Teoria dei Nodi 7/12/16

$A, \circ, /$ $\qquad (a_n)_{n=1}^{\infty}$

(U) $\quad W(\bigcirc \cdots \bigcirc) = a_n$

(S) $\quad W(L_+) = W(L_-) \circ W(L_0)$

$\qquad W(L_-) = W(L_+) / W(L_0)$

$\exists!\ W_k : \mathcal{D}_k \to A$ che soddisfa (U), (S) ed è inv. per $R_{I/II/III}$ dentro $\mathcal{D}_k$.

$k=0$ ok

P.I. $(b_1, \dots, b_n)$ pt base sulle componenti

incrocio cattivo se ci si passa prima sopra

$\ell = \#$ incroci cattivi : $\ell = 0 \longrightarrow a_1$

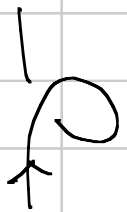
$\ell > 0$ usa (5)

Visto: (1) Vale (5) (con "ritorno" b su $L_\pm$)

(2) Indip. da b_j su K_j (resta l'ordine).

Claim 3 : $W_{k+1}(L, b)$ non cambia con $R_{I/II/III}$ dentro D_{k+1}.

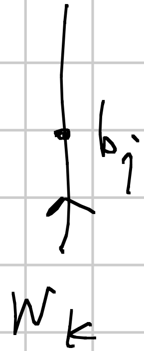
$\boxed{R_I}$



$\rightsquigarrow$ scelgo

P buono

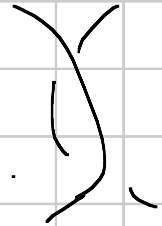
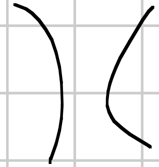
$\Rightarrow$ nel calcolo a cascata di



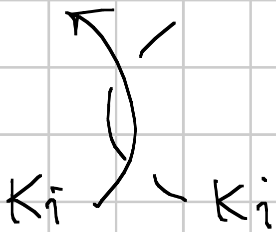
si fanno esattamente gli stessi usi di (S) per ridurre # incroci cattivi e nei casi corrisp. alla fine si ha lo stesso # di componenti. (Altri casi R_I analoghi.)

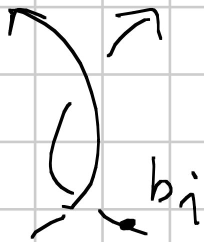
$\boxed{R_{II}}$ (i) 

$l < j$ P, Q buoni $\Rightarrow$ come sopra

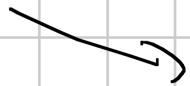
$\Rightarrow$ il calcolo di W_{k+1} e W_{k-1} di  e 

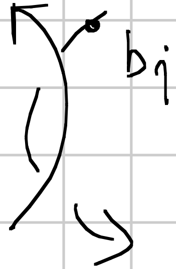
uso (S) allo stesso modo .... come sopra.

(ii) 

$\longrightarrow$ 

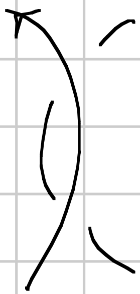
incroci buoni





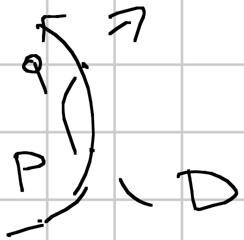
$\Longrightarrow$ come sopra

(iii)



$j < \ell$

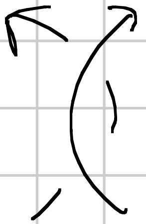
(iii-1)

