Geometria 18/5/17

$\omega = f(x,y)dx + f(x,y)dy$ su $\Omega \subset \mathbb{R}^2$

$\int_\alpha \omega$; ω esatta se $\omega = d\bar{U}$

$$dU = \frac{\partial U}{\partial x}dx + \frac{\partial \bar{U}}{\partial y}dy$$

($\bar{U}$ = potenziale ; se ω e' esatta rappresenta il lavoro di una forza conservativa).

<u>Fatto:</u> ω esatta $\Longrightarrow \omega$ chiusa, cioè

$$d\omega = 0$$
$$\shortparallel$$
$$\left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dx\,dy$$

infatti se $\omega = dU$ ho $f = \dfrac{\partial U}{\partial x}$, $g = \dfrac{\partial U}{\partial y}$

$$\Rightarrow \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} = \frac{\partial^2 U}{\partial x \partial y} - \frac{\partial^2 U}{\partial y \partial x} = 0 .$$

Prop: ω è esatta $\Longleftrightarrow$ $\int_\alpha \omega$ dipende solo

degli estremi per ogni

$\alpha : [a,b] \to \Omega$ _

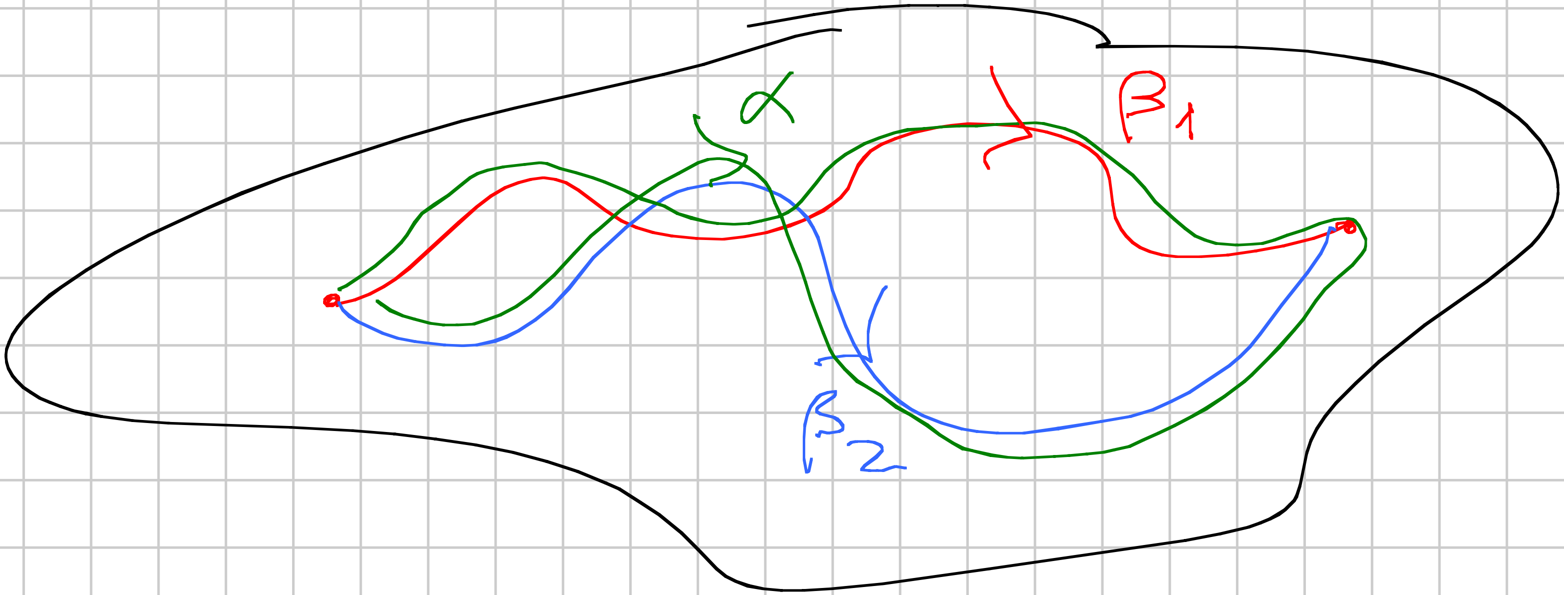
Cor: ω è esatta $\Longleftrightarrow$ $\int_\alpha \omega = 0$ $\forall \alpha : [a,b] \to \Omega$ curve chiuse

Dim: devo vedere

$$\int_\alpha \omega = 0 \ \forall \alpha \text{ chiusa} \Longleftrightarrow \int_\beta \omega = F(\text{estremi di } \beta) \ \forall \beta$$

