Geometrie 25/5/17

$\boxed{36}$

1. Calcolare angolo tra $\begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix}$ e $\underbrace{\begin{pmatrix} -3 \\ 2 \\ 1 \end{pmatrix} \wedge \begin{pmatrix} 6 \\ -2 \\ -7 \end{pmatrix}}_{= \begin{pmatrix} 4 \\ 9 \\ -6 \end{pmatrix}}$

$$\arccos\left(\frac{\left\langle \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} \middle| \begin{pmatrix} 4 \\ 9 \\ -6 \end{pmatrix} \right\rangle}{\left\| \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} \right\| \cdot \left\| \begin{pmatrix} 4 \\ 9 \\ -6 \end{pmatrix} \right\|} \right)$$

$$= \frac{12 + 18 + 6}{\sqrt{9+4+1} \cdot \sqrt{16+81+36}} = \dots$$

2. Per quali $k \in \mathbb{R}$ la $\begin{pmatrix} -11 & 2+k \\ 2k-1 & 1+k^4 \end{pmatrix}$

$$\underbrace{\text{ammette base ortonormale di autovettori}}_{\Vert} \; ?$$

Te simmetrica

Se $2k-1 = 2+k$ cioè se $k=3$

3. Esiste $X \in M_{3\times 3}(\mathbb{C})$ antihermitiana invertibile?

Sappiamo: $X \in M_{3\times 3}(\mathbb{R})$ antisimm ha autoval $0, \pm i\alpha$. Dunque su $\mathbb{R}$ la risposta sarebbe stata no.

Su $\mathbb{C}$:

$$\begin{pmatrix} i & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & i \end{pmatrix}$$

Sì.

4. Per quali $k \in \mathbb{R}$ i punti di $\mathbb{P}^2(\mathbb{R})$

$$[\,2k-1 : 1+k : 1-2k\,] \quad e \quad [\,k-1 : 3-k : 2k-5\,]$$

coincidono?

$\|$

i vettori $\begin{pmatrix} 2k-1 \\ 1+k \\ 1-2k \end{pmatrix}$ e $\begin{pmatrix} k-1 \\ 3-k \\ 2k-5 \end{pmatrix}$ sono proporzionali.

I svolg

$$6k-3-2k^2$$
$$+k$$
$$-k+1-k^2$$

$$\frac{+k}{7k-2-3k^2}$$

$$3k^2-7k+2 = (3k-1)(k-2)$$

$k = 1/3 \qquad k = 2$

$k = 1/3$: $\begin{pmatrix} -1/3 \\ 4/3 \\ 1/3 \end{pmatrix} \quad \begin{pmatrix} -2/3 \\ 8/3 \\ -13/3 \end{pmatrix}$ No

$k = 2$: $\begin{pmatrix} 3 \\ 3 \\ -3 \end{pmatrix} \quad \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$ Sì

II Stolp :

$$\begin{pmatrix} 2k-1 \\ 1+k \\ 1-2k \end{pmatrix} \quad \begin{pmatrix} k-1 \\ 3-k \\ 2k-5 \end{pmatrix}$$

$2^e + 3^e$ rije :

$$\begin{array}{l} 2k - 5 + 2k^2 \\ -5k \quad -3 - 2k^2 \\ +k \\ +6k \\ \hline 4k - 8 \end{array} \quad \longrightarrow \quad k = 2$$

Sostituisco ...

5. Determinare tipo affine di

$$x^2+5y^2-3z^2+6xy+2xz-2yz+2z=0$$

$$\left(\begin{array}{ccc|c} 1 & 3 & 1 & 0 \\ 3 & 5 & -1 & 0 \\ 1 & -1 & -3 & 1 \\ \hline 0 & 0 & 1 & 0 \end{array}\right)$$

$d_2 < 0$ $\qquad$ $d_3 = -15 + 27 - 3 - 3 - 5 - 1 = 0$

Se $d_4 \neq 0$ è un par. iperb.

(se $d_4 = 0$ è degenere) :

