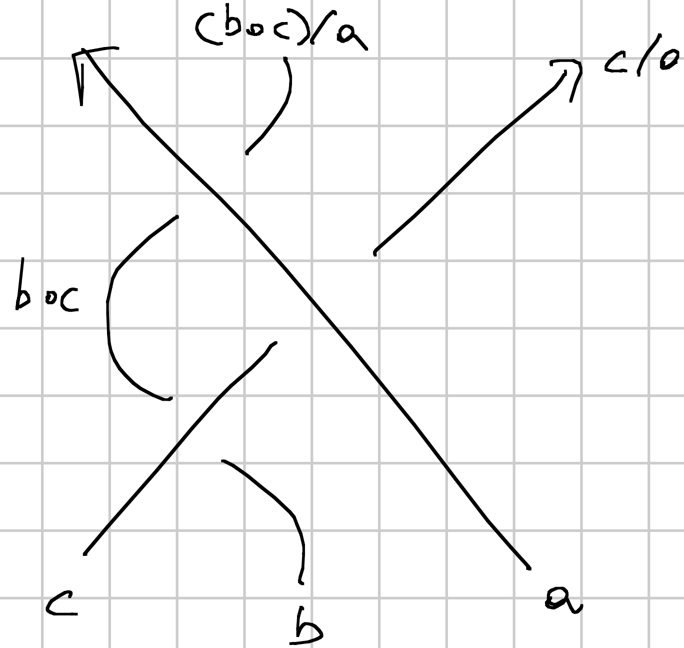


$a \leftrightarrow x$

$b \leftrightarrow y$

$R_{\square}$:

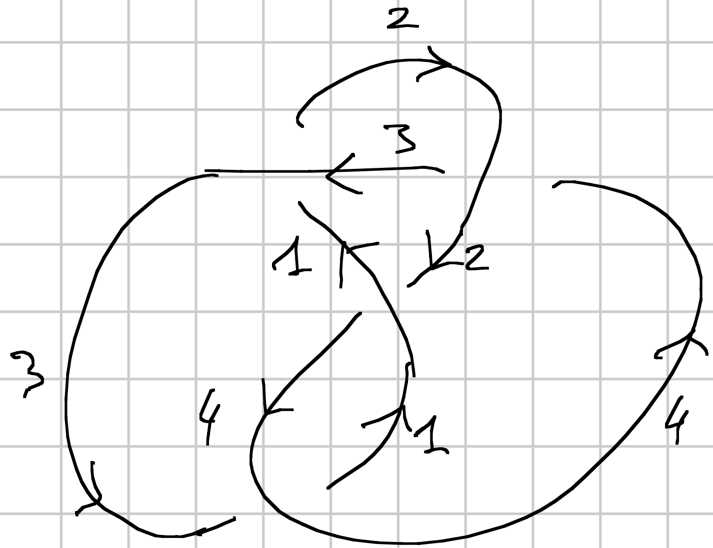




∎

Es:

$\circ$	1	2	3	4
1	1	3	4	2
2	4	2	1	3
3	2	4	3	1
4	3	1	2	4



$3 \circ 4 = 1$

Quandle universale di L

$\bar{U}(L)$ = intorno

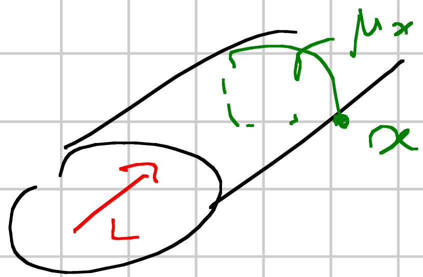
$E(L) = S^3 \setminus \mathring{U}(L)$.

$x_0 \in E(L)$

$$\Pi_L^{(x_0)} = \{\text{cammini in } E(L) \text{ da } x_0 \text{ a } \partial U(L)\} \big/ \simeq$$

$$[\alpha] \cdot [\beta] = [\beta \cdot \mu_{\beta(1)} \cdot \beta^{-1} \cdot \alpha]$$

dove μ_x = meridiano negativo di $U(L)$ basato in x



Prop: è quandle.

Dim: (1) $a \circ a = [\alpha \cdot \mu_{\alpha(1)} \cdot \alpha^{-1} \cdot \alpha]$ $\qquad a = [\alpha]$

$= [\alpha \cdot \mu_{\alpha(1)}] = [\alpha] = a$

(2) $x \circ a = b \iff [\alpha \cdot \mu_{\alpha(1)} \cdot \alpha^{-1} \cdot \xi] = [\beta]$

$\iff [\xi] = [\beta^{-1} \cdot \alpha \cdot \mu_{\alpha(1)} \cdot \alpha^{-1}]$

(3) $(a \circ b) \circ c = [\gamma \cdot \mu_{\gamma(1)} \cdot \gamma^{-1} \cdot \beta \cdot \mu_{\beta(1)} \cdot \beta^{-1} \cdot \alpha]$

$(a \circ c) \circ (b \circ c) = \ldots$

$\boxtimes$

Indipendenza da x_0:

$$[\alpha] \longmapsto [u \cdot \alpha]$$

$$\Gamma^{(y_0)} \longrightarrow \Gamma^{(x_0)} \quad \text{isomorfismo}$$

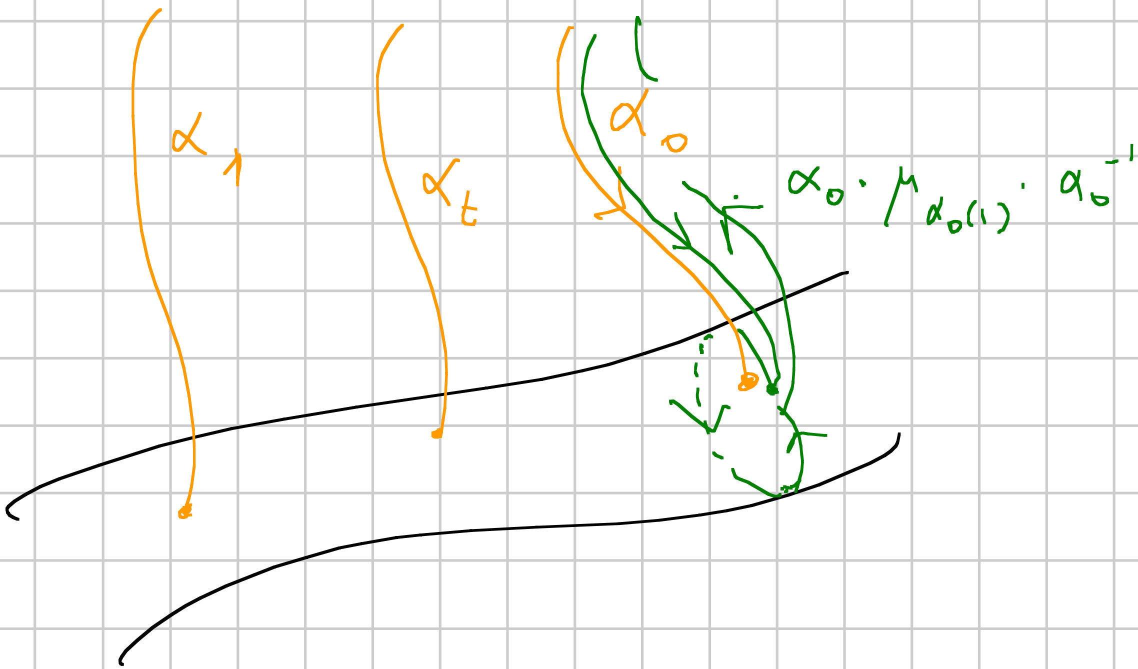
<u>Prop</u>: $\psi : \Gamma_L \to \pi_1(S^3 \setminus L)$

