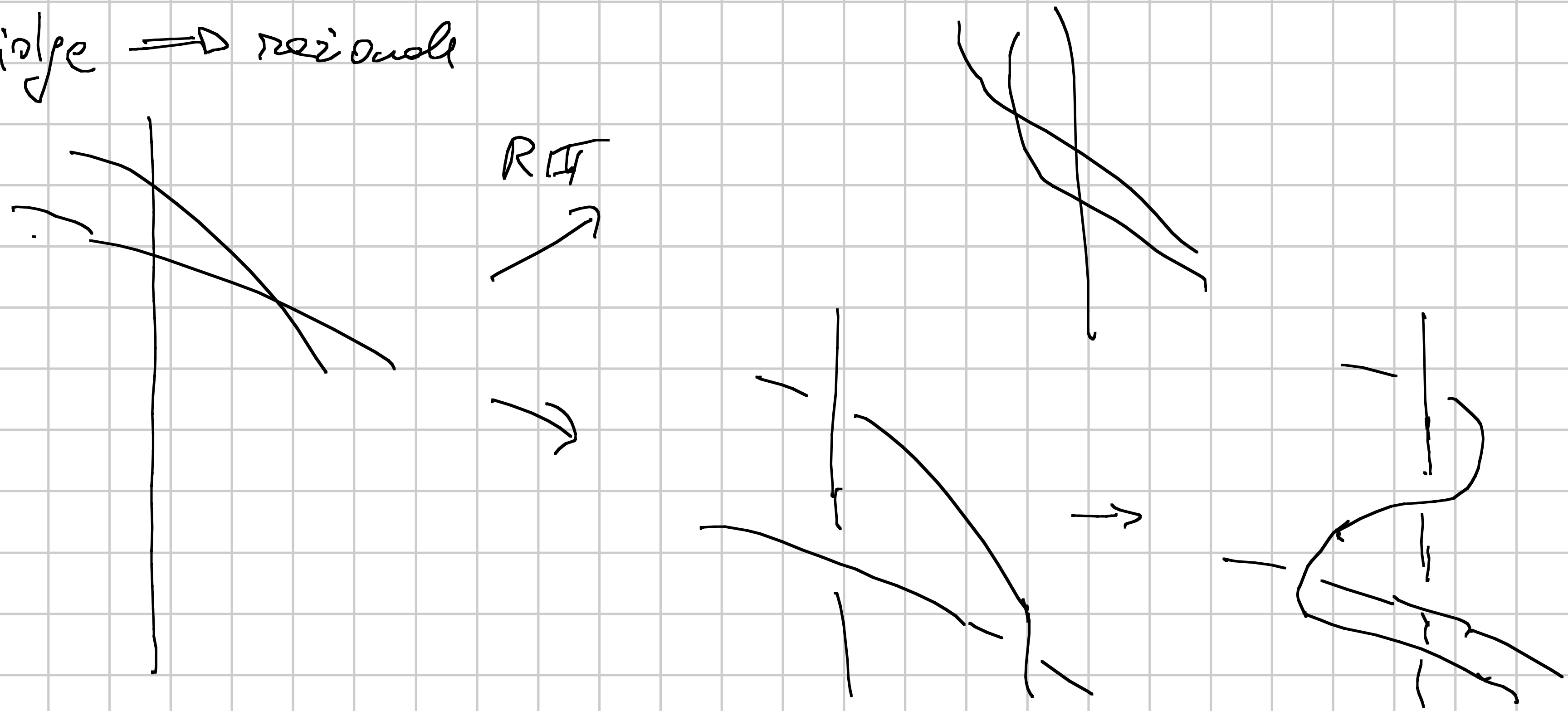
Teoria dei nodi 12/10/16

2-bridge $\Longrightarrow$ razionale



Thm: $\langle x_1,\dots,x_n \mid r_1,\dots,r_k\rangle \cong \langle y_1,\dots,y_m \mid s_1,\dots,s_h\rangle$