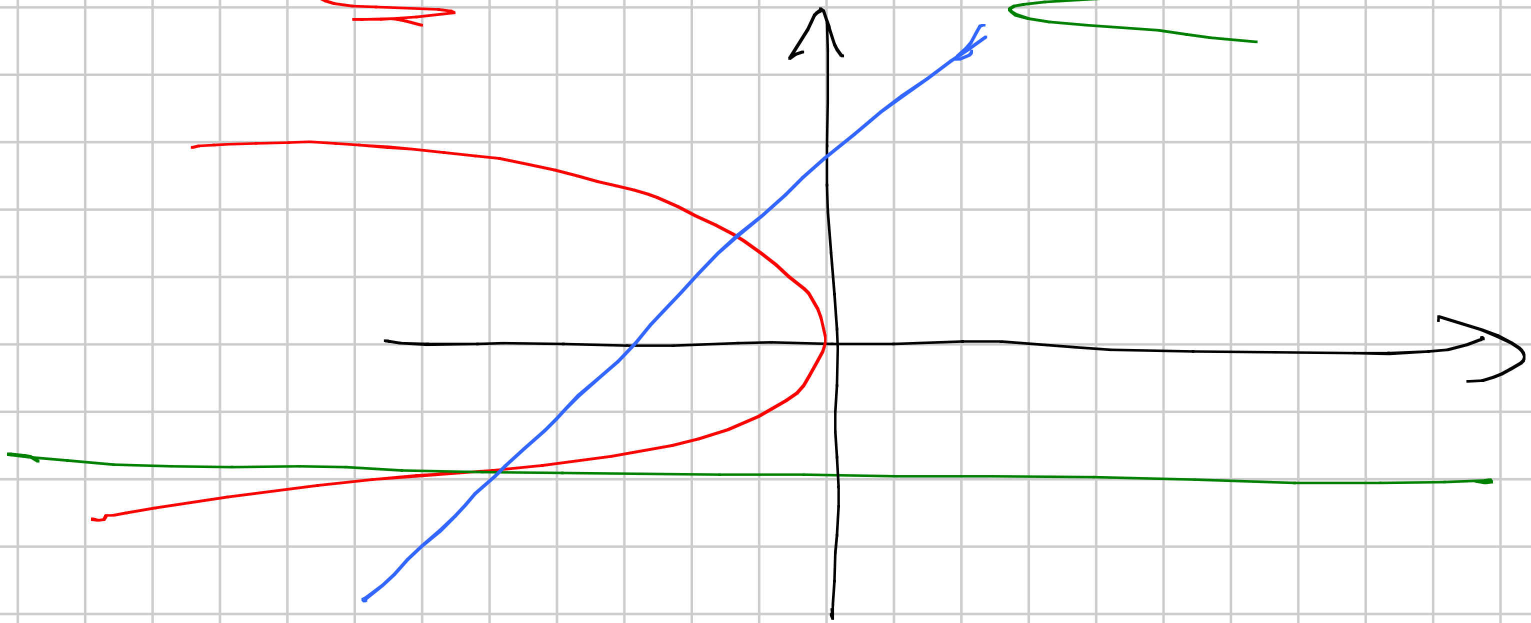
$$\det\begin{pmatrix} 1 & 3 & 1 & 0 \\ 3 & 5 & -1 & 0 \\ 1 & -7 & -3 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \neq 0 .$$

✓

6. Trovare pt. all'∞ di

$$A = \{ (x,y) \in \mathbb{R}^2 : (x+y^2)(y+\sqrt[3]{5})(2x-3y+11) = 0 \}$$

$$\overline{A} = \{x+y^2=0\} \cup \{y+\sqrt[3]{5}=0\} \cup \{2x-3y+1=0\}$$

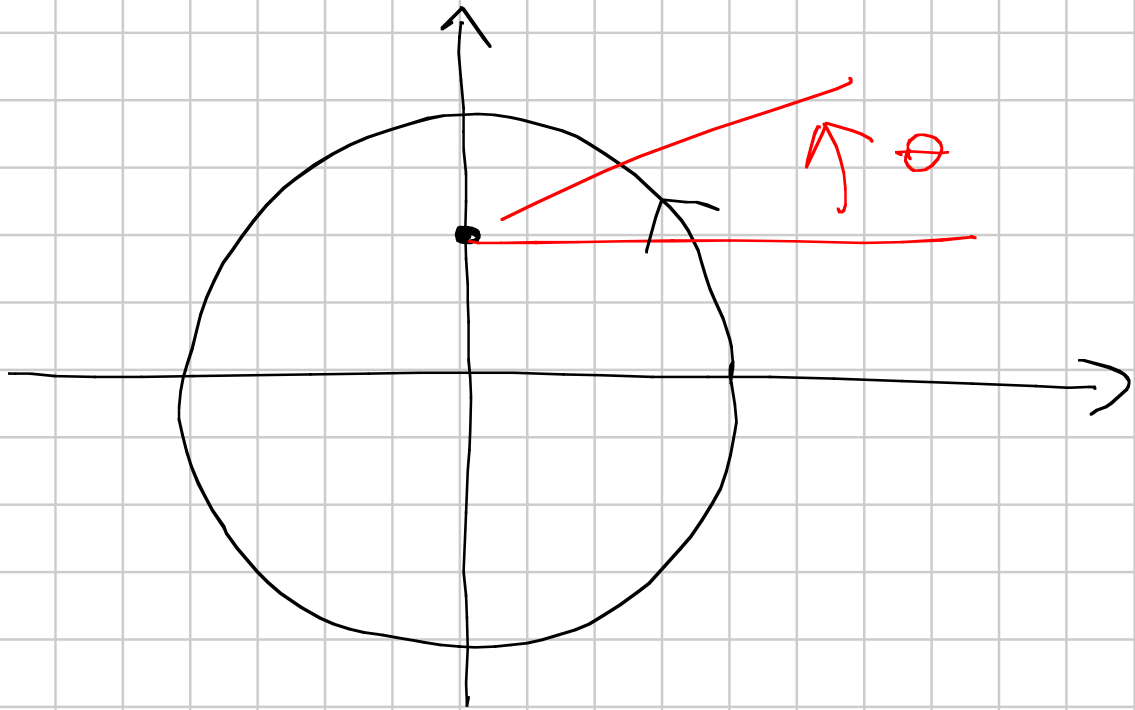


$[1:0:0]$ $\qquad$ $[1:0:0]$ $\qquad$ $[3:2:0]$

$\{y^2=0\} \cap \{z=0\}$ $\qquad$ $\{y=0\} \cap \{z=0\}$ $\qquad$ $\{2x-3y=0\} \cap \{z=0\}$