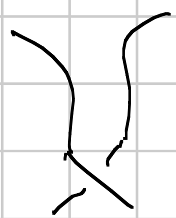


$$D = D_{\substack{(P,+)\\(Q,-)}}$$

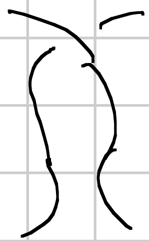
$$W_{k+1}(D) = W_{k+1}\left(D_{\substack{(P,+)\\(Q,-)}}\right) = W_{k+1}\left(D_{\substack{(P,-)\\(Q,-)}}\right) \circ W_k\left(D_{\substack{(P,0)\\(Q,-)}}\right)$$

$$= \left(W_{k+1}\left(D_{\substack{(P,-)\\(Q,+)}}\right) / W_k\left(D_{\substack{(P,-)\\(Q,0)}}\right)\right) \circ W_k\left(D_{\substack{(P,0)\\(Q,-)}}\right)$$





uguali



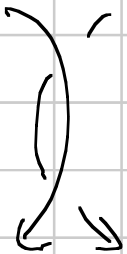
$$= W_{k+1}\Bigg($$ 

$$\Bigg) = W_{k-1}\Bigg($$  $$\Bigg)$$

|

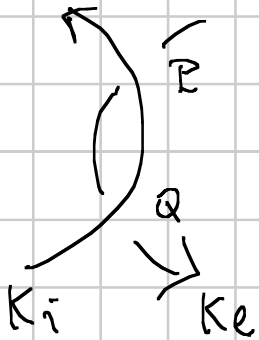
caso (i)

(iii-2)



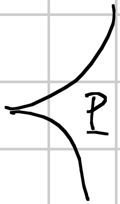
analogo

(iii-3)

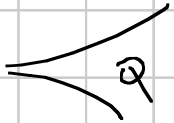


$j < l$

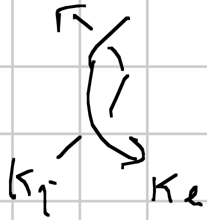
$D = D_{(P,+)(Q,-)}$



$D_{(P,-)(Q-)}$

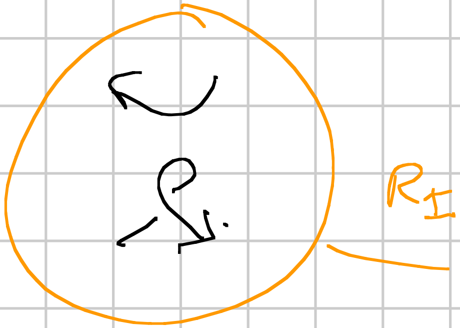


$D_{(P,-)(Q,+)} =$



$j < \ell$ incroci buoni

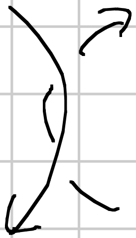
$D_{(P,0)(Q-)} =$



$D_{(P,-)(Q,0)} =$

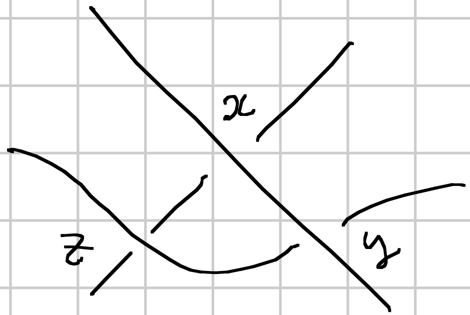
$\Longrightarrow$ ok.

(iii-4)

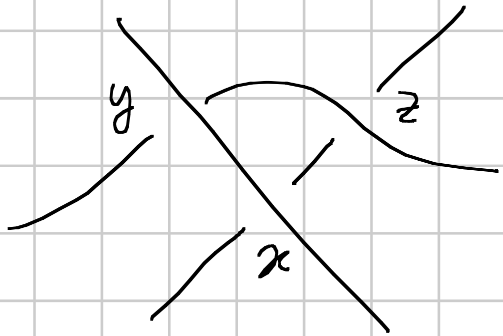


analogo

$\boxed{R_{III}}$



D



D'

xy
sopra
yz
sopra
xz

Suppongo i pti base lontani. In D:

- x buono, y, z cattivi è impossibile. "$<$" = "viene prima di percorrendo L"
  poiché comportano $xz < xy < yz < xz$ <u>NO</u>
- x cattivo, y, z buoni impossibile <u>NO</u> (stessa disug. finale)

- x, y, z buoni $\Rightarrow$ anche in D'

$\Rightarrow$ come sopra: le (S) da usare a concorda per ottenere tutti buoni sono le stesse per D e D' $\Rightarrow \dots$ ✓

- x, y buoni, z cattivo

$$W_{k+1}(D_{(z,\varepsilon)}) = W_{k+1}(D_{(z,-\varepsilon)}) \tilde{\varepsilon} W_k(D_{(z,0)})$$

$$= W_{k+1}(D'_{(z,-\varepsilon)}) \tilde{\varepsilon} W_k(D'_{(z,0)}) = W_{k+1}(D'_{(z,\varepsilon)}).$$

- x, z buoni, y cattivo analogo.