$\Longrightarrow$ collegati da mosse $I^{\pm 1}$, $II^{\pm 1}$

$$I: \quad \langle x_1,\dots,x_n \mid r_1,\dots,r_k\rangle \longrightarrow \langle x_1,\dots,x_n \mid r_1 \dots r_k\, r\rangle$$

$$r \in N(r_1,\dots,r_k)$$

$$II: \quad \langle x_1 \dots x_n \mid r_1 \dots r_k\rangle \longrightarrow \langle x_1 \dots x_n\, x \mid r_1 \dots r_k\, x^{-1} g\rangle$$

$$g \in F(x_1 \dots x_n)$$

Lem: $\rho : F(x_i, y_j) \longrightarrow F(x_i) \qquad \rho(x_i) = x_i$

$\phi : F(x_i) \longrightarrow \langle x_i \mid r_p \rangle$ proiezione

$\Longrightarrow \mathrm{Ker}(\phi \circ \rho) = N(r_p, y_j^{-1} \cdot \rho(y_j))$

Dim: $\supseteq$ $\quad (\phi \circ \rho)(r_p) = \phi(\rho(\underset{\substack{\cup\!\!\shortmid \\ F(x_i)}}{r_p})) = \phi(r_p) = 1$

$$(\phi \circ \rho)(y_j^{-1} \cdot \rho(y_j)) = \phi(\rho(y_j^{-1})) \cdot \phi(\underset{\substack{\| \\ \rho}}{\rho^2}(y_j)) = \phi(\rho(y_j))^{-1} \cdot \phi(\rho(y_j)) = 1$$

$\subseteq$ $\quad H = N(r_t, y_j^{-1} \rho(y_j))$

$\gamma : F(x_i, y_j) \longrightarrow F(x_i, y_j)/H$ proiezione

$$\eta = \gamma|_{F(x_i)}$$

Affermo:

$$\begin{tikzcd}
F(x_i, y_j) \arrow[rr, "\rho"] \arrow[dr, "\gamma"'] & \arrow[d, phantom, "\circlearrowleft"] & F(x_i) \arrow[dl, "\eta"] \\
& F(x_i, y_i)/H &
\end{tikzcd}$$

$$(\eta \circ \rho)(x_i) = \eta(x_i) = \gamma(x_i) \quad \checkmark$$

$$\gamma(y_j^{-1}) \cdot \eta(\rho(y_j)) = \gamma(y_j^{-1}) \cdot \gamma(\rho(y_j)) = \gamma(\underbrace{y_j^{-1} \cdot \rho(y_j)}_{\in H}) = 1$$

$$u \in F(x_i, y_j) \Rightarrow \gamma(u \cdot \rho(u)^{-1}) =$$

$$= \eta(\rho(u) \cdot \rho(u)^{-1}) = \eta(\rho(u)) \cdot \eta(\rho^2(u)^{-1}) = 1$$

Se $u \in \operatorname{Ker}(\phi \circ \rho) \Rightarrow \rho(u) \in \operatorname{Ker}\phi = H$

ma $u \cdot \rho(u)^{-1} \in H \Rightarrow u \in H$. $\blacksquare$

<u>Dimo</u> (Teo): $f : \langle x_i \mid r_p \rangle \longrightarrow \langle y_j \mid s_l \rangle$ isomorfismo

$g : \langle y_j \mid s_l \rangle \longrightarrow \langle x_i \mid r_p \rangle$ $\quad g = f^{-1}$

Esistono sollevamenti:

$$\begin{tikzcd}
F(x_i) \arrow[r, shift left, "F"'] \arrow[d, "\phi"'] & F(y_j) \arrow[l, shift left, "G"'] \arrow[d, "\psi"] \\
\langle x_i \mid r_p \rangle \arrow[r, shift left, "f"] & \langle y_j \mid s_l \rangle \arrow[l, shift left, "g"]
\end{tikzcd}$$

$$
\begin{tikzcd}
 & F(x_i, y_j) \arrow[dl, "\rho"'] \arrow[dr, "\sigma"] & \\
F(x_i) \arrow[rr, shift right, "F"'] \arrow[d, "\phi"'] & & F(y_j) \arrow[ll, shift right, "G"'] \arrow[d, "\psi"] \\
\langle x_i \mid r_p \rangle \arrow[rr, shift left, "f"] & & \langle y_j \mid s_\ell \rangle \arrow[ll, shift left, "g"]
\end{tikzcd}
$$