7. Esibire 1-forma chiusa ω definita su $\mathbb{R}^2 \setminus \{(0,1)\}$ t.c. $\displaystyle\int_{\partial\Delta(0,2)} \omega = -1$.

$$-\frac{1}{2\pi}\underbrace{\frac{-(y-1)\,dx + x\,dy}{x^2+(y-1)^2}}_{d\theta}$$



$\boxed{37}$ 1. $A = \begin{pmatrix} 1 & i \\ -i & 3 \end{pmatrix}$

Ortonormalizzare (ie_1, e_2) risp. a $\langle \cdot | \cdot \rangle_A$.

(Verifico che $\langle \cdot | \cdot \rangle_A$ è prod. scal. herm.:

A hermitiana: ✓

$$\lambda_1, \lambda_2 > 0 \iff d_1 > 0, \; d_2 > 0 \qquad \checkmark \qquad)$$

$$v_1 = \frac{ie_1}{\|ie_1\|_A} \qquad \|ie_1\|_A = (-i, 0) \cdot \begin{pmatrix} 1 & i \\ -i & 3 \end{pmatrix} \begin{pmatrix} i \\ 0 \end{pmatrix}$$

$$= 1$$

$$v_1 = ie_1$$

$$\tilde{v}_2 = e_2 - \langle e_2 | ie_1 \rangle_A \cdot ie_1$$

$$= \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & i \\ -i & 3 \end{pmatrix} \begin{pmatrix} i \\ 0 \end{pmatrix} \cdot \begin{pmatrix} i \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} \cdots \\ 1 \end{pmatrix} \cdot \begin{pmatrix} i \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} -i \\ 1 \end{pmatrix}$$

$$v_2 = \frac{\tilde{v}_2}{\|\tilde{v}_2\|_A} \qquad \|\tilde{v}_2\|_A = \begin{pmatrix} i & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & i \\ -i & 3 \end{pmatrix} \begin{pmatrix} -i \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 2 \end{pmatrix} = 2$$

$$v_2 = \frac{1}{2} \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

2. Trovare autovalori di $\begin{pmatrix} -2 & 1 \\ 6 & 3 \end{pmatrix}$ e base di $\mathbb{R}^2$ che la diagonalizza.

$$\lambda_1 + \lambda_2 = \operatorname{tr} = 1$$

$$\lambda_1 \cdot \lambda_2 = \det = -12$$

$$\lambda_1 = 4 \qquad \lambda_2 = -3$$

$$v_1 : \quad \begin{cases} -2x + y = 4x \\ \cancel{6x + 3y = 4y} \end{cases} \qquad \begin{cases} -6x + y = 0 \\ \cancel{6x - y = 0} \end{cases}$$

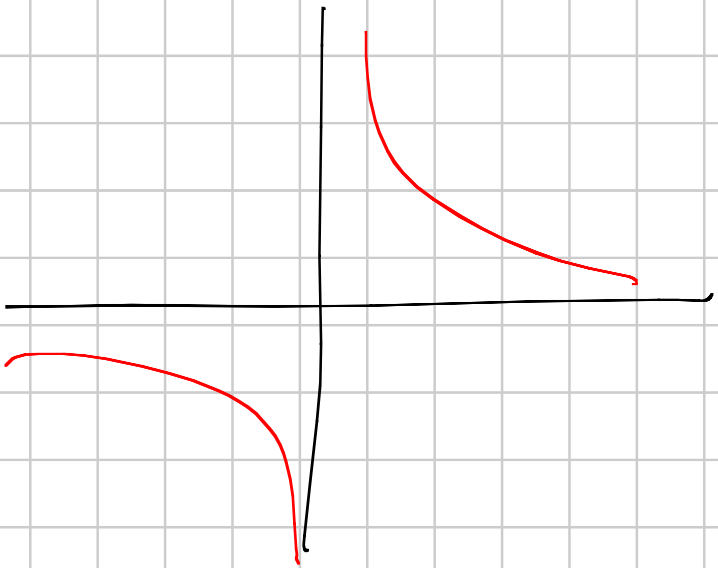
$$v_1 = \begin{pmatrix} 1 \\ 6 \end{pmatrix}$$

v_2 : $\quad -2x+y=-3x \qquad x+y=0 \quad v_2=\begin{pmatrix}1\\-1\end{pmatrix}$

3. Se $\mathbb{R}^2=\{[x:y:z]\in\mathbb{P}^2(\mathbb{R}) : z=1\}$

cos'è $\mathbb{R}^2\cap\{[t^2:1:t] : t\in\mathbb{R}\}$?