$$[\alpha] \mapsto [\alpha \cdot \mu_{\alpha(1)} \cdot \alpha^{-1}]$$

ben def. surj e $\psi(a \circ b) = \psi(b) \cdot \psi(a) \cdot \psi(b)^{-1}$

(ψ anche bigettiva $\Longrightarrow$ I_L è il quandle associato a $\pi_1(S^3 \setminus L)$)

<u>Dim</u>: ben def



Surgettiva: immagine contiene generatori di Wirtinger ...

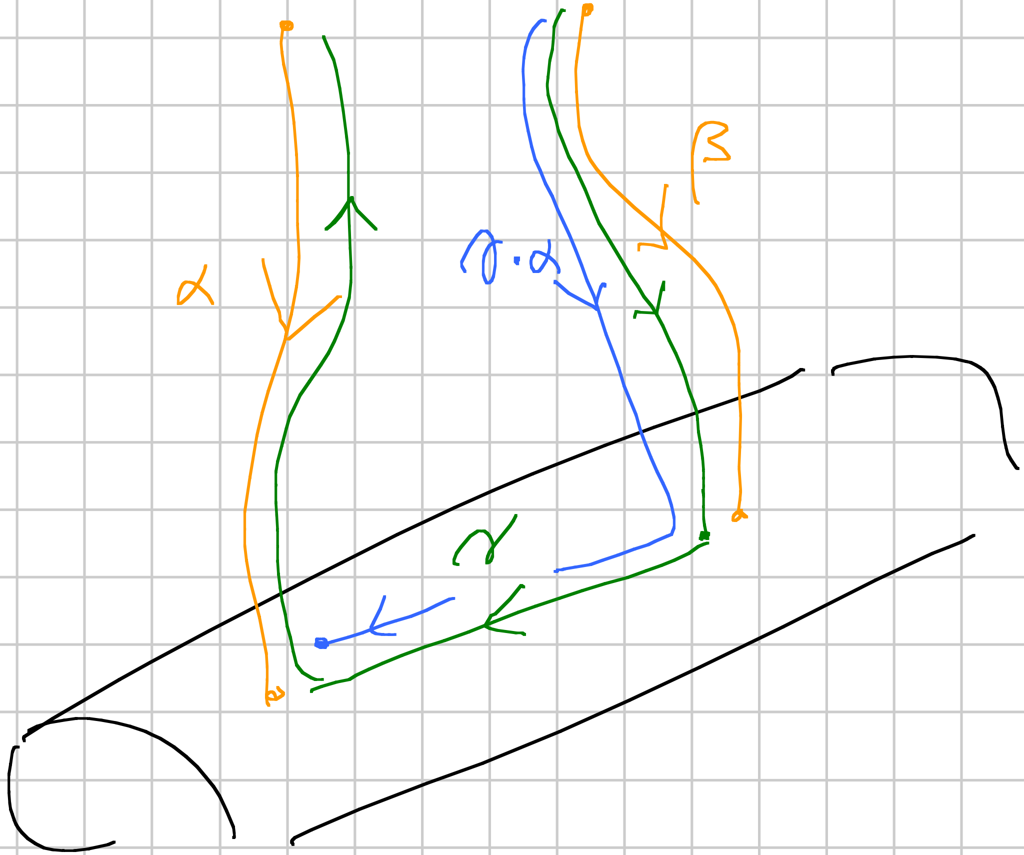
$$[\psi(a \circ b)] = \psi([\beta \cdot \mu_{\beta(1)} \cdot \beta^{-1} \cdot \alpha])$$

$$= [\underbrace{\beta \cdot \mu_{\beta(1)} \cdot \beta^{-1}}_{\psi(b)} \cdot \underbrace{\alpha \cdot \mu_{\alpha(1)} \cdot \alpha^{-1}}_{\psi(a)} \cdot \underbrace{\beta \cdot \mu_{\beta(1)}^{-1} \beta^{-1}}_{\psi(b)^{-1}}] .$$

<u>Fatto</u>: c'è un'azione transitiva senza pti fissi di $\pi_1(S^3 \setminus L)$ su T_L data

$$[\gamma] \cdot [\alpha] = [\gamma \cdot \alpha] .$$

Transitive:



Versione combinatoria di Π_L : fissando D diag. di L

$\overline{W}_D =$ parole nelle lettere
- overarc di D
- $($, $)$, $\circ$, $/$

generate da:

$a \in \overline{W}_D \quad \forall a$ overarc

$u, v \in \overline{W}_D \Rightarrow (u)\circ(v)\,,\ (u)/(v) \in \overline{W}_D$

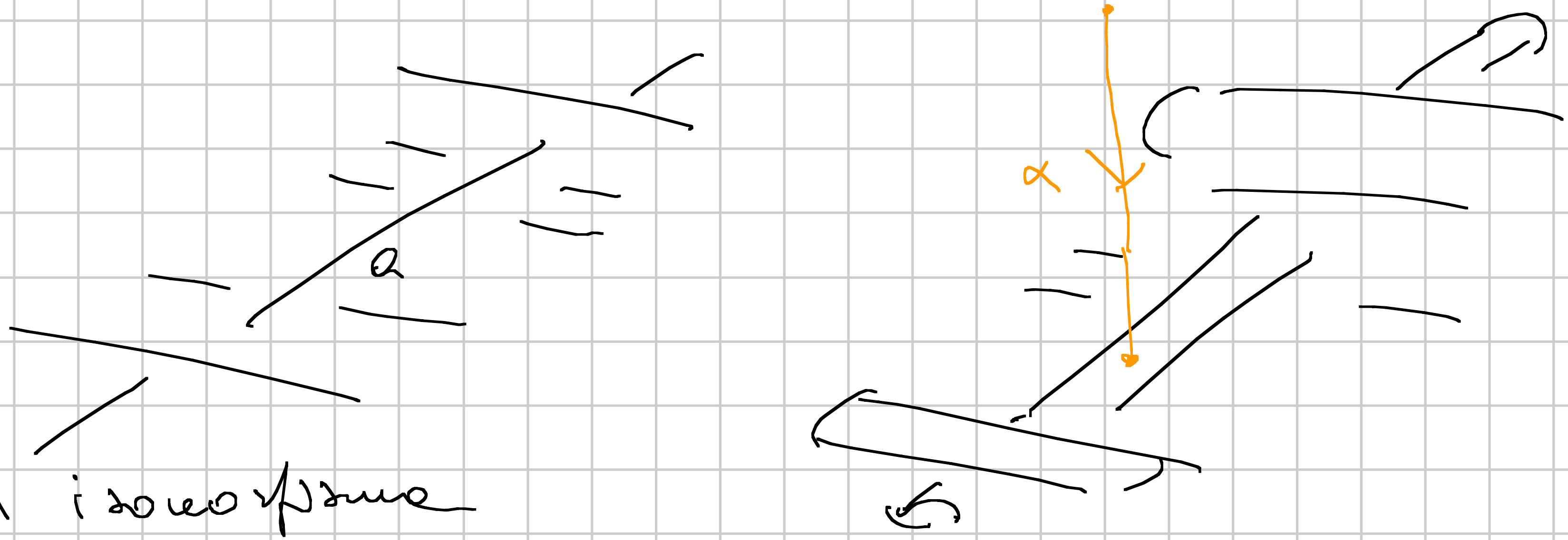
$$\Pi_D = \frac{W_D}{\begin{array}{c} w \circ w \sim w \\ ((u)\circ(w))/(w) \sim u \\ ((u)/(w))\circ(w) \sim u \end{array}}$$