- x, z cattivi (y come gli pare) (x, y cattivi sarà analogo)

$$W_{k+1}\left(D_{\substack{(z,\varepsilon_z)\\(x,\varepsilon_x)}}\right) = \left(W_{k+1}\left(D_{\substack{(z,-\varepsilon_z)\\(x,-\varepsilon_x)}}\right) \tilde{\varepsilon}_x \, W_k\left(D_{\substack{(z,-\varepsilon_z)\\(x,0)}}\right)\right) \tilde{\varepsilon}_z \, U_k\left(D_{\substack{(z,0)\\(x,\varepsilon_x)}}\right)$$

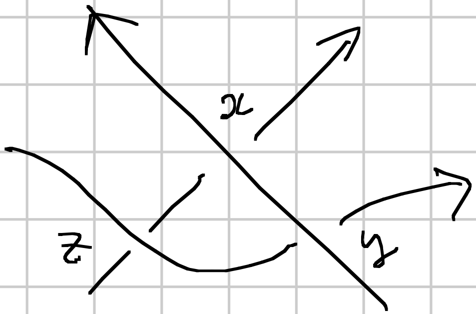
$\Vert$ x, z buoni $\qquad \updownarrow \qquad \updownarrow$

$$W_{k+1}\left(D'_{\substack{(z,\varepsilon_z)\\(x,\varepsilon_x)}}\right) = \left(W_{k+1}\left(D'_{\substack{(z,-\varepsilon_z)\\(x,-\varepsilon_x)}}\right) \check{\varepsilon}_x \, W_k\left(D'_{\substack{(z,-\varepsilon_z)\\(x,0)}}\right)\right) \tilde{\varepsilon}_z \, W_k\left(D'_{\substack{(z,0)\\(x,\varepsilon_x)}}\right)$$

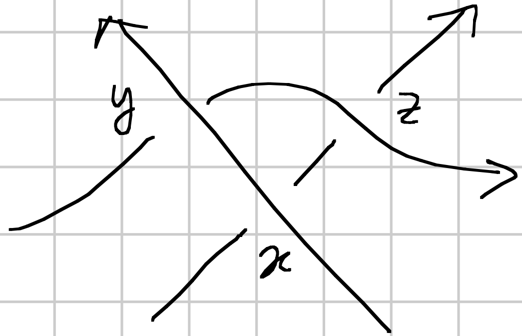
Affermo che $\updownarrow$ sono sempre uguali via $R_{I/II}$:

Vanno esaminati tutti i casi per le orientazioni:

*)

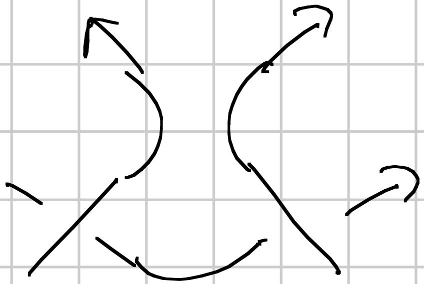


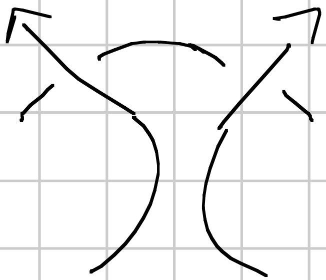
D



D'

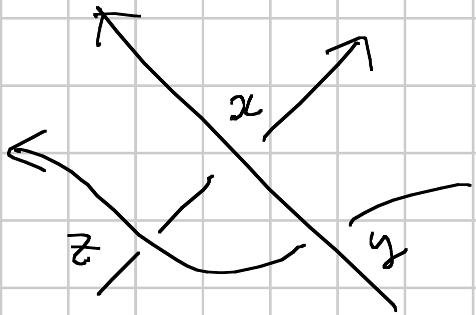
$(z,-\varepsilon_z)$
$(x,0)$;