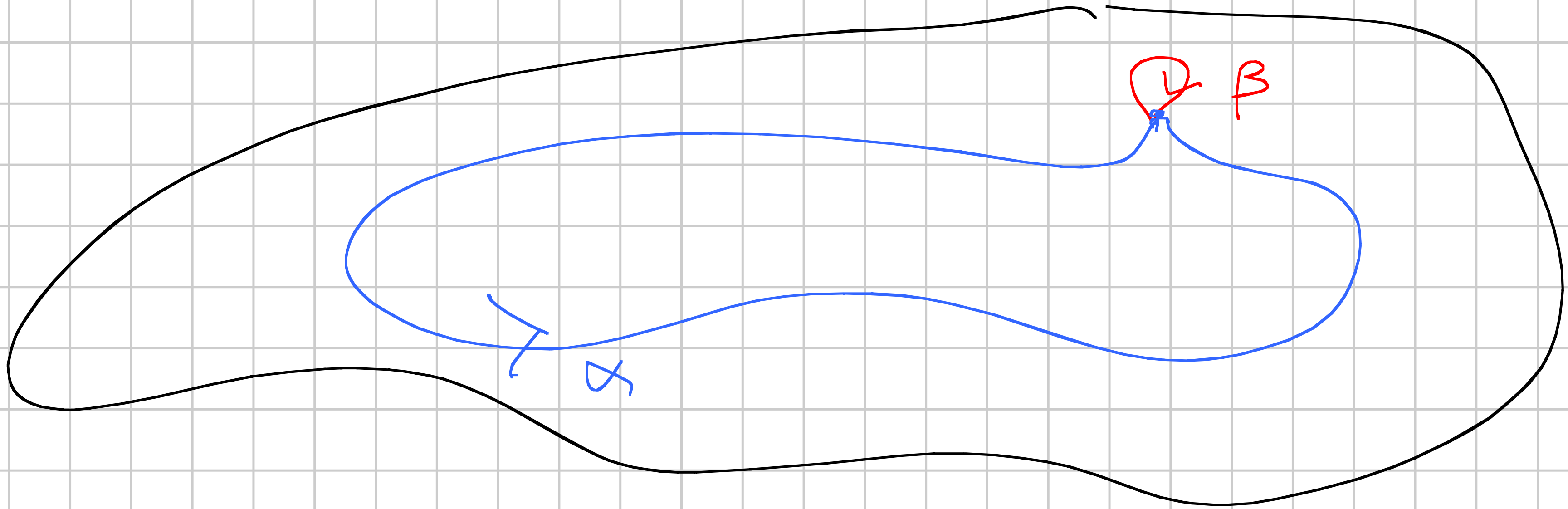
Infatti:

$\Rightarrow$



$$\int_{\alpha} \omega = 0 \implies \int_{\beta_1} \omega - \int_{\beta_2} \omega = 0 \quad \checkmark$$