$\rho(x_i) = x_i$

$\rho(y_j) = G(y_j)$

$\sigma(y_j) = y_j$

$\sigma(x_i) = F(x_i)$

<u>Ora</u>:

$$\langle x_i \mid r_p \rangle \xrightarrow{\mathrm{II}} \langle x_i, y_j \mid r_p,\ y_j^{-1} \cdot \rho(y_j) \rangle$$

$$\langle y_j \mid s_\ell \rangle \xrightarrow{\mathrm{II}} \langle x_i, y_j \mid s_\ell,\ x_i^{-1} \cdot \sigma(x_i) \rangle$$

Lem $\Longrightarrow \operatorname{Ker}(\phi\circ\rho) = N(r_p, y_j^{-1}\cdot\rho(y_j))$

ma $\operatorname{Ker}(\phi\circ\rho) = \operatorname{Ker}(g\circ\psi\circ\sigma) = \operatorname{Ker}(\psi\circ\sigma)$

$(\psi\circ\sigma)(s_l) = \psi(s_l) = 1$

$(\psi\circ\sigma)(x_i^{-1}\cdot\sigma(x_i)) = \psi(x_i)^{-1}\cdot\psi(\underset{\substack{\| \\ \sigma}}{\sigma^2}(x_i)) = 1$

$\Longrightarrow s_l,\ x_i^{-1}\cdot\sigma(x_i) \in N(r_p, y_j^{-1}\cdot\rho(y_j))$

Simm: $r_p, y_j^{-1}\cdot\rho(y_j) \in N(s_l, x_i^{-1}\cdot\sigma(x_i))$

$$\begin{tikzcd}
\langle x_i, y_j \mid r_p, y_j^{-1}\rho(y_j)\rangle \arrow[dr, "I"] & \\
& \langle x_i, y_j \mid r_p, s_l, y_j^{-1}\cdot\rho(y_j), x_i^{-1}\cdot\sigma(x_i)\rangle. \\
\langle x_i, y_j \mid s_l, x_i^{-1}\cdot\sigma(x_i)\rangle \arrow[ur, "I"'] &
\end{tikzcd}$$

$\blacksquare$

Def: G, H gruppi ; $G * H = \left(\text{parole con lettere in } G \cup H\right) \big/ \text{cancellazione}$

$$1 = \emptyset$$

$$u \cdot 1_G \cdot w = u \cdot w$$

$$u \cdot 1_H \cdot w = u \cdot w$$

$$u \cdot x \cdot x^{-1} \cdot w = u \cdot w \quad \forall x \in G \cup H$$

Es: $\langle x, y, z \mid xyz = yzx \rangle \cong \mathbb{Z}^2 * \mathbb{Z}$

$\xrightarrow{II} \langle x, y, z, a \mid xyz = yzx,\ a = yz \rangle$

$\xrightarrow{I} \langle x, y, z, a \mid xyz = yzx,\ a = yz,\ x \cdot a = a \cdot x \rangle$

$\xrightarrow{I^{-1}} \langle x, y, z, a \mid a = yz,\ xa = ax \rangle$

$$\xrightarrow{I} \langle x,y,z,a \mid a = yz \quad xa = ax \quad z = y^{-1}a \rangle$$

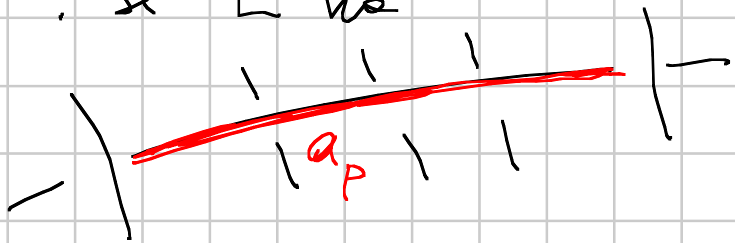
$$\xrightarrow{I^{-1}} \langle x,y,z,a \mid xa = ax \ , \ z = y^{-1}a \rangle$$

$$\xrightarrow{II^{-1}} \langle x,y,a \mid xa = ax \rangle = \langle x,a \mid xa = ax \rangle * \langle y \mid - \rangle$$