$\Rightarrow \quad u \circ v \sim w$

Fact: con $[u]\circ[v] = [(u)\circ(v)]$ è quandle.

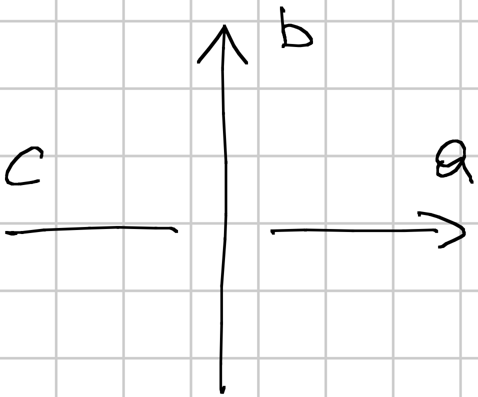
Teo: $\phi : \Gamma_D \longrightarrow \Gamma_L$

$[a] \longmapsto$ cammino che da ∞ scende dritto e un pto di $\partial U(L)$ sopra a



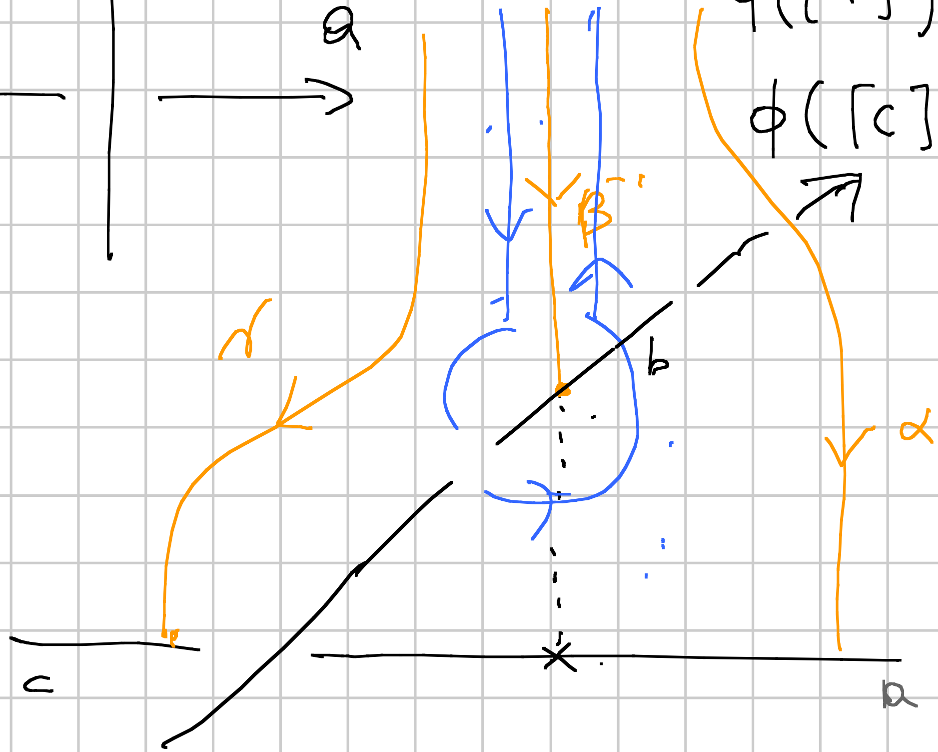
è un isomorfismo

"Dim" Provo che è rispettata la relazione dell'incrocio.



$$\phi([a]) \circ \phi([b]) = [\alpha] \circ [\beta] = [\beta \cdot \mu_{\beta(1)} \cdot \beta^{-1} \cdot \alpha]$$

$$\phi([c]) = [\gamma]$$



Inverso: $\Theta : \Gamma_L \to \Gamma_D$

$\Theta([u])$ $\qquad u : [0,1] \to E(L)$ $\qquad 0 \mapsto \infty$, $1 \mapsto \partial U(L)$

poniamo u in posizione generica rispetto a D

supponiamo che passi sotto overarc $a_n, \dots, a_1$ con segni $\varepsilon_n, \dots, \varepsilon_1$ e finisce in a_0

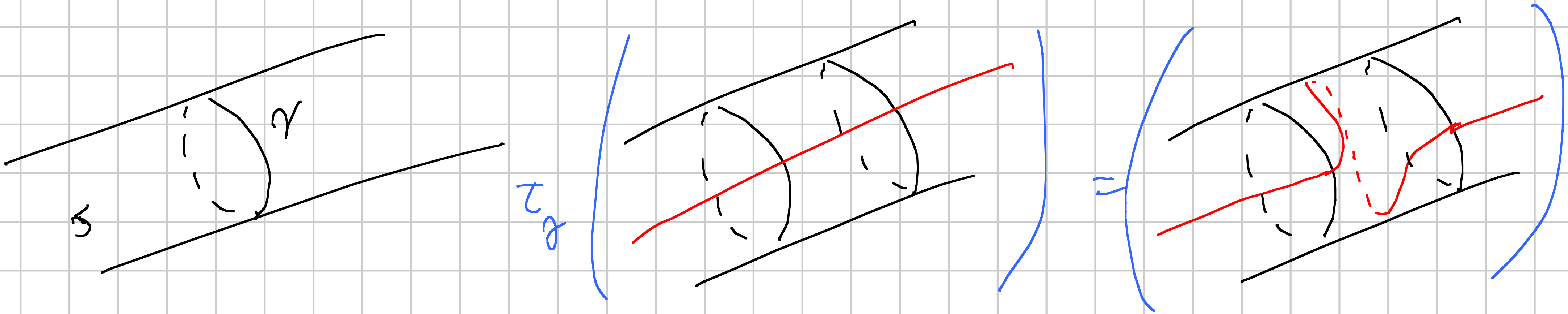
$$\Theta([u]) = \left[\quad \cdots \quad ((a_0 \tilde{\varepsilon}_1 a_1) \tilde{\varepsilon}_2 a_2) \tilde{\varepsilon}_3 a_3 \quad \cdots \right]$$

$$\tilde{\varepsilon}_j = \begin{cases} 0 & \varepsilon_j = + \\ 1 & \varepsilon_j = - \end{cases}$$