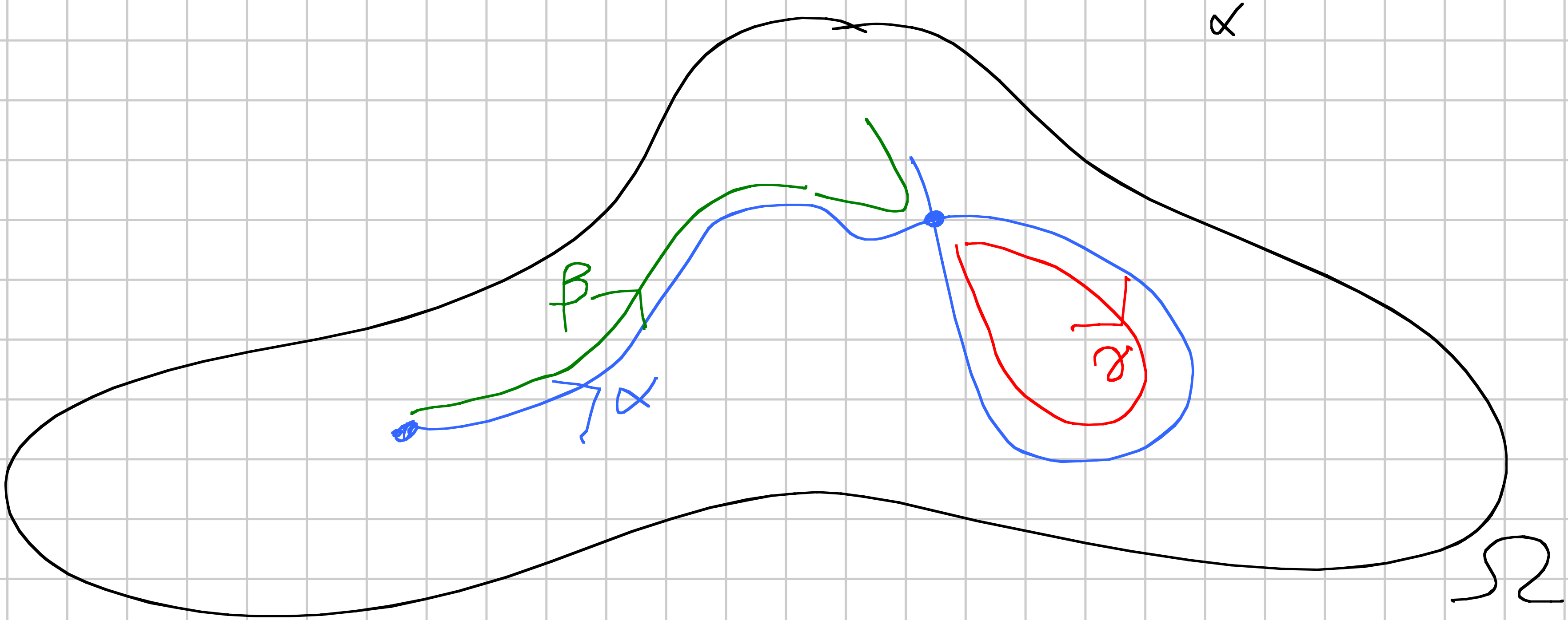
$\Leftarrow$



$$\int_\alpha \omega = \int_\beta \omega \xrightarrow[\beta \to pt]{} 0 .$$

$\blacksquare$

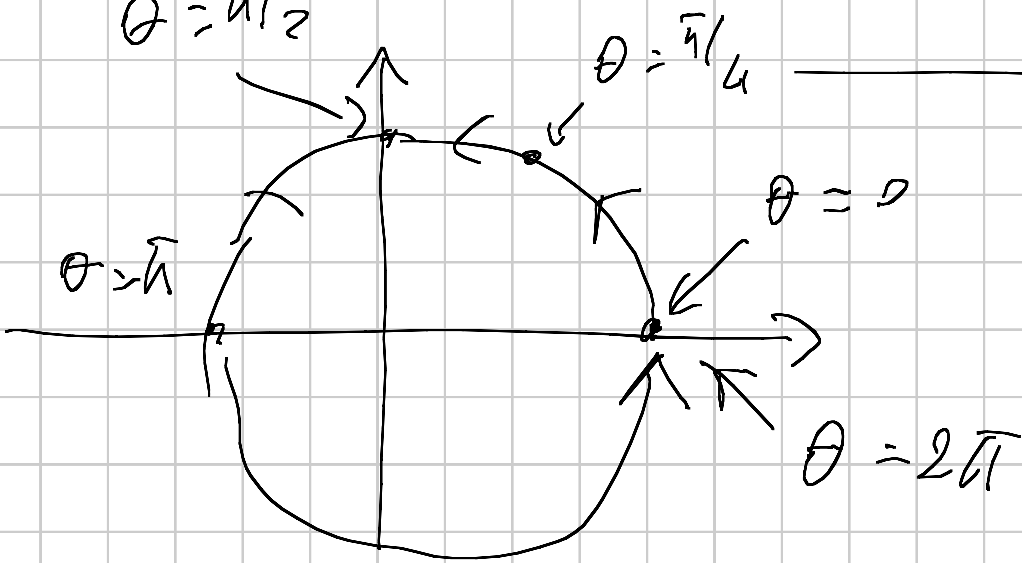
Oss: È equivalente dire che $\int_\alpha \omega = 0 \quad \forall \alpha$ semplice e chiusa

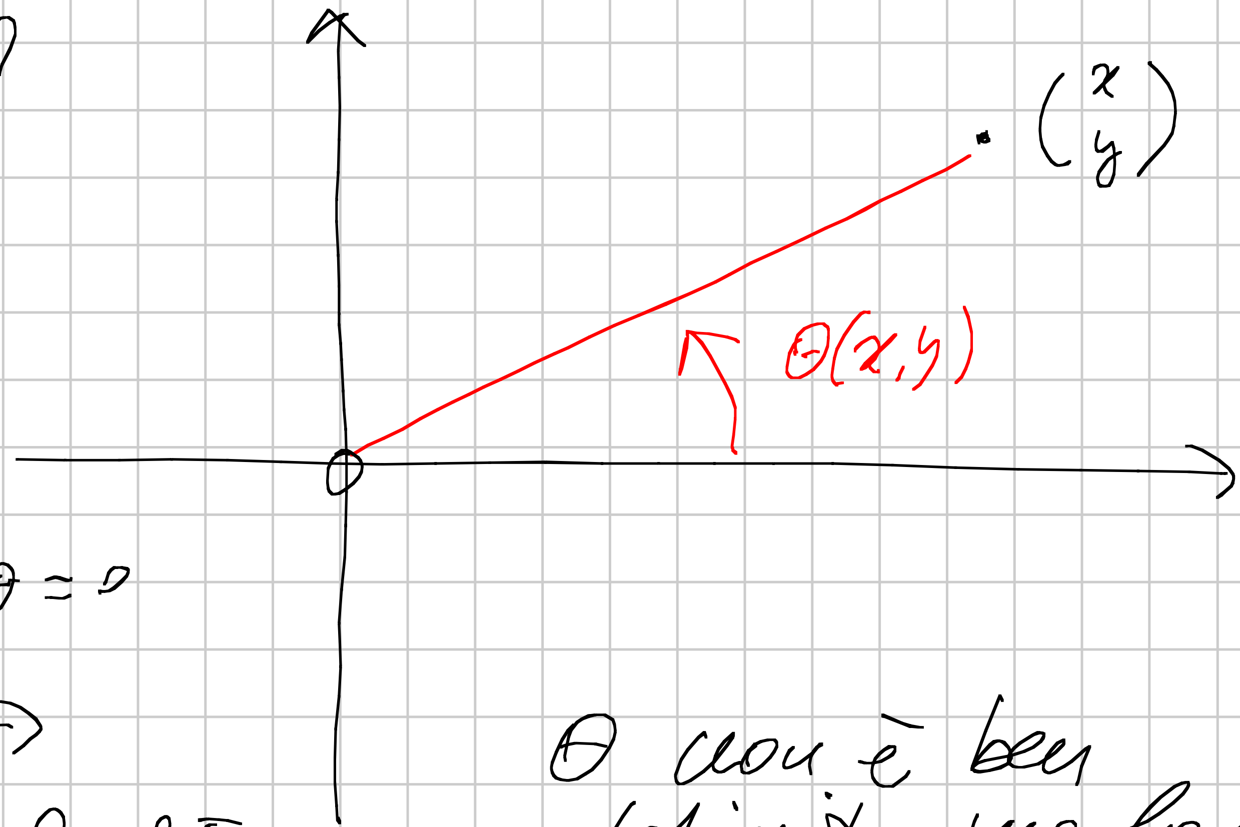


$$\int_\alpha \omega = \int_\beta \omega + \underbrace{\int_\gamma \omega}_{0} \Longrightarrow \text{mi riconduco a curve semplice.}$$

<u>Esempio</u> (forme chiuse non esatte).