$$= \mathbb{Z}^2 * \mathbb{Z}. \qquad \square$$

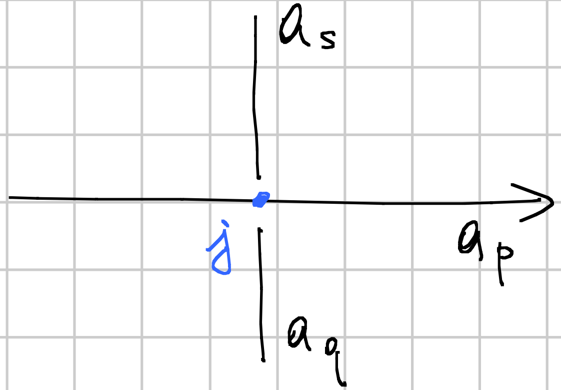
Invariante di $L \subset S^3$ link è $\pi_1(S^3 \setminus L)$.

<u>Teorema</u> (presentazione di Wirtinger): se L ha diagramma D con n sovrarchi $a_1, \dots, a_n$ allora $\pi_1(S^3 \setminus L) =$



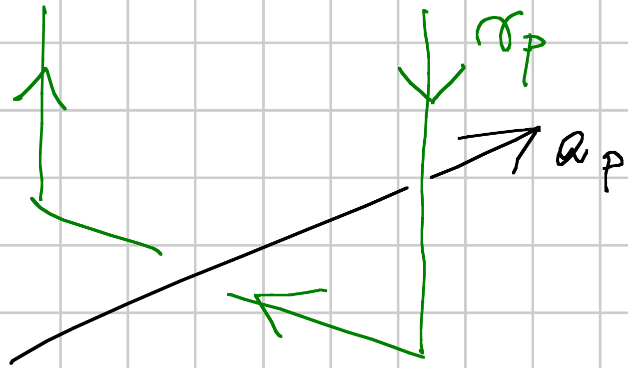
$\langle x_1, \dots, x_n \mid r_1, \dots, r_k \rangle$ ($k = n$, $k = n-1$ vedremo)

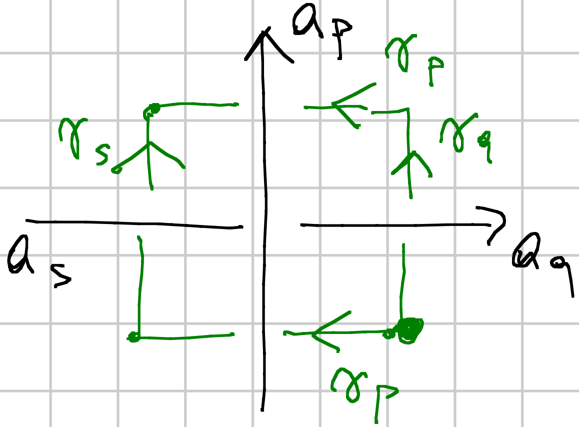
dove r_j viene dal j-esimo vertice così: orientato D a caso (una volta per tutte):



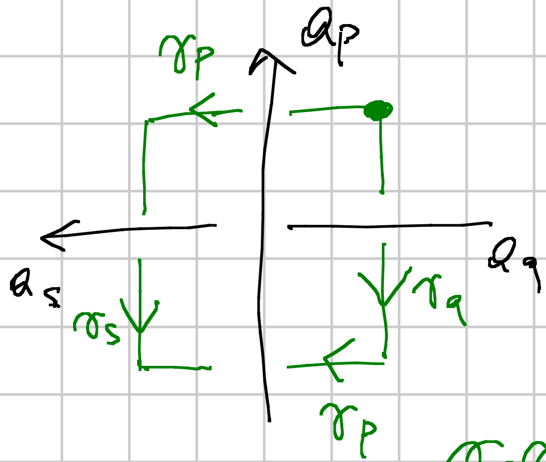
$\rightsquigarrow r_j : \quad x_q \cdot x_p = x_p \cdot x_s$