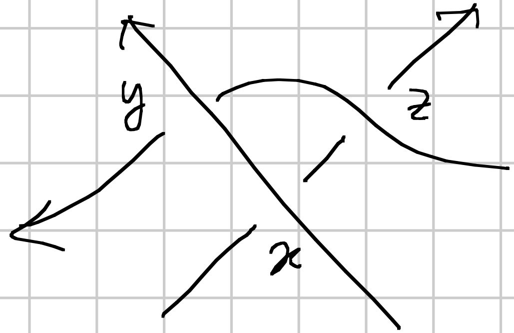


isotopi

*)

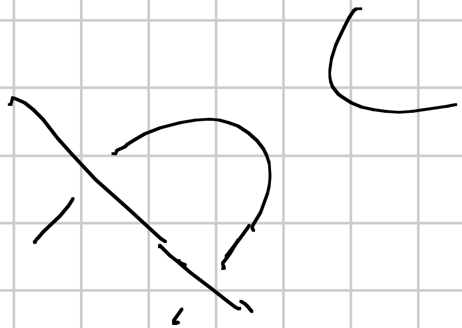


D



$(z, 0)$

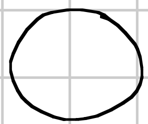
(x, ε_x)



RII

Finito elenco orientazioni.

Resta: indip. da ordine componenti

Claim 4: ogni diagramma diventa  con sequenza di mosse:

S: switch di crossing

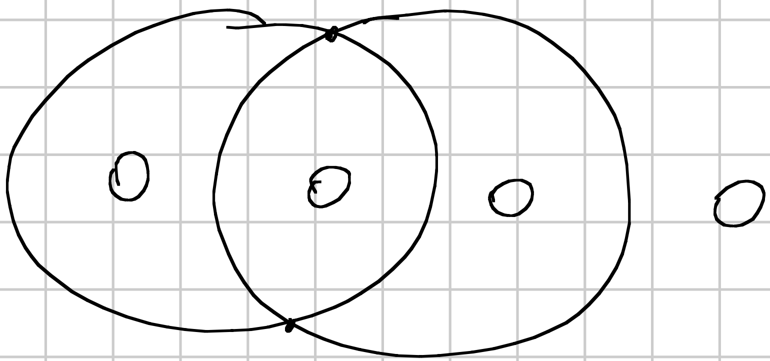
U: rimozione compo. senza incroci

R_I in discesa

R_{II} in discesa

R_{III}

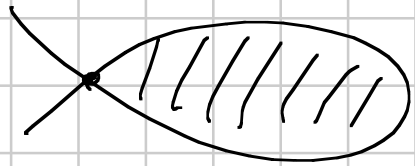
[Manturov: ogni diagramma diventa senza incroci con mosse
S, $R_I \downarrow$, $R_{II} \downarrow$, R_{III} :

Falso :  .]

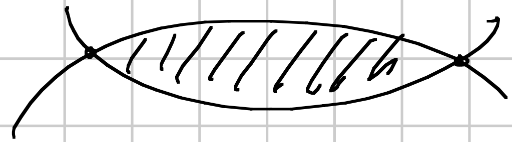
Dim : grazie a S disegno $\times$.

Suppongo D connesso : altrimenti parto da un connesso innermost.

Esamino i monogoni:

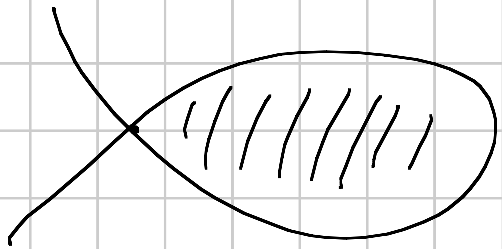


bigoni:



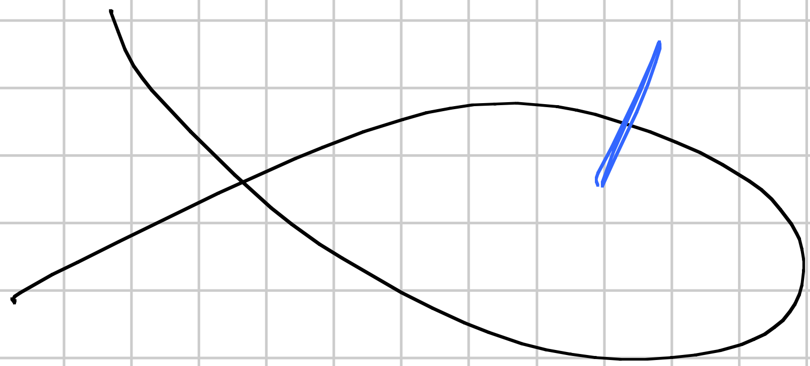
Fatto: se ci sono incroci esistono monogon o bigon. (facile).

Prendo un monogon/bigon innermost. Se è monogon:

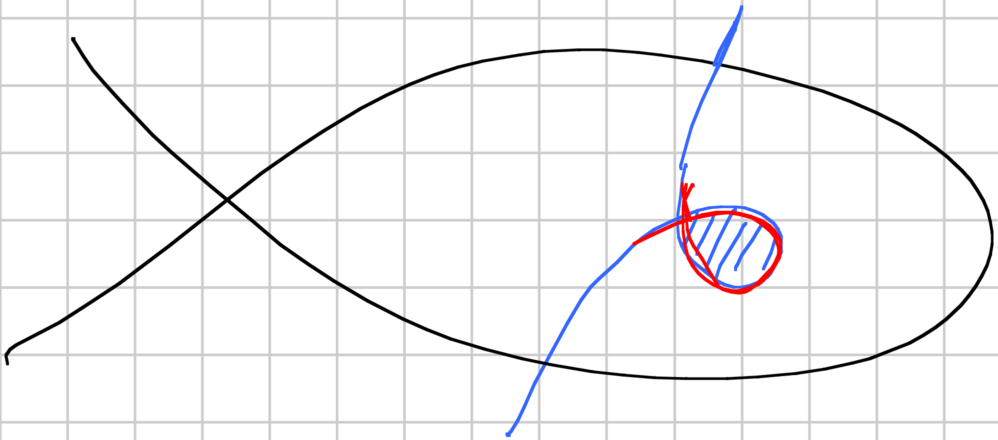


se dalla regione non passa nulla applico R_I ↓

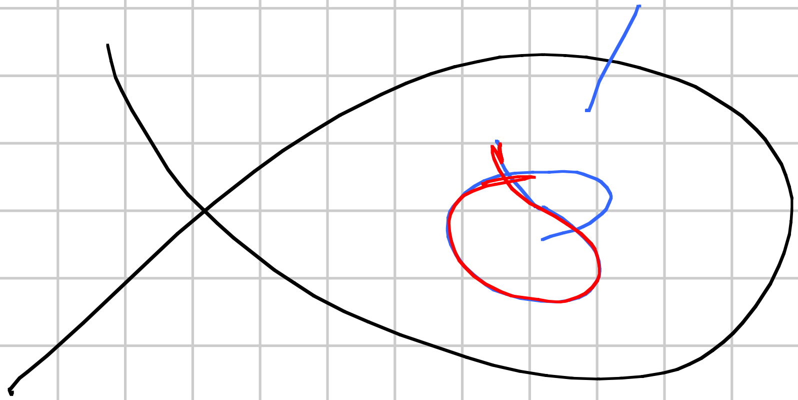
Se invece qualcosa entra:



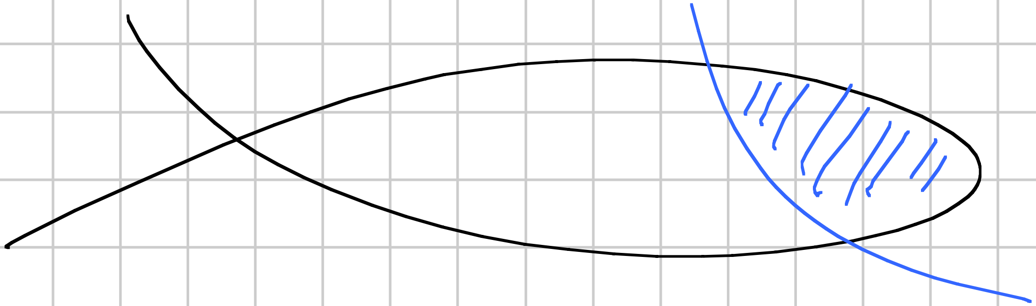
otteniamo che non può autointersecarsi prima di uscire:
altrimenti ho un raggio più piccolo.