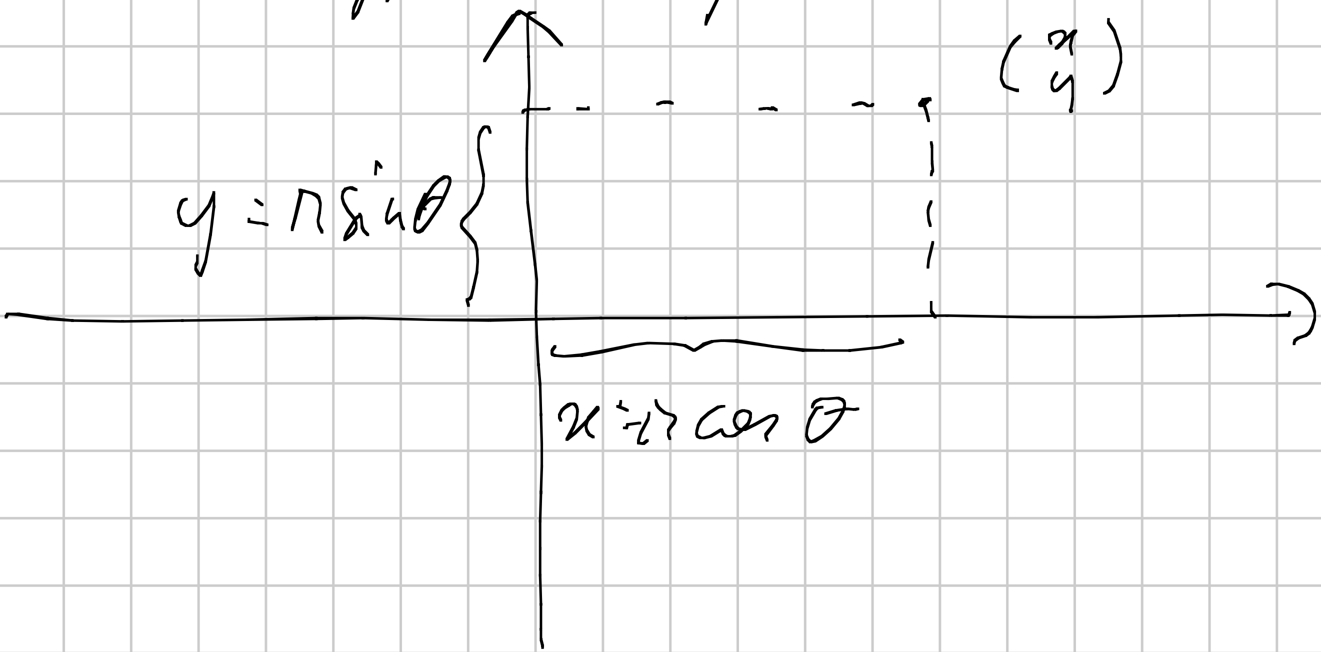
$$\Omega = \mathbb{R}^2 \setminus \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}$$