__Dim__: fisso pto base ∞ e pongo $\gamma_p =$
pensato come $(\cdot, \cdot, +\infty)$



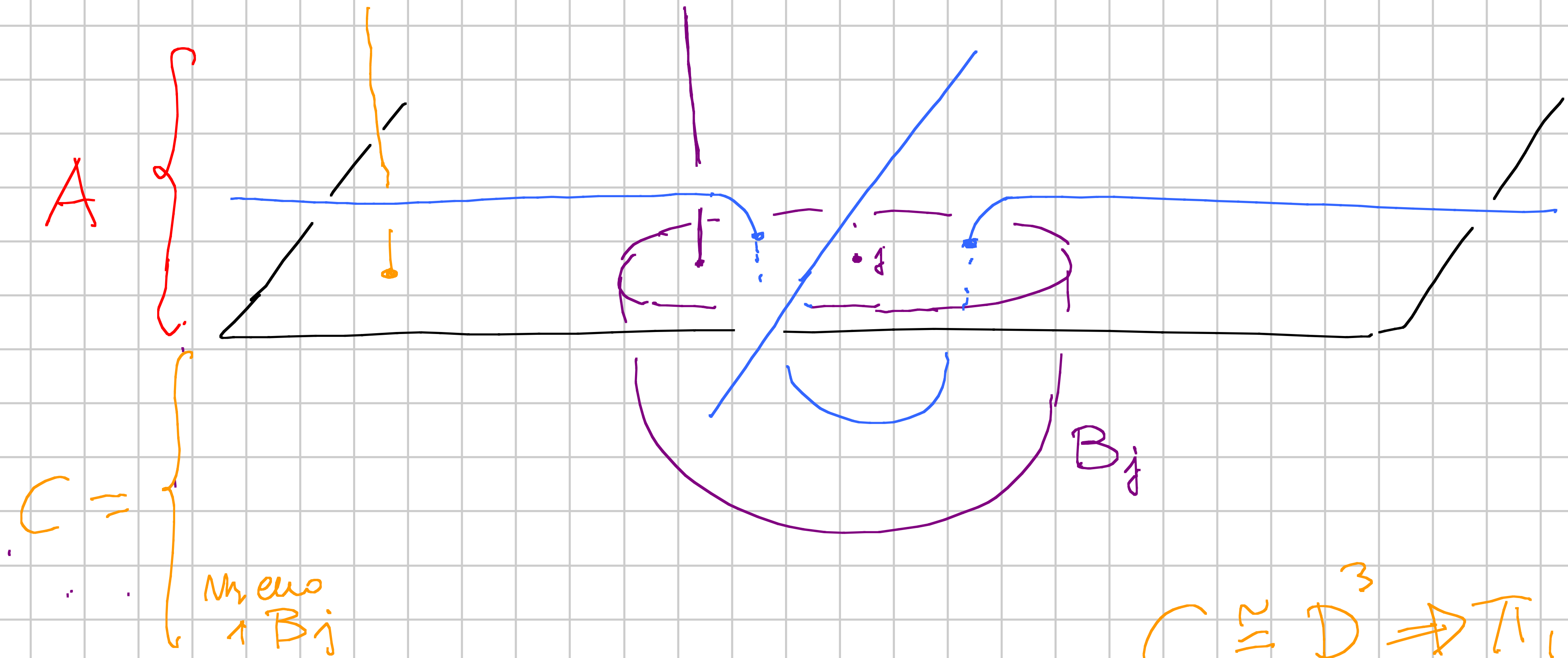


$$\gamma_q \gamma_p = \gamma_p \gamma_s$$



$$\gamma_q \gamma_p = \gamma_p \gamma_s$$

Formalmente: Van Kampen:



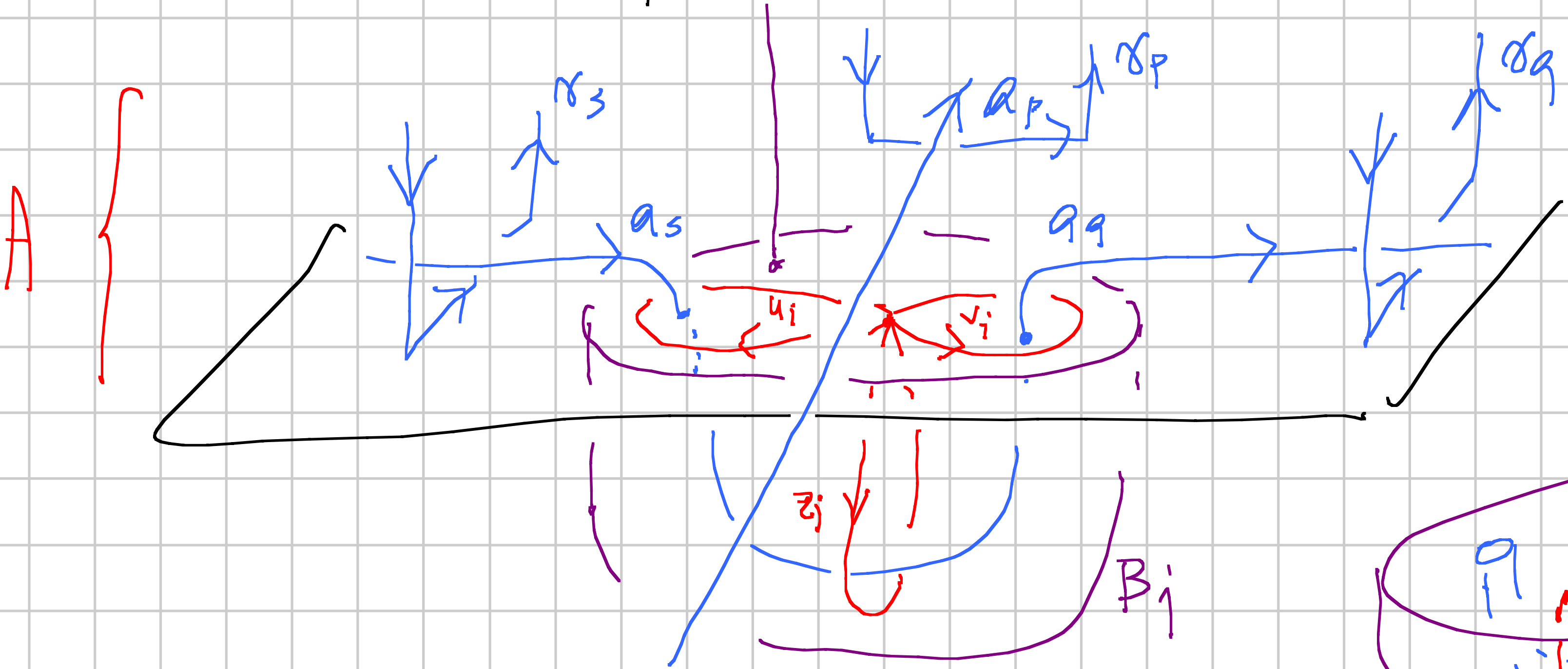
$$C \cong D^3 \Rightarrow \pi_1(C) = \{1\}$$

$A =$ 

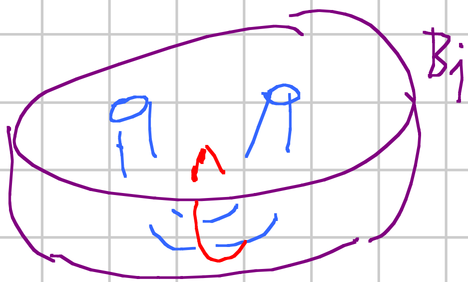
$\Rightarrow \pi_1(A) = \langle x_1, \dots, x_n \mid \longrightarrow \rangle \qquad x_j = [\gamma_j]$

Aggiungo i B_j uno alla volta (non si intersecano fra loro) mostrando che ogni volta l'effetto è introdurre x_j

$$\pi_1(A\cup B_i) = \frac{\pi_1(A) * \pi_1(B_i)}{N\left(i_A^*(u)\cdot i_{B_i}^*(u)^{-1}\ ;\ u\in\pi_1(A\cap B_i)\right)}$$



$\pi_1(B_i)=\mathbb{Z}$



V.K:

$$\pi_1(A \cup B_j) = \frac{\langle x_1, \dots x_p, z_j \rangle}{\dots \qquad u_j, v_j \in \pi_1(A \cap B_j)}$$

$A \cap B_j =$ 

$\Rightarrow \pi_1(A \cap B_j) = \mathbb{Z} * \mathbb{Z}$

$$\begin{aligned} \pi_1(A \cap B_j) &\longrightarrow \pi_1(B_j) \\ u_j &\longmapsto z_j \\ v_j &\longmapsto z_j \end{aligned}$$

$$\begin{aligned} \pi_1(A \cap B_j) &\longrightarrow \pi_1(A) \\ u_j &\longmapsto x_s \\ v_j &\longmapsto x_p^{-1} x_q x_p \end{aligned}$$