o

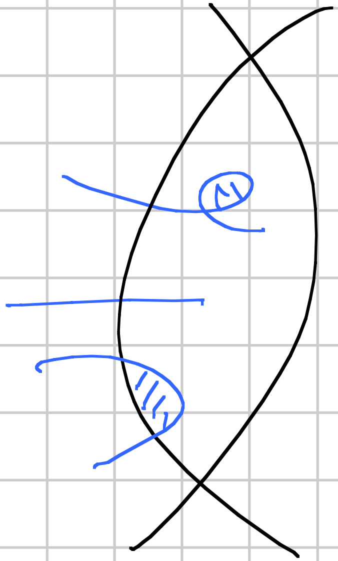


$\Rightarrow$ esce direttamente



NO

Se è bipolare

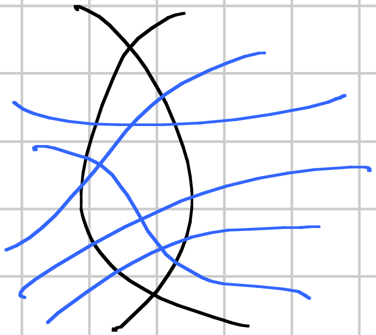


Se / entra

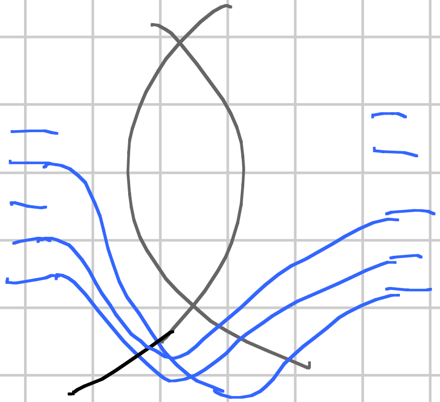
→ non può autointersecarsi prima di uscire (se no monogono)

→ deve uscire dall'altra parte (se no bigono)

Dunque



$\overrightarrow{R_{III}}$



$\overrightarrow{R_{II}} \downarrow$

$\boxed{4}$

Claim 5: W_{k+1} indipendente dall'ordine delle componenti.

Per induzione sulla minima lunghezza m di una successione di mosse come nel claim 4.

$m = 0 \Rightarrow L = \bigcirc$ ✓

$m > 0$; esaminiamo prima mossa di una successione di m come nel Claim 4.

Dividiamo a seconda del tipo:

$\boxed{S}$ $L_+ \longrightarrow L_-$

$$W_{k+1}(L_+, b) = W_{k+1}(L_-, b) \circ W_k(L_0)$$
$$W_{k+1}(L_+, b') = W_{k+1}(L_-, b') \circ W_k(L_0)$$

Per L_- m vale $m-1$ $\longrightarrow$ ok.

$\boxed{U}$ $L \rightsquigarrow \tilde{L}$ cioè $\tilde{L} = L \setminus$ (una corpo senza incroci).

L'ordine b su L induce $\tilde b$ su $\tilde L$ per restrizione.

$W_{k+1}(\tilde L, \tilde b)$ indip. da $\tilde b$ per induz. su m

Il calcolo di $W_{k+1}(\tilde L, \tilde b)$ coinvolge esattamente

e nello stesso ordine le stesse relazioni (S) che per $W_{k+1}(L, b)$

salvo che nelle formule finali (tutti incroci buoni) per L

c'è una compo. senza incroci in più.

Per $\tilde L$ non conta $\tilde b$ "$\Longrightarrow$" per L non conta b.