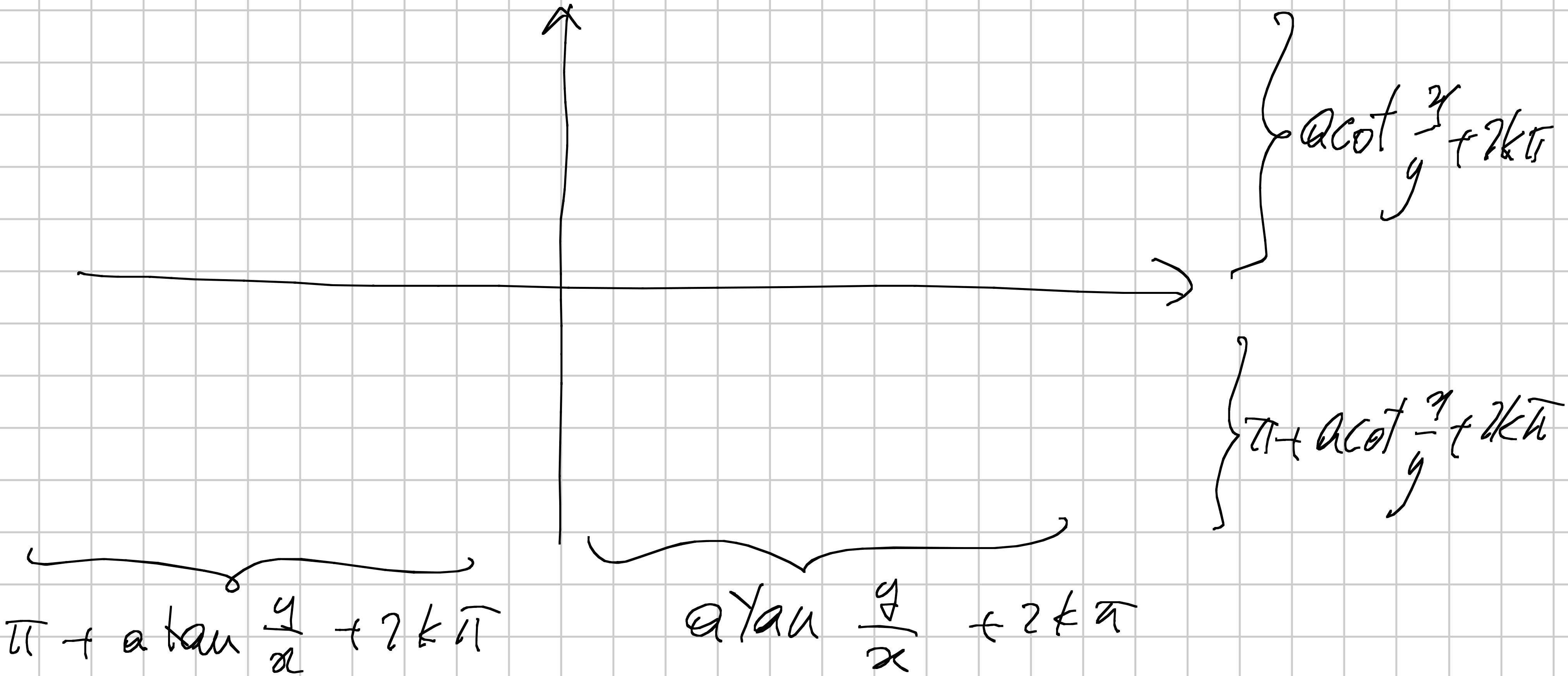




θ non è ben definita, ma lo è solo a meno di addendi $2k\pi$.

Però $d\theta$ è ben def perché due "rami" differiscono per costante.




$\operatorname{acot}\frac{x}{y}+2k\pi$
$\pi+\operatorname{acot}\frac{x}{y}+2k\pi$
$\pi+\operatorname{atan}\frac{y}{x}+2k\pi$
$\operatorname{atan}\frac{y}{x}+2k\pi$

$$\omega_{(0,0)} = \text{"}d\theta\text{"} = d\left(\operatorname{atan}\frac{y}{x}\right)$$

$$= \frac{1}{1+\left(\frac{y}{x}\right)^2}\left(-\frac{y}{x^2}\,dx + \frac{1}{x}\,dy\right)$$

$$= \boxed{\frac{-y\,dx + x\,dy}{x^2+y^2}}$$

$$\left(d\left(\operatorname{acot}\frac{x}{y}\right) = -\frac{1}{1+\left(\frac{x}{y}\right)^2}\cdot\left(\frac{1}{y}\,dx - \frac{x}{y^2}\,dy\right)\right)$$

$$= \frac{-y\,dx + x\,dy}{x^2+y^2} \Big)$$

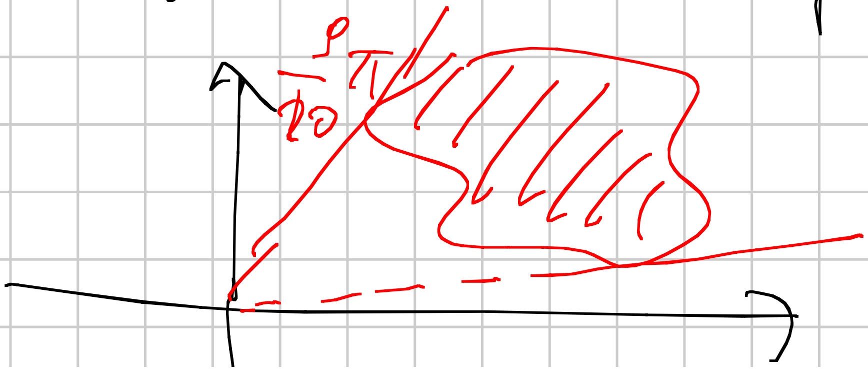
Per costruzione $d\omega_{(0,0)} = $ "$dd\theta$" $= 0$; verifichiamolo

$$f = -\frac{y}{x^2+y^2} \qquad g = \frac{x}{x^2+y^2}$$

$$\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} = \frac{x^2+y^2-2x\cdot x}{(x^2+y^2)^2} + \frac{x^2+y^2-2y\cdot y}{(x^2+y^2)^2} = 0.$$

$\Rightarrow \omega_{(0,0)}$ è forma chiusa non esatta.

Oss: dipende dal fatto che la penso definita su $\mathbb{R}^2 \setminus \{(0,0)\}$. Se la considero definita ad es



qui è esatta: posso scegliere Θ a valori in

$\left(\frac{\pi}{10}, \frac{9}{20}\pi\right)$

<u>Idea</u>: su insiemi particolari è vero che "chiuse $\Longrightarrow$ esatte".