$$E^{c} : \left\{ \left[t : \frac{1}{t} : 1 \right] : t \in \mathbb{R} \setminus \{0\} \right\}$$



$$= \left\{ (x,y) \in \mathbb{R}^2 : xy = 1 \right\}$$

4. Trovare tipo affine di $x^2 - 3xy + 2y^2 - \sqrt{3}x + 11 = 0$

$$\begin{pmatrix} 1 & -3/2 & -\sqrt{3}/2 \\ -3/2 & 2 & 0 \\ -\sqrt{3}/2 & 0 & 11 \end{pmatrix}$$

$d_2 = 2 - \frac{9}{4} < 0$. Se $d_3 \neq 0$ è iperbole.

$$d_3 = 22 - \frac{3}{2} - \frac{99}{4} = \frac{88-6-99}{4} \neq 0.$$

5. Trovare tipo affine di $2xy + y^2 - z^2 - 4x + 2 = 0$.

$$\begin{pmatrix} 0 & 1 & 0 & -2 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ -2 & 0 & 0 & 2 \end{pmatrix}$$

$d_2 < 0 \qquad d_3 > 0$

autoval delle parte quad: $- \ - \ +$

$$d_4 = -2 \det \begin{pmatrix} 0 & 1 & -1 \\ 1 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix} = -2(-2-1) > 0$$

$\Rightarrow$ autoval di A $\quad - \ - \ + \ +$

$$-x^2 - y^2 + z^2 + 1 = 0$$

$$x^2 + y^2 = 1 + z^2$$

iperboloide 1 falda
(iperbolico)

6. Troverai autovalori di $H_{(0,0)} f$

$$f(x,y) = y^2 + e^{2x + \sin(y)}$$

$$\frac{\partial f}{\partial x} = 2e^{2x+\sin(y)} \qquad \frac{\partial f}{\partial y} = 2y + \cos(y) \cdot e^{2x + \sin(y)}$$

$$\frac{\partial^2 f}{\partial x^2} = 4e^{\cdots} \qquad \frac{\partial^2 f}{\partial x \partial y} = 2\cos(y) \cdot e^{\cdots}$$

$$\frac{\partial^2 f}{\partial y^2} = 2 - \sin(y)\, e^{\cdots} + \cos^2(y) \cdot e^{\cdots}$$

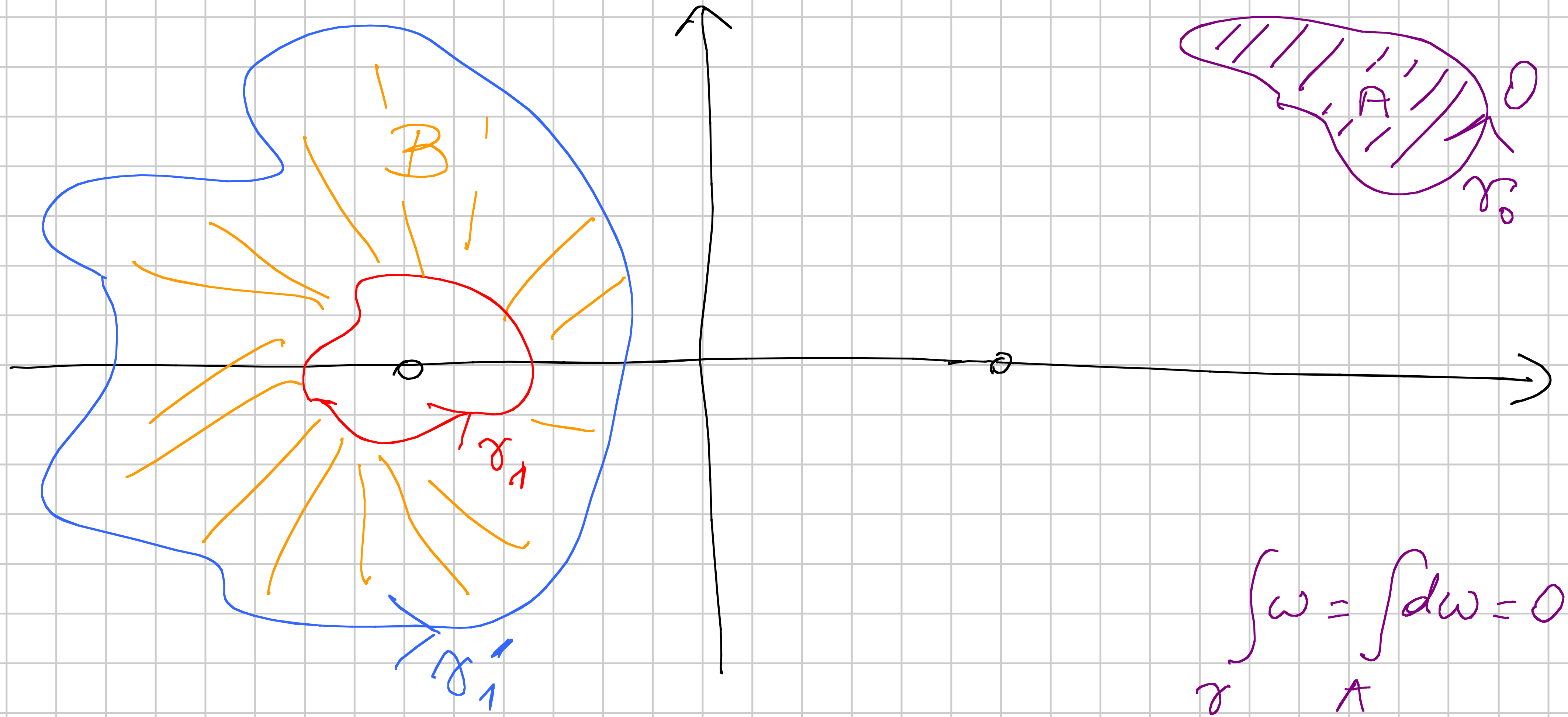
$$H_{(0,0)} f = \begin{pmatrix} 4 & 2 \\ 2 & 3 \end{pmatrix}$$