Verificare che è ben def. (richiede nozione tipo Reidemeister di α wrt a D). ☐

Presentazione per diffeomorfismi

Def: se Σ è sup. orientata e $\gamma \subset \mathring{\Sigma}$ è curva semplice chiamo T_γ (twist di Dehn) $\in \mathrm{Aut}(\Sigma)$ supportato in $U(\gamma)$ che agisce così:

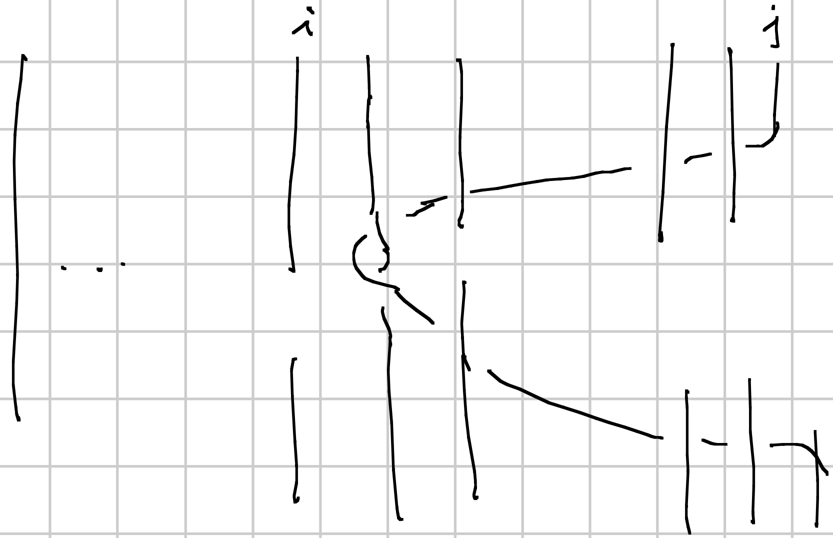


Teo: Σ sup. cpt orientata (anche con ∂)

$\Rightarrow \mathrm{Aut}^+_\partial(\Sigma) = \{ f \in \mathrm{Aut}(\Sigma),\ f|_{\partial\Sigma} = id \} \big/ \text{isotopie}$

è generato da Dehn twist.

$\tau_{i,j} \in P_n$



$\sigma_{j-1}\sigma_{j-2}\cdots\sigma_i^2\cdots\sigma_{j-2}^{-1}\sigma_{j-1}^{-1}$

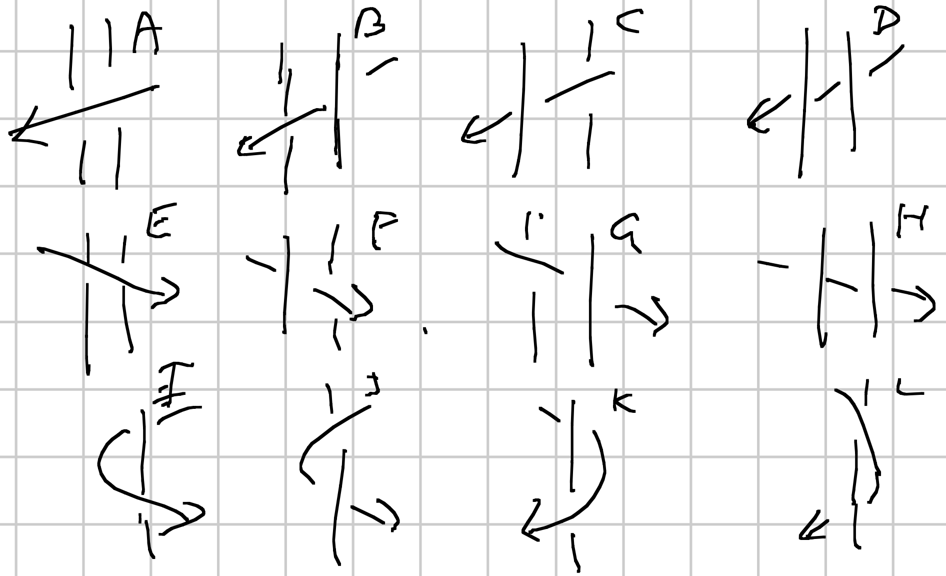
Prop: P_n è generato dai $\tau_{ij}^{\pm 1}$.

Dim: induz. su n. $n=2$ ✓

Sia vero per n e $\beta \in P_{n+1}$. Sia $\alpha \in P_n$ ottenuto da β cancellando l'ultimo filo. Considero $\gamma = i_n(\alpha)^{-1} \cdot \beta$

$i_n : P_n \hookrightarrow P_{n+1}$. Cancellando ultimo filo di γ ho $1 \in P_n$ dunque posso disegnare γ con i primi n fili dritti.

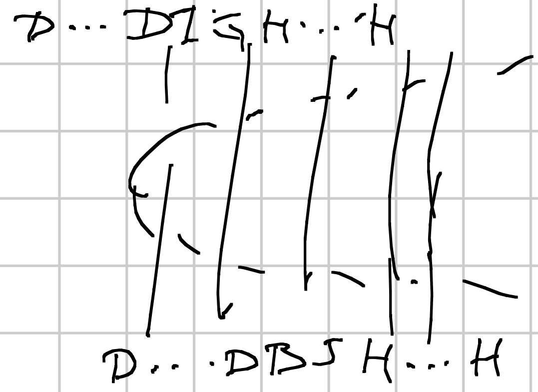
Induttivamente $i_n(\alpha) \in \langle \tau_{ij} : 1 \leq i,j \leq n-1 \rangle$. Basta vedere che $\gamma \in \langle \tau_{in} \rangle$. Disegno di γ si compone di:



ci piace:

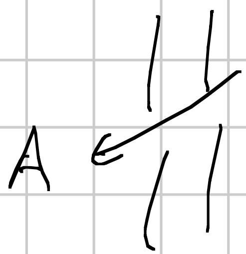
τ_{ij}

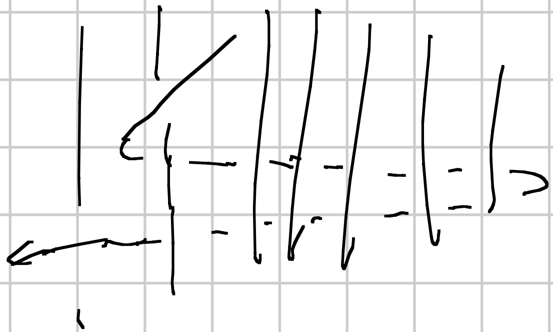
- con B, D, G, H, I, J ottengo solo T_{ij}^{+1}



T_{ij}^{-1}

- gli altri si possono sostituire con loro aggiungendo di "ritorni a dx sotto"





J H ⋯ H D ⋯ D B

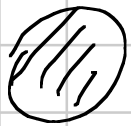
altri analoghi.