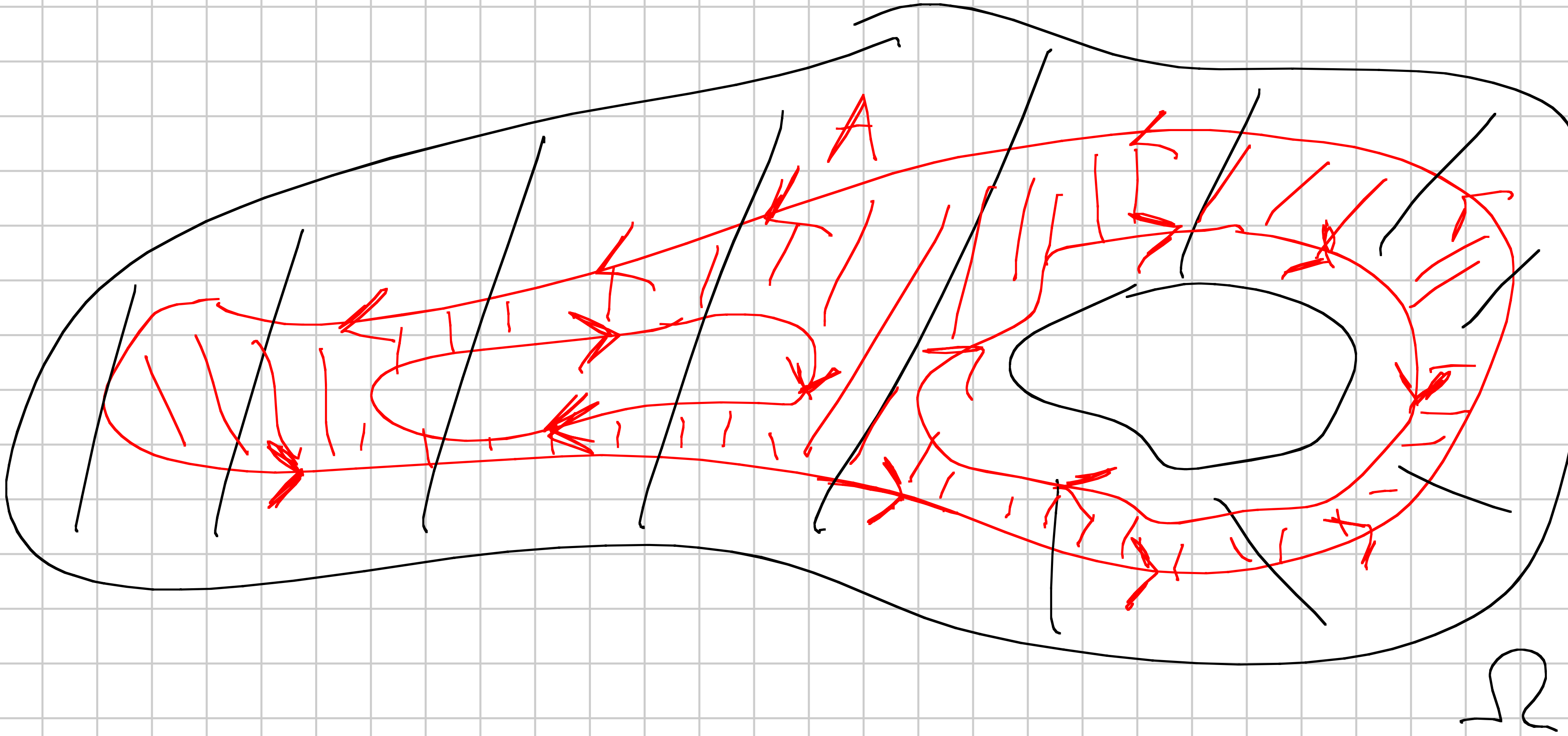
<u>Teo</u> (formula di Gauss-Green): sia $\omega = f\,dx + g\,dy$ su $\Omega \subset \mathbb{R}^2$;

sia $A \subset \Omega$ con ∂A = unione di curve

$\partial A \subset \Omega$

$$\Longrightarrow \underbrace{\int_{\partial A} \omega}_{} = \int_A d\omega = \int_A \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dx\,dy$$

$\Rightarrow$ dove ∂A è orientato in modo da lasciare A a sinistra:


A
Ω

Ripeto:
$$\int_{\partial A} \omega = \int_A d\omega .$$

<u>Oss</u>: è del tutto analogo al Teo Fond. del calcolo integrale:
$$\int_a^b F'(t)\,dt = F(b) - F(a)$$