$$t^2 - 7t + 8$$

$$\lambda_{1,2} = \frac{7 \pm \sqrt{49 - 32}}{2} = \frac{7 \pm \sqrt{17}}{2}$$

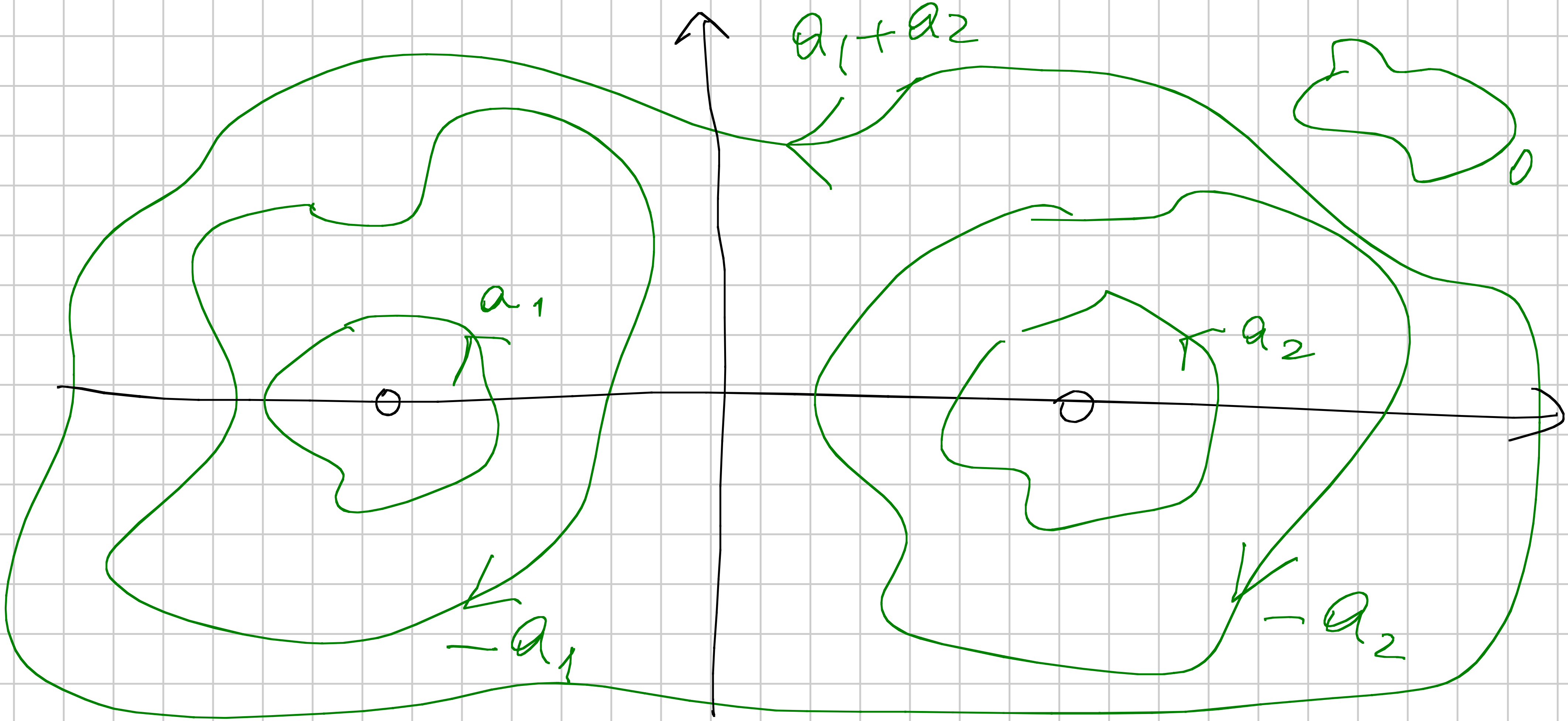
7. Sia ω una forma chiusa su $\mathbb{R}^2 \setminus \{(-1,0),(1,0)\}$.

Dire quanti valori diversi può assumere $\int_\gamma \omega$ con γ curva semplice chiusa orientata.