$\boxed{R_I \downarrow,\ R_{II} \downarrow,\ R_{III}}$ $\qquad L \rightsquigarrow \tilde{L}$

So pié $\quad W_{k+1}(L, b) = W_{k+1}(\tilde{L}, \tilde{b})$

per induz su m, $\tilde{b}$ non conta $\Rightarrow$ ok. "$\boxtimes$"

Prop: Se A è anello commut. con 1, $a_1, \alpha, \beta \in A$, α, β invertibili allora $(A, \circ, /, a_m)$ soddisfa ipotesi del Teo:

$$x \circ y = \alpha x + \beta y$$

$$x / y = \alpha^{-1} x - \alpha^{-1} \beta y$$

$$a_n = \left(\beta^{-1}(1-\alpha)\right)^{n-1} a_1$$

Dim: (1) $(x \circ y)/y = x$;

$$\alpha^{-1}(x \circ y) - \alpha^{-1}\beta y = \alpha^{-1}(\alpha x + \beta y) - \alpha^{-1}\beta y = x \quad \checkmark$$

$(x/y) \circ y = x$ simile

(2) $a_n \circ a_{n+1} = a_n / a_{n+1} = a_n$

$$\alpha^{-1}\left(\beta^{-1}(1-\alpha)\right)^{n-1} a_1 - \alpha^{-1}\beta\left(\beta^{-1}(1-\alpha)\right)^{n} a_1$$

$$= \left(\beta^{-1}(1-\alpha)\right)^{n-1}\left(\underbrace{\alpha^{-1} - \alpha^{-1}\beta\cdot\beta^{-1}(1-\alpha)}_{1}\right) a_1$$

...

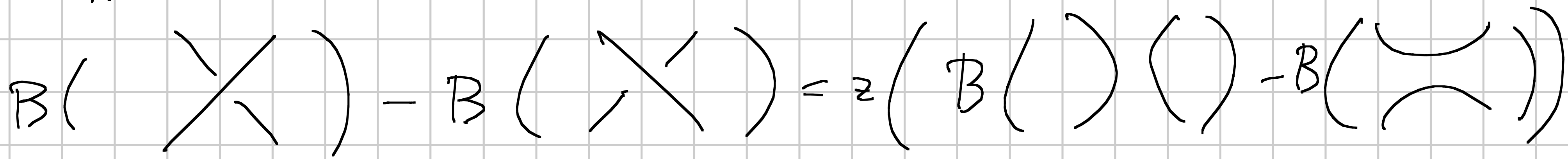
(5) $$(x/y)\circ(z/w) = (x\circ z)/(y\circ w)$$

$$(x/y)\circ(z/w) = \alpha(x/y) + \beta(z/w) = \alpha(\alpha^{-1}x - \alpha^{-1}\beta y) + \beta(\alpha^{-1}z - \alpha^{-1}\beta w) = x - \beta y + \alpha^{-1}\beta z - \alpha^{-1}\beta^2 w$$

$$(x\circ z)/(y\circ w) = \alpha^{-1}(x\circ z) - \alpha^{-1}\beta(y\circ w) = \alpha^{-1}(\alpha x + \beta z) - \alpha^{-1}\beta(\alpha y + \beta w) = x + \alpha^{-1}\beta z - \beta y - \alpha^{-1}\beta^2 w.$$

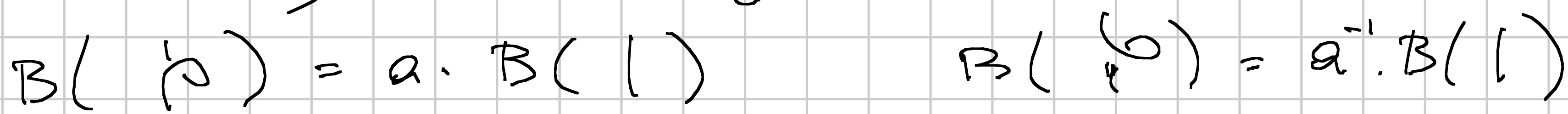
$\blacksquare$

<u>Fatto</u>: con tecnica analoga a quella del teorema si prova che esiste unico invariante $B : \{\text{link con framing}\} \to \mathbb{Z}[a^{\pm 1}, z^{\pm 1}]$ (Kauffman bracket a 2 variabili) t.c.