$$\parallel \qquad\qquad\qquad\qquad \parallel$$

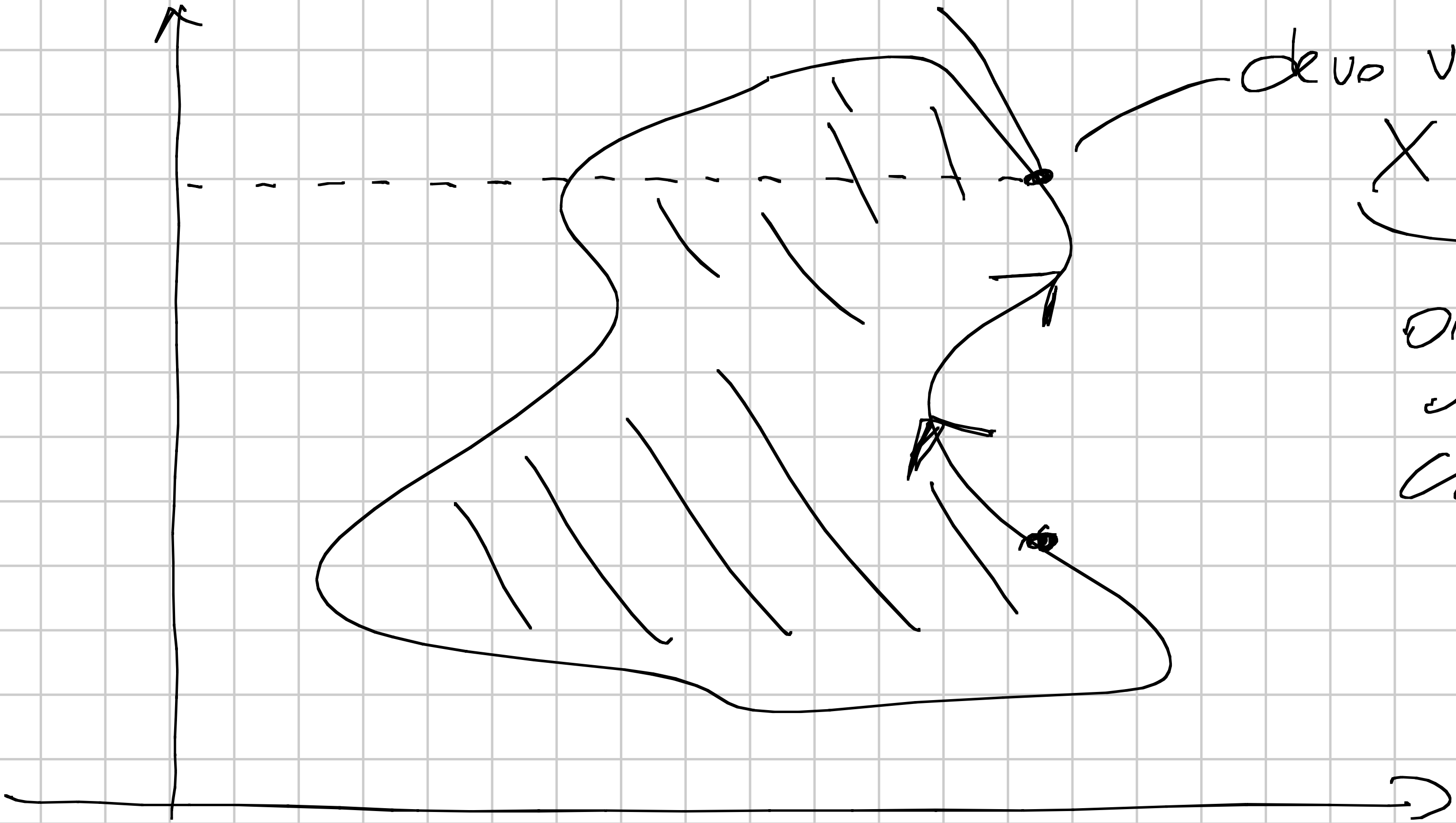
$$\int_{[a,b]} F' \qquad\qquad\qquad\qquad \int_{\partial [a,b]} F$$