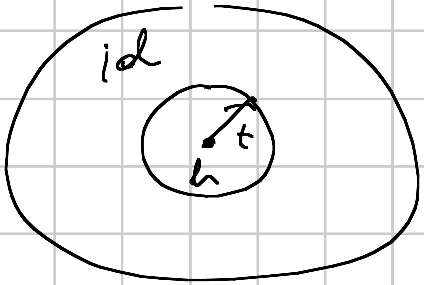
$\boxed{\checkmark}$

$H_n = \mathrm{Aut}_\partial^+ \Big($ 

$\Big) \big/ \text{isotope}$

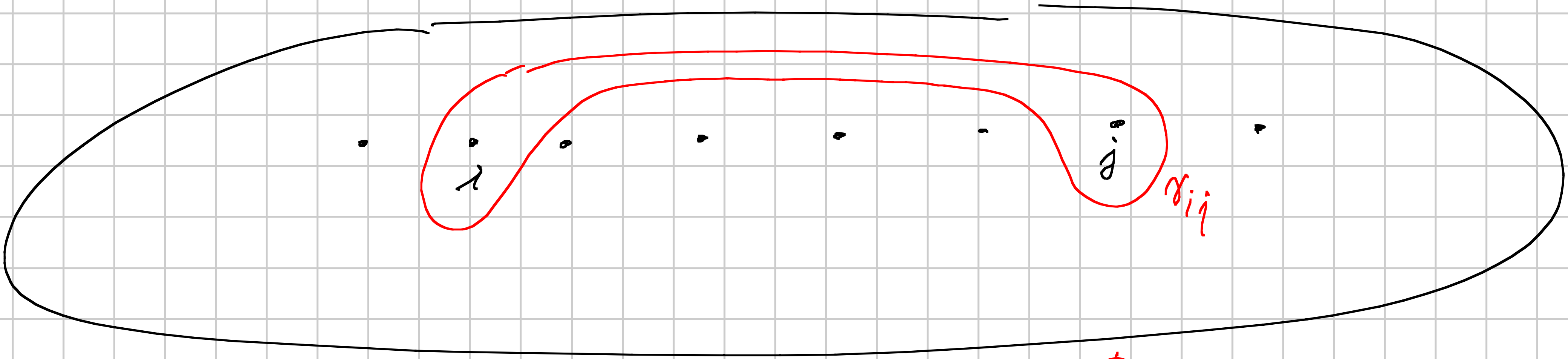
Prop: $H_n = \langle \text{Dehn twist} \rangle$

Dim: $H_0 = 1$ $\qquad h \in \mathrm{Aut}_\partial \Big($  $\Big)$

$h_t =$ 

$$h_t(z) = \begin{cases} z & \text{se } |z| \geq t \\ t \cdot h\left(\frac{z}{t}\right) & \text{se } |z| < t \end{cases}$$

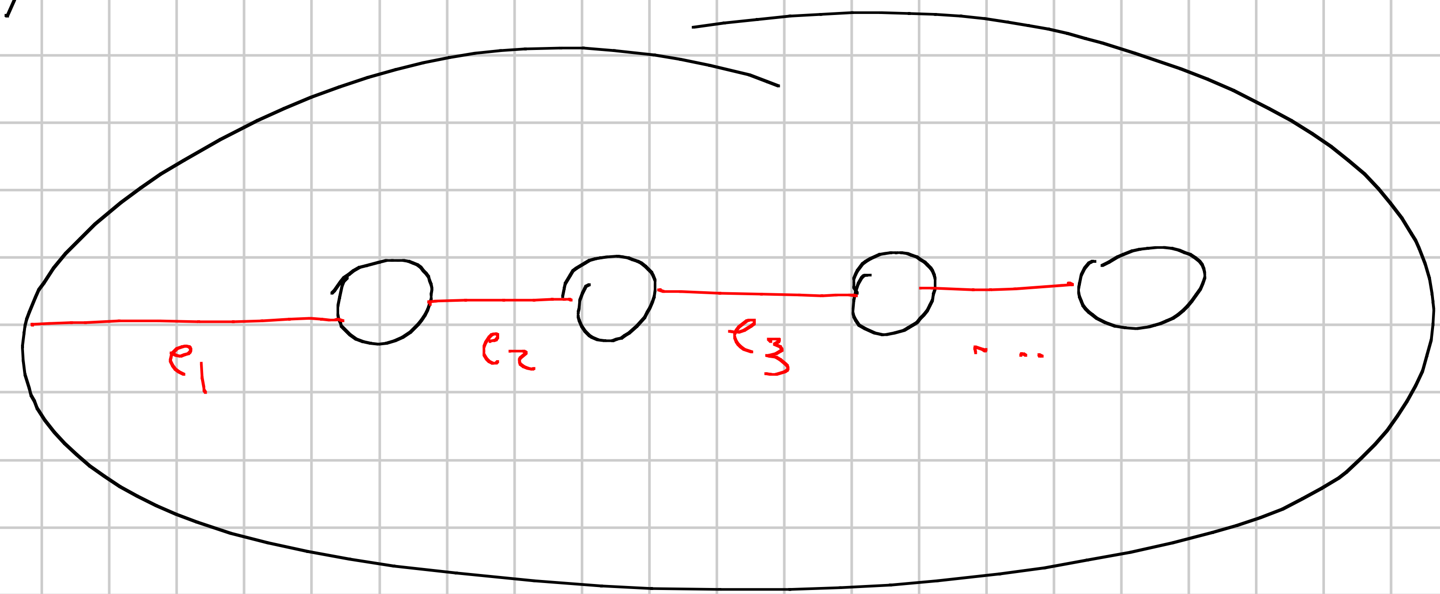
Sia ora $h \in H_n$; estendiamo ad $\bar{h} \in H_0$ come id nei buchi.
"Collassando i buchi" a punti e prendendo l'isotopia a id
ho una treccia pura, generata dai τ_{ij}