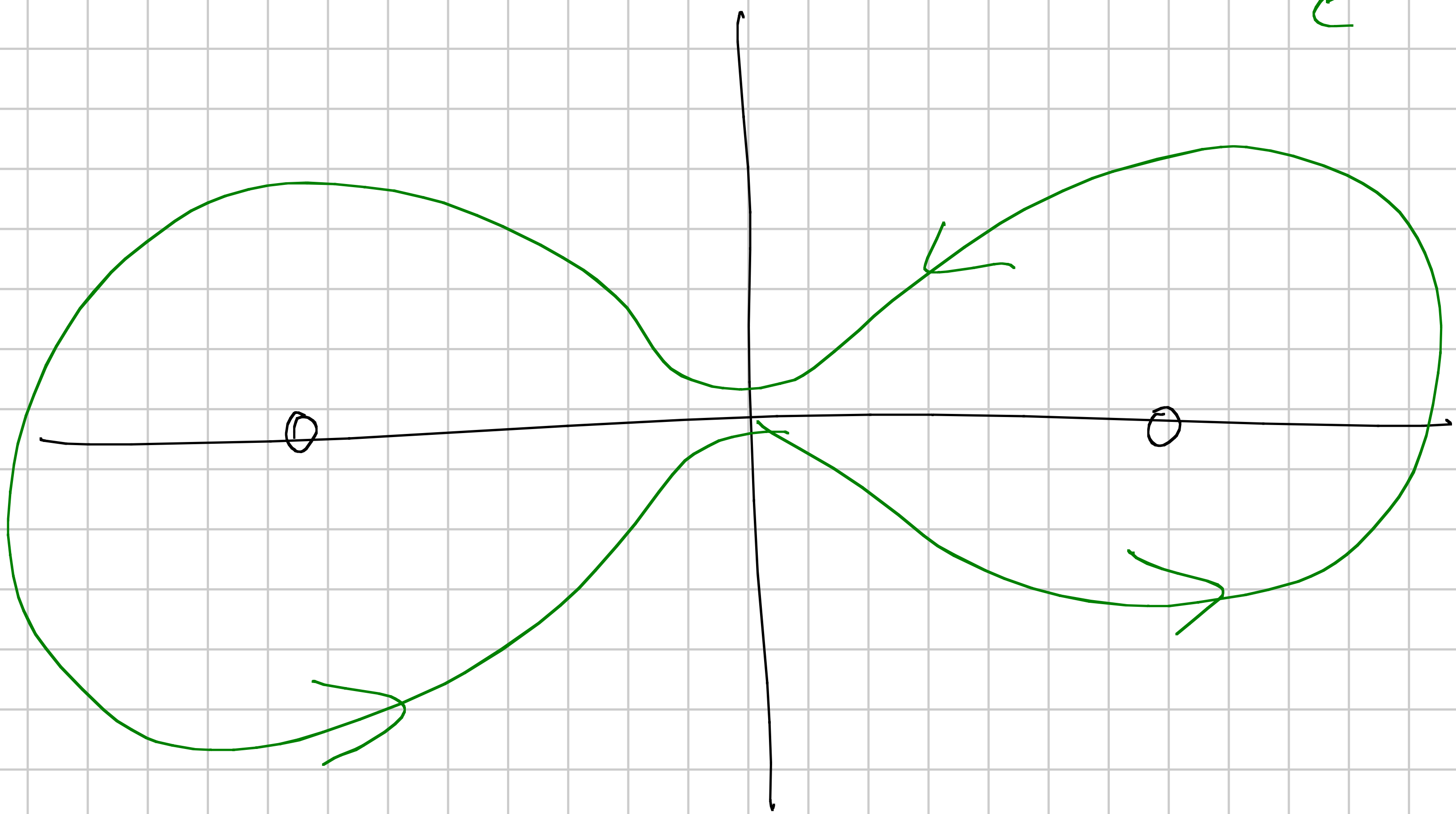
$$\int_\gamma \omega = \int_A d\omega = 0$$

$$0 = \int_B d\omega = \int_{\gamma_1'} \omega - \int_{\gamma_1} \omega \Rightarrow \int_{\gamma_1'} \omega = \int_{\gamma_1} \omega$$