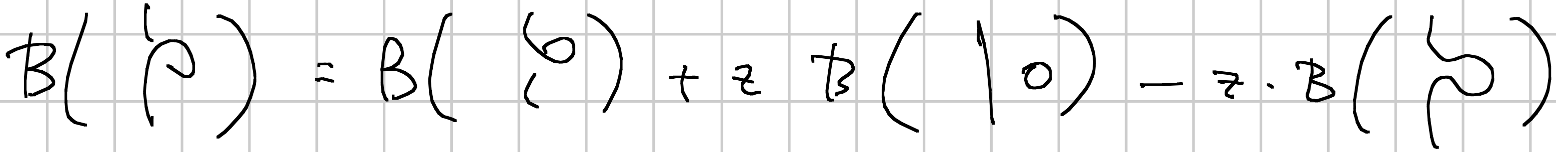


$$B(D_1 \sqcup D_2) = B(D_1) \cdot B(D_2)$$

$$B(\bigcirc) = 1 + \frac{a - a^{-1}}{z}$$



Provo solo che le ultime due si ricavano l'una dall'altra sapendo le precedenti.



$$= a^{-1} \cdot B(\,|\,) + z \cdot B(\,|\,) \cdot \left(1 + \frac{a - a^{-1}}{z}\right) - z\, B(\,|\,)$$

$$= B(\,|\,)\left(a^{-1} + z + a - a^{-1} - z\right) = a \cdot B(\,|\,)$$

Invarianti via Teo + Prop:

HOMFLY-PT

H

$$A = \mathbb{Z}[l^{\pm 1}, m^{\pm 1}]$$

$$\alpha = -\frac{1}{l^2} \qquad \beta = -\frac{m}{l} \qquad a_1 = 1$$

$$a_n = \left(-\frac{l}{m}\left(1+\frac{1}{l^2}\right)\right)^{n-1} = \left(-\frac{l}{m}+\frac{1}{lm}\right)^{n-1}$$

(S) $$H(L_+) = -\frac{1}{l^2} H(L_-) - \frac{m}{l} H(L_0)$$

$$\Rightarrow \quad l \cdot H(L_+) + \frac{1}{l} H(L_-) + m H(L_0) = 0$$

JONES

$A = \mathbb{Z}(q^{\pm 1/2})$ a priori

$\alpha = q^2 \qquad \beta = q^{3/2} - q^{1/2} \qquad a_1 = 1$

$$a_n = \left(\frac{1-q^2}{q^{3/2}-q^{1/2}}\right)^{n-1} = \left(\frac{(1-q)(1+q)}{q^{1/2}(q-1)}\right)^{n-1} = \left(-q^{1/2}-q^{-1/2}\right)^{n-1}$$

$$= \left(-A^2 - A^{-2}\right)^{n-1} \Big|_{A = q^{-1/4}}$$

valore noto di V su

$O \; \cdots \; O$

$$V(L_+) = q^2\, V(L_-) + (q^{3/2} - q^{1/2})\, V(L_0)$$

$$\Longrightarrow q^{-1} V(L_+) - q V(L_-) + (q^{1/2} - q^{-1/2}) V(L_0) = 0$$

skein note per V

$(\Longrightarrow \mp V)$

inoltre questo implica che V ha valori in $\mathbb{Z}[q^{\pm 1/2}]$

<u>Alexander - Conway</u> $\qquad A = \mathbb{Z}(t^{\pm 1/2})$ a priori

Δ

$\alpha = 1 \qquad \beta = t^{1/2} - t^{-1/2} \qquad a_1 = 1 \qquad a_n = 0 \ \forall n \geq 2$

$$\Delta(L_+) - \Delta(L_-) = (t^{1/2} - t^{-1/2})\,\Delta(L_0)$$

$(\Rightarrow$ valori in $\mathbb{Z}[t^{\pm 1/2}])$

<u>Oss</u>: esistono altre costruzioni di H e V, Δ si ricavano dalli:

$$V = H\Big|_{\substack{l = i \cdot q^{-1} \\ m = i(q^{-1/2} - q^{1/2})}} \qquad \Delta = H\Big|_{\substack{l = i \\ m = i(t^{1/2} - t^{-1/2})}}$$

Esercizio: provare che tali $H/\ldots$ soddisfano le (U) e (S) di τ e Δ.