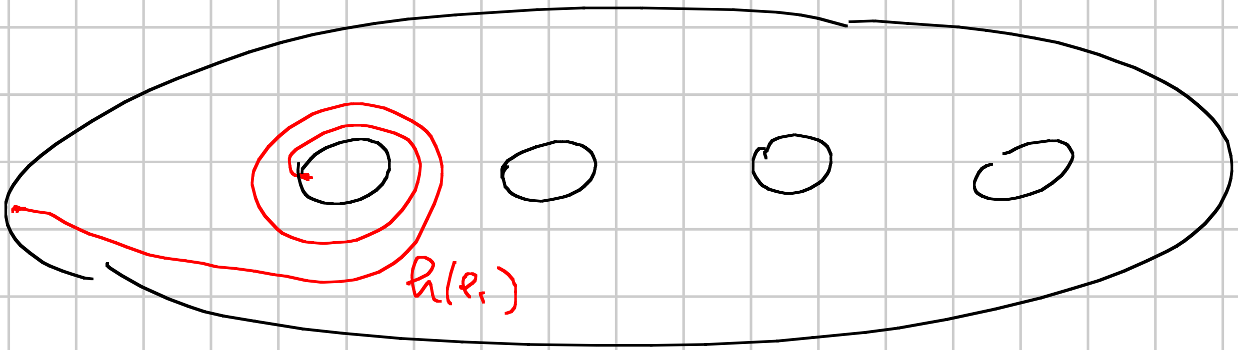
$\tau_{ij}^{\pm 1}$ è la treccia corrispondente a $\tau^{\pm}_{\gamma_{ij}}$

⟹ a meno di comporre h con Dehn twist posso supporre che l'isotopia di $\bar{h}$ a id non permuti i bordi dei buchi.