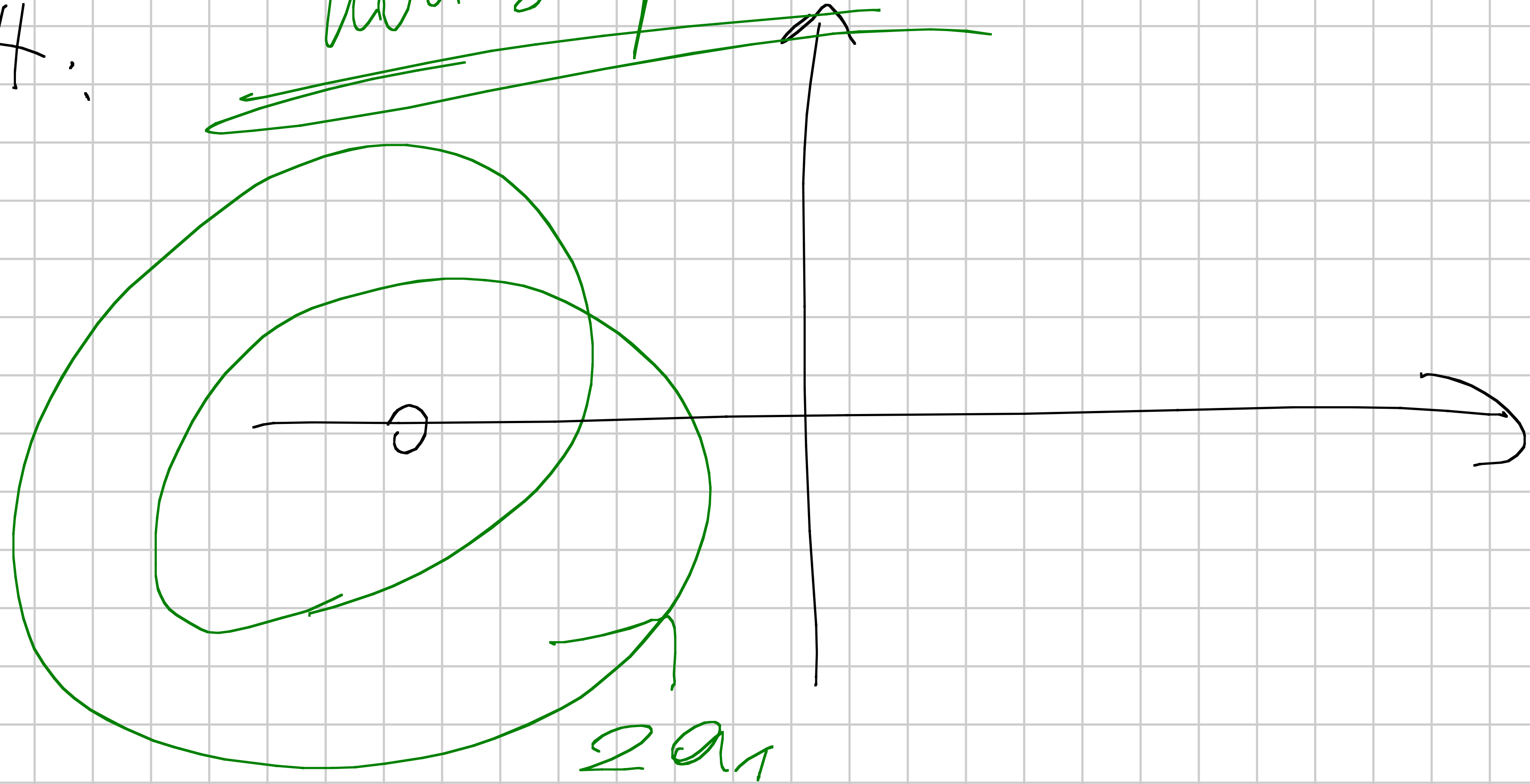
$a_1 + a_2$
a_1
$-a_1$
a_2
$-a_2$
0

$C - q_1 - q_2$



Att:

non semplice



$\Longrightarrow$ al massimo

$0, \pm a_1, \pm a_2, \pm(a_1 + a_2)$

7 valori diversi.

$\boxed{39}$ 1 Trovare autoval. di $\begin{pmatrix} 2+i & 4i-3 \\ i & -1 \end{pmatrix}$

e una base di $\mathbb{C}^2$ che la diagonalizza.

$$\mathrm{tr} = 1+i$$
$$\det = -2-i+4+3i = 2+2i$$

$$t^2-(1+i)\,t+2+2i$$

$$\Delta = (1+i)^2 - 8(1+i)$$
$$= 1+2i-1-8-8i$$
$$= -8-6i$$

cerco $\sqrt{\Delta} = a + i b$

$$\begin{cases} a^2 - b^2 = -8 \\ ab = -3 \end{cases} \qquad a = 1 \qquad b = -3$$

$$x_{1,2} = \frac{1 + i \pm (1 - 3i)}{2} = \begin{cases} 1 - i \\ 2i \end{cases}$$

$$r_1 : \begin{cases} \cancel{(2+i)x + (4i-3)y = (1-i)x} \\ i\,x - y = (1-i)y \end{cases}$$

$$\begin{cases} \cancel{(1+2i)x + (4i-3)y = 0} \\ i\,x + (i-2)y = 0 \end{cases}$$

$$\det\begin{pmatrix} 1+2i & 4i-3 \\ i & i-2 \end{pmatrix} = i - 2 - 4i - 2 + 3i + 4 = 0 \quad ✓$$

$$v_1 = \begin{pmatrix} 2-i \\ i \end{pmatrix}$$

v_2:

$$ix - y = 2iy$$

$$ix - (1+2i)y = 0 \qquad v_2 = \begin{pmatrix} 1+2i \\ i \end{pmatrix}$$

2. Trovare i vett. di $\mathbb{C}^2$ unitari, ortog. a $\begin{pmatrix} 1+2i \\ 3-i \end{pmatrix}$ con I coord. in $i\mathbb{R}$.