$$\parallel$$

pto b − pto a

<u>Cor</u>: $$\int_{\partial A} x\,dy = \text{area}(A)$$

infatti: $\int_{\partial A} x\,dy = \int_A dx\,dy =$ area

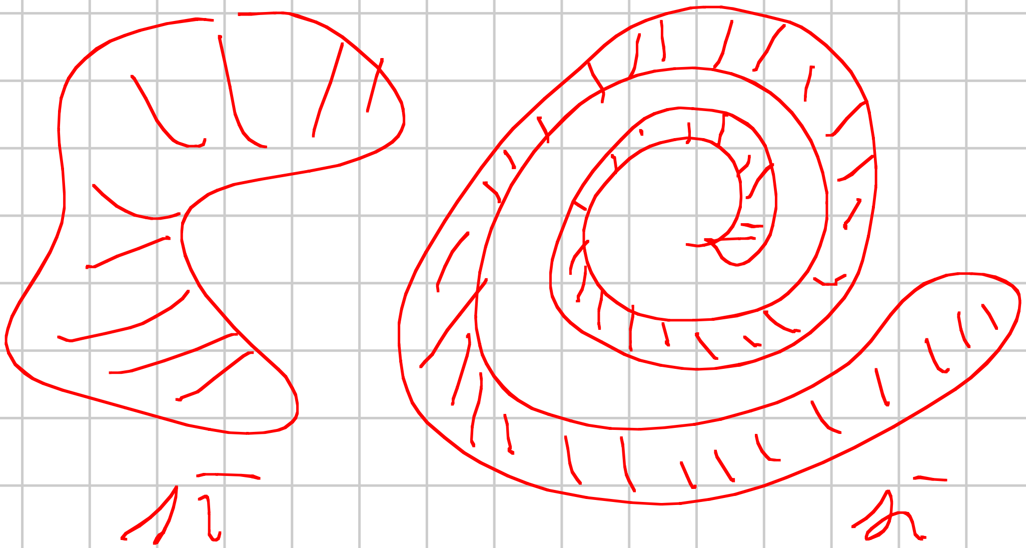


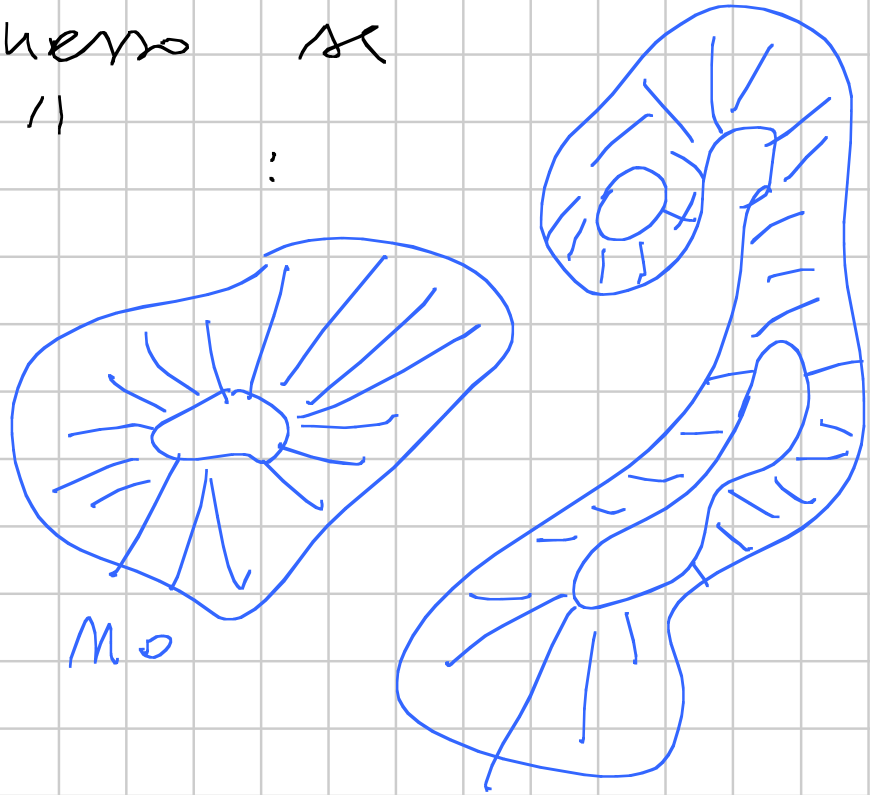
devo valutare

$X(t) \cdot Y'(t)$

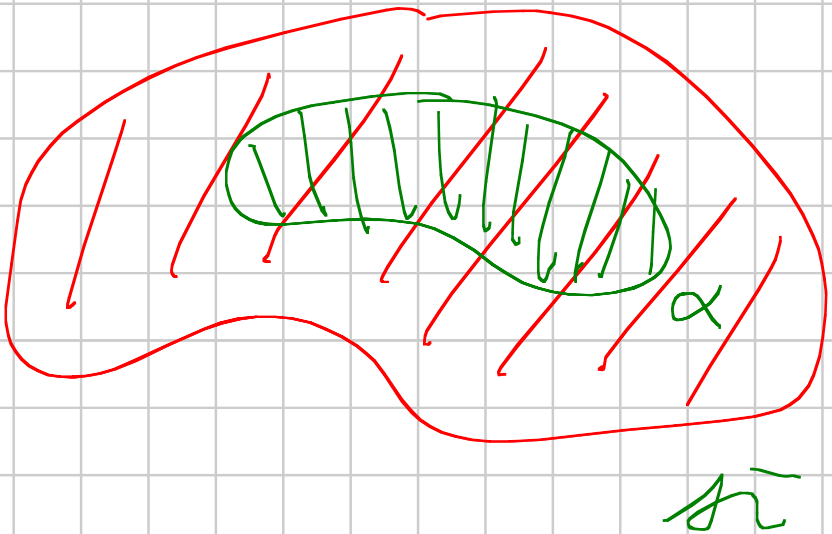
oggetti calcolabili

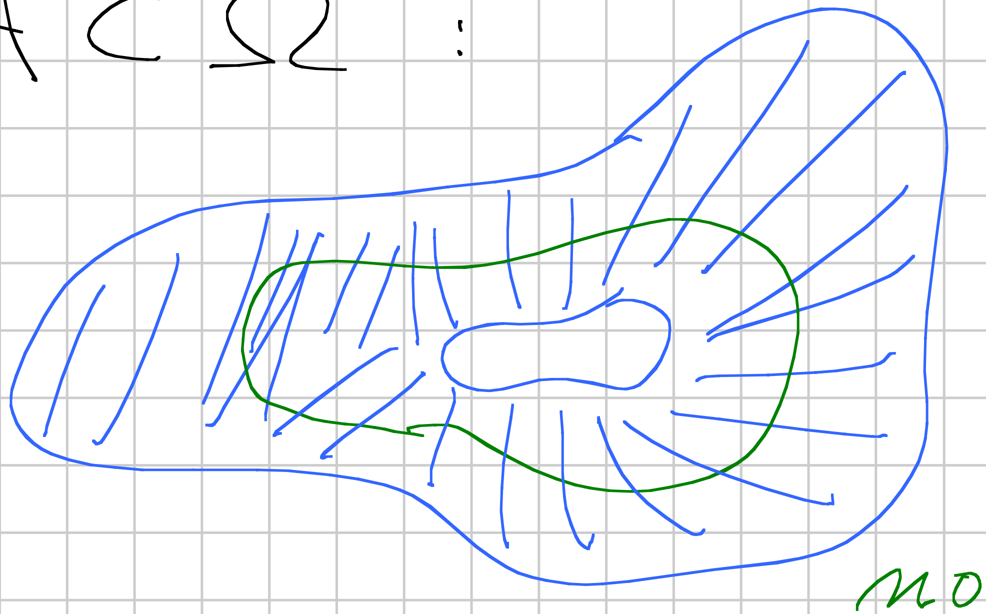
Def: dico che Ω è un insieme semplicemente connesso se "non ha buchi":





cioè: se ogni curva semplice chiusa α contenuta in Ω è il bordo di un sottoinsieme $A \subset \Omega$:





Teo: se ω è chiusa su Ω semplicemente connesso allora ω è esatta su Ω.