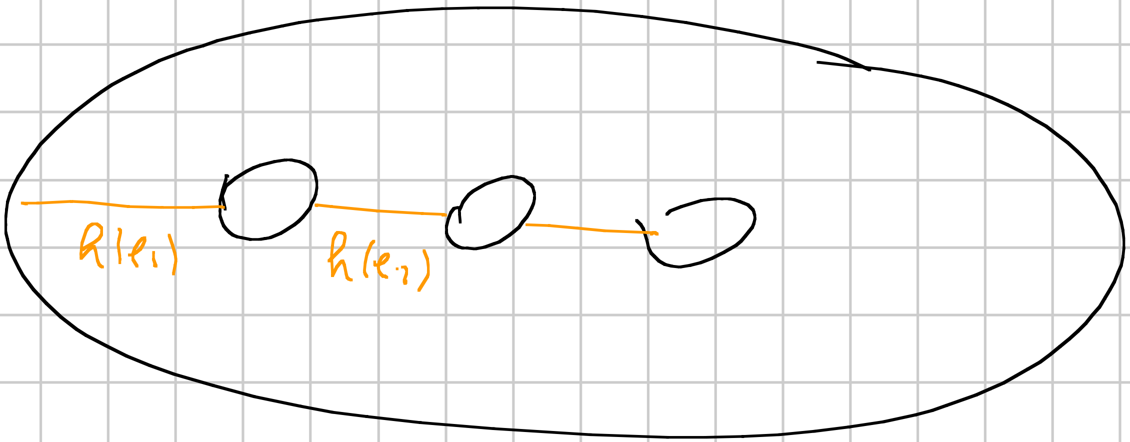
Considero



poiché h_t lascia invarianti i bordi interni ho



$\Rightarrow$ a meno di comporre con D-T ho

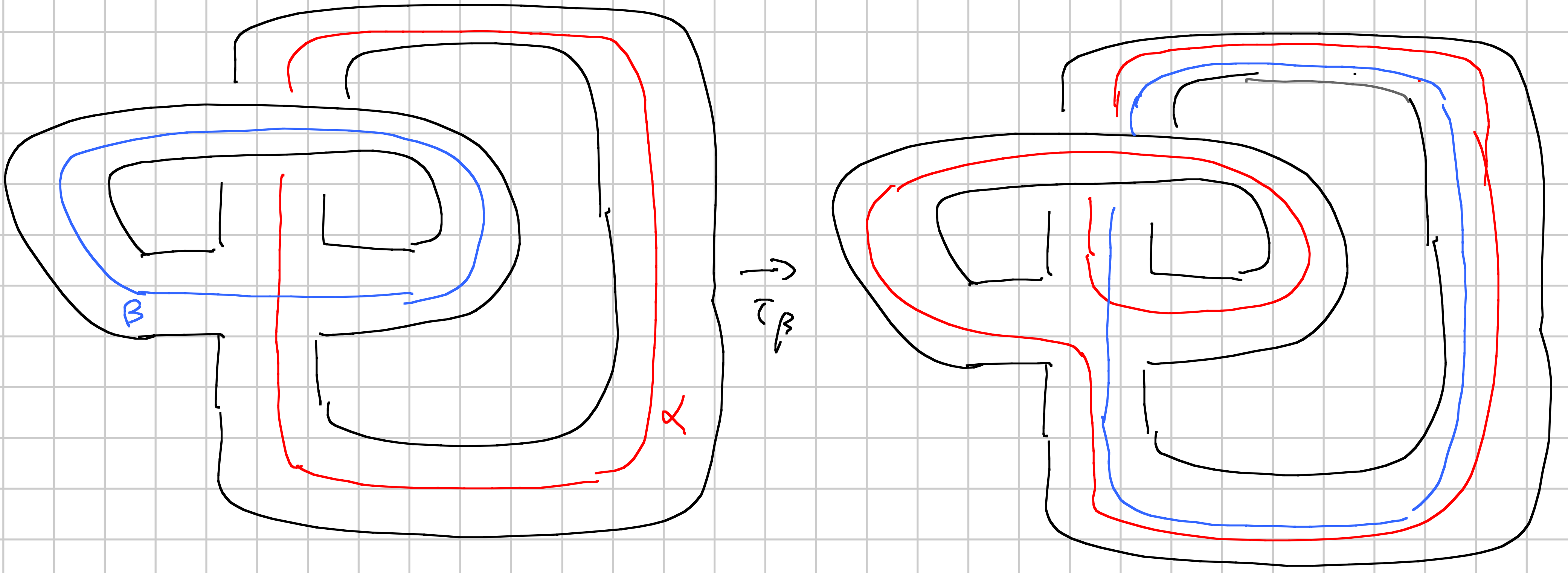


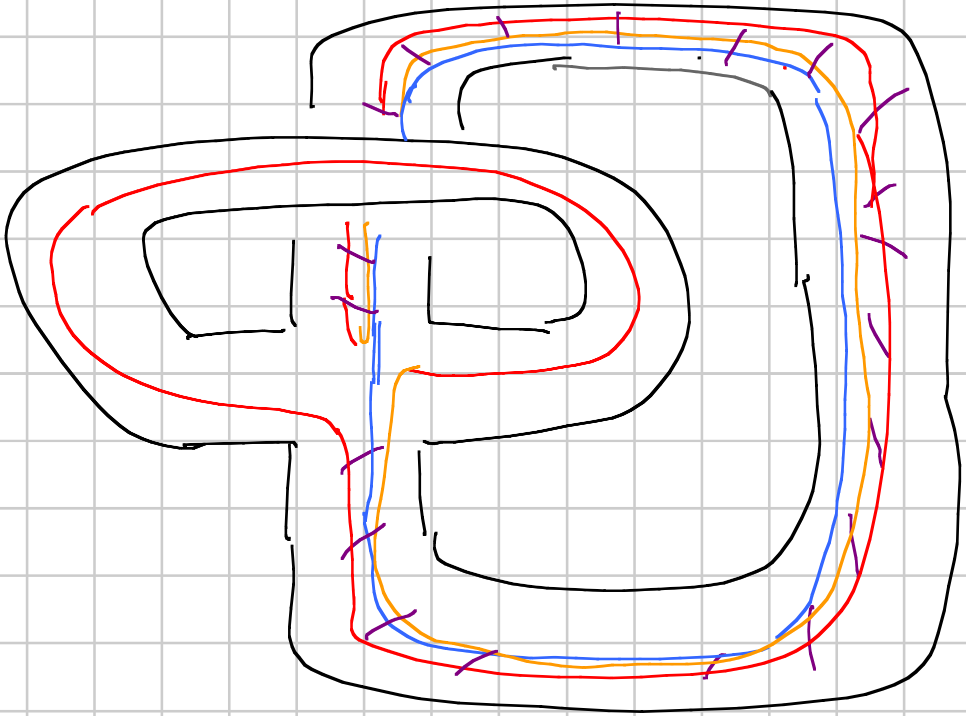
e unico di coppie con D-T ho $h = id/_{e_1, \dots, e_n}$ $\Rightarrow$ induce $\tilde{h} \in H_0 = \{1\}$
bord. interni $\Rightarrow h = 1$. $\boxtimes$

<u>Dim</u> (Teo). $\mathcal{D} = \langle D\text{-}T \rangle$

<u>Lem</u>: $\alpha, \beta \subset \Sigma$, c.s.c. non sep $\Rightarrow \exists f \in \mathcal{D}$ t.c. $f(\alpha) = \beta$

<u>Dim</u>: se $\alpha \pitchfork \beta = 1$ pto


β
α
τ_β

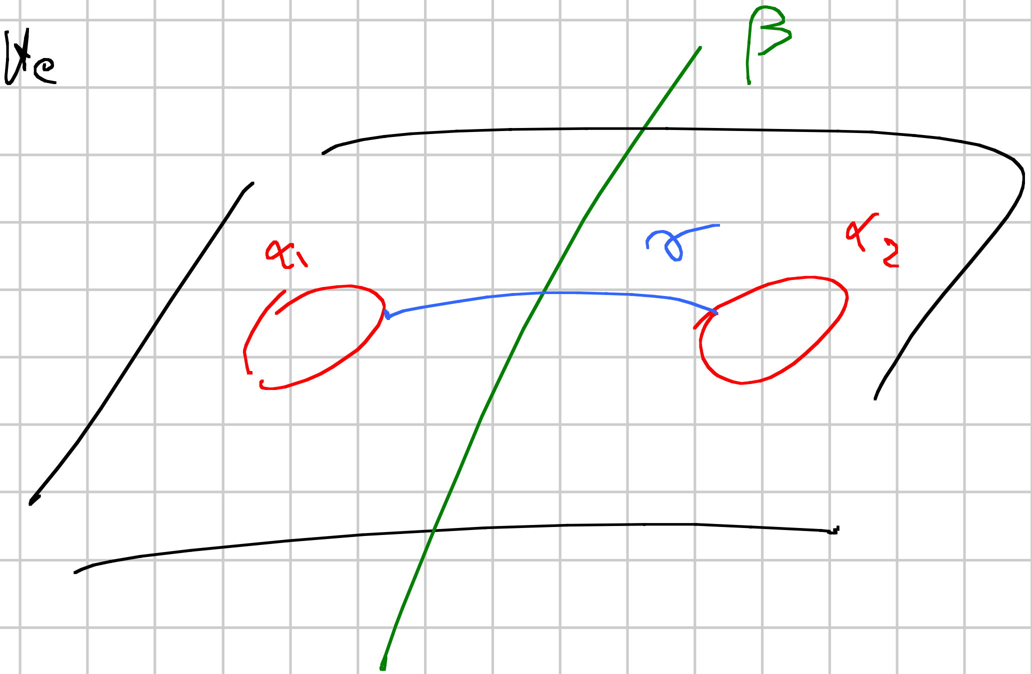


$\swarrow \tau_\alpha$

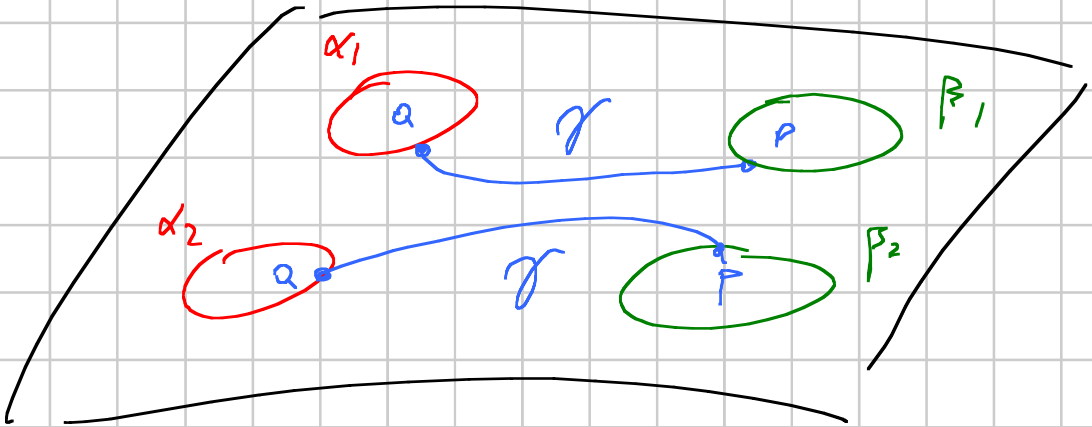
$\Rightarrow (\tau_\alpha \circ \tau_\beta)(\alpha) = \beta$

· $\alpha \cap \beta = \emptyset$ affermo che esiste γ c.s.r. non sep. con $\alpha \pitchfork \gamma = \alpha \pitchfork \beta = 1$ punto

$\alpha \cup \beta$ sconnette



$\alpha \cup \beta$ non sconnette

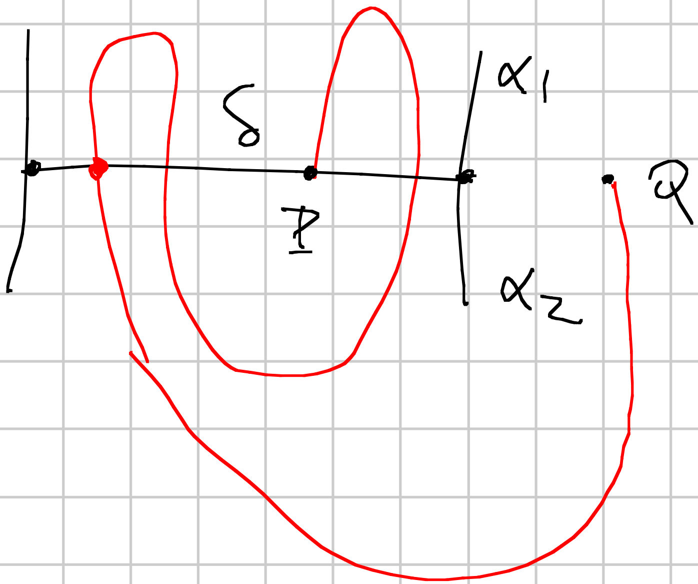


Se $\alpha \cap \beta = n \geq 2$ pti allora che esiste γ t.c.