$$\left\langle \begin{pmatrix} i \\ z \end{pmatrix} \middle| \begin{pmatrix} 1+2i \\ 3-i \end{pmatrix} \right\rangle = 0$$

$$(1-2i)i + (3+i)z = 0$$

$$z = -\frac{2+i}{3+i} = -\frac{(2+i)(3-i)}{10}$$

$$= -\frac{1}{10}(6-2i+3i+1) = -\frac{1}{10}(7+i)$$

$$\pm \frac{1}{5\sqrt{6}} \begin{pmatrix} -10i \\ 7+i \end{pmatrix}$$

3. Per quali $z \in \mathbb{C}$ la $\begin{pmatrix} 1+z & z-i \\ z+i & z \end{pmatrix}$

$$\underbrace{\text{ammette base ortonormale di autovettori?}}_{\Updownarrow}$$

è normale, cioè

$$\begin{pmatrix} 1+z & z-i \\ z+i & z \end{pmatrix}\begin{pmatrix} 1+\bar{z} & \bar{z}-i \\ \bar{z}+i & \bar{z} \end{pmatrix}=\begin{pmatrix} 1+\bar{z} & \bar{z}-i \\ \bar{z}+i & \bar{z} \end{pmatrix}\begin{pmatrix} 1+z & z-i \\ z+i & z \end{pmatrix}$$

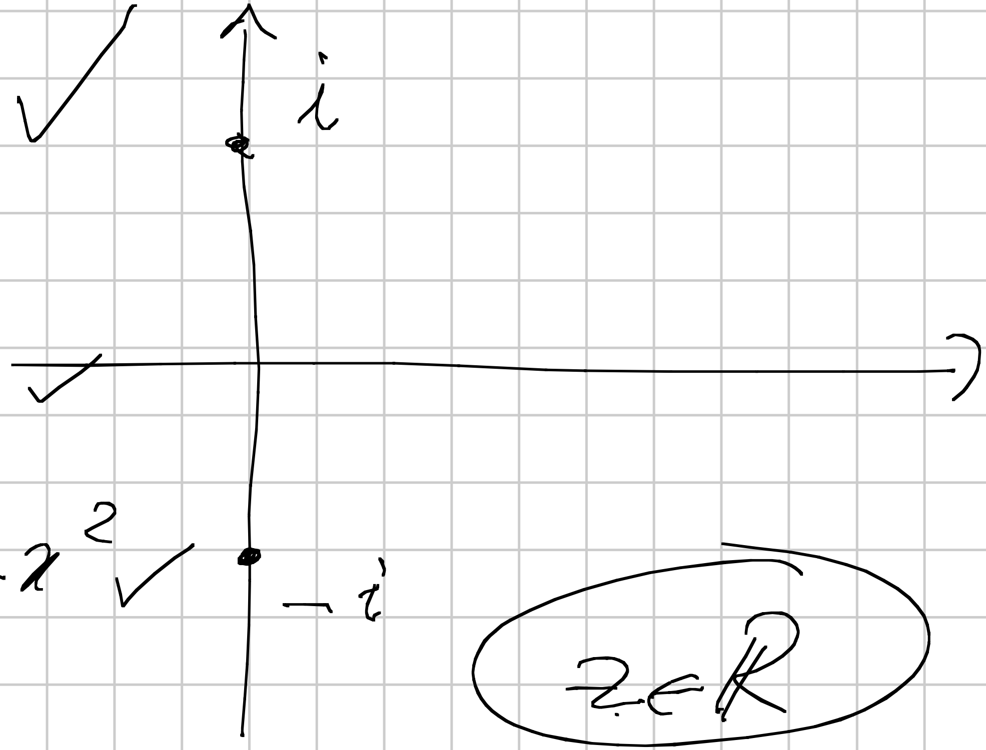
$$\cancel{|1+z|^2} + |z-i|^2 = \cancel{|1+z|^2} + |z+i|^2 \Longleftrightarrow z \in \mathbb{R}$$

$z = x$

$$(1+x)(x-i) + (x-i)x$$
$$= (1+x)(x-i) + (x-i)x \checkmark$$

$$(x+i)(x+i) + x(x+i)$$
$$= (x+i)(x+i) + x(x+i) \checkmark$$

$$(x+i)(x-i) + x^2 = (x+i)(x-i) + x^2 \checkmark$$



$z \in \mathbb{R}$

4. $C_1 = \{ [x_0 : x_1 : x_2] \in \mathbb{P}^2(\mathbb{R}) : x_0^2 = x_1 x_2 \}$.

$E_j = \{ [x_0 : x_1 : x_2] \in \mathbb{P}^2(\mathbb{R}) : x_j = 1 \}$

Trovare tipo affine di $C_1 \cap E_j$.

$C \cap E_0 : \quad x_1 x_2 = 1 \qquad$ iperbole

$C \cap E_1 \ : \quad x_2 = x_0^2 \qquad$ parabole

$C' \cap E_2 \ : \quad x_1 = x_0^2 \qquad$ parabole.