- γ c.s.c. non sep
- $\gamma \cap \alpha \leq 1$ pto
- $\gamma \cap \beta < n$ pti

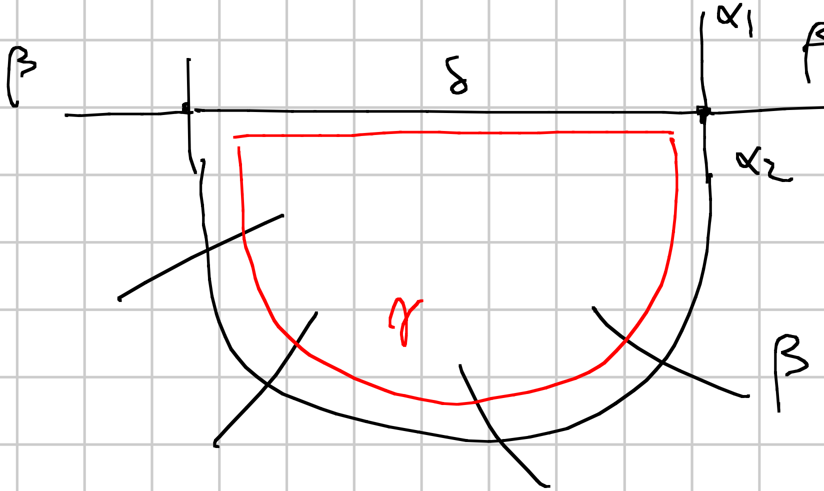
Sia δ un arco su β tra due intersez. consecutive con α.

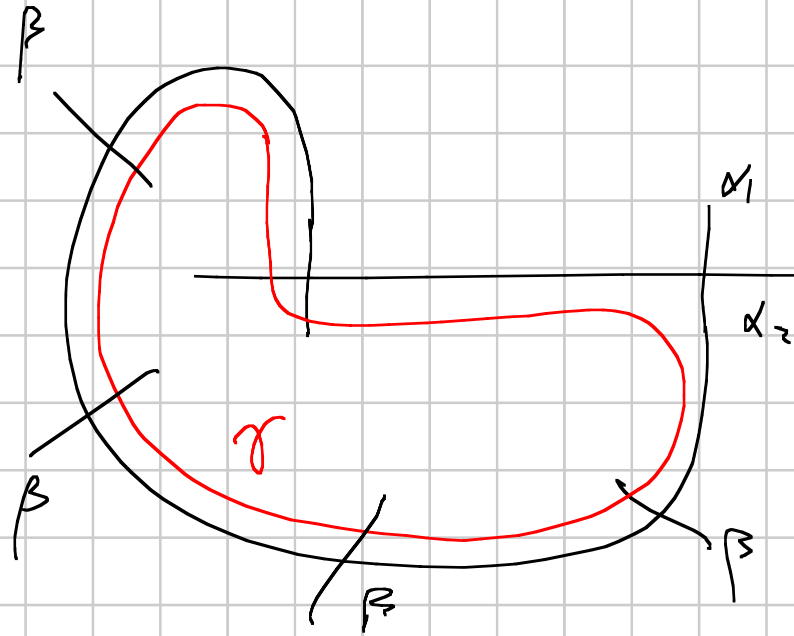
$\gamma\beta$ separa α in α_1 e α_2; affermo che $\delta \cup \alpha_1$ o $\delta \cup \alpha_2$ non sep:



$\Rightarrow$ $\delta \cap \alpha_2$ non sep.

Modifico $\delta \cup \alpha_2$ alla γ che voglio: due config:





$\blacksquare$

<u>Dim</u> (Dehn-Lickorish) : induzione su $g(\Sigma)$.

$g=0 \Rightarrow H_n$ <u>OK</u>

$g>0$ sia $f \in \mathrm{Act}$.

Prendo μ non separante $\Rightarrow f(\mu)$ non sep.

$\exists\, \tau \in \mathcal{D}$ t.c. $(\tau \circ f)(\mu) = \mu$.

Se $(\tau \circ f)(+\mu) = +\mu$ wlog $\tau \circ f|_{\mu} = \mathrm{id}_{\mu}$

$\Rightarrow$ taglio lungo μ $\Rightarrow$ concludo per induzione.

Se $(\tau \circ f)(+\mu) = -\mu$ prendo γ con $\gamma \cap \mu = 1$ pto.

<u>Esercizio</u>: $(\tau_{\gamma} \circ \tau_{\mu}^{2} \circ \tau_{\gamma})(+\mu) = -\mu$

$\Rightarrow$ sostituisco τ con $\tau_\gamma \circ T_\mu^{2} \circ \tau_\gamma \circ \tau$. $\blacksquare$

Def: Sia $M^{(3)}$ orientata e $T \subset \partial M$ toro.
Se $f : \partial(D^2 \times S^1) \to T$ chiamo Dehn-filling di M lungo T e f

$$M \cup_f (D^2 \times S^1)$$

Prop: conta solo $f(\underbrace{\partial D^2 \times \{*\}}_{\text{meridiano di } D^2 \times S^1 = \mu}) \subset T$ come curva/isotopia

Dim: $f_1, f_2 : \partial D^2 \times S^1 \to T$ $\quad f_1(\cancel{\partial D^2 \times \{x\}}) = f_2(\cancel{\partial D^2 \times \{x\}})$
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \mu \qquad\qquad\qquad\qquad \mu$

ho $(f_2^{-1} \circ f_1)(\mu) = \mu$

$\Rightarrow f_2^{-1} \circ f_1$ si estende a $F : D^2 \times S^1 \circlearrowleft$

$$\begin{array}{ccc} M \cup_{f_1} (& D^2 \times S^1 &) \\ \cup & \cup & \\ x & y & \\ \downarrow & \downarrow & \\ x & F(y) & \\ \cap & \cap & \\ M \cup_{f_2} (& D^2 \times S^1 &) \end{array}$$

omeo

Chirurgia di Dehn lungo L = Dehn filling di $E(L)$.
La chiamo intera se ogni $f_j(\mu)$ è una longitudine della j-esima compo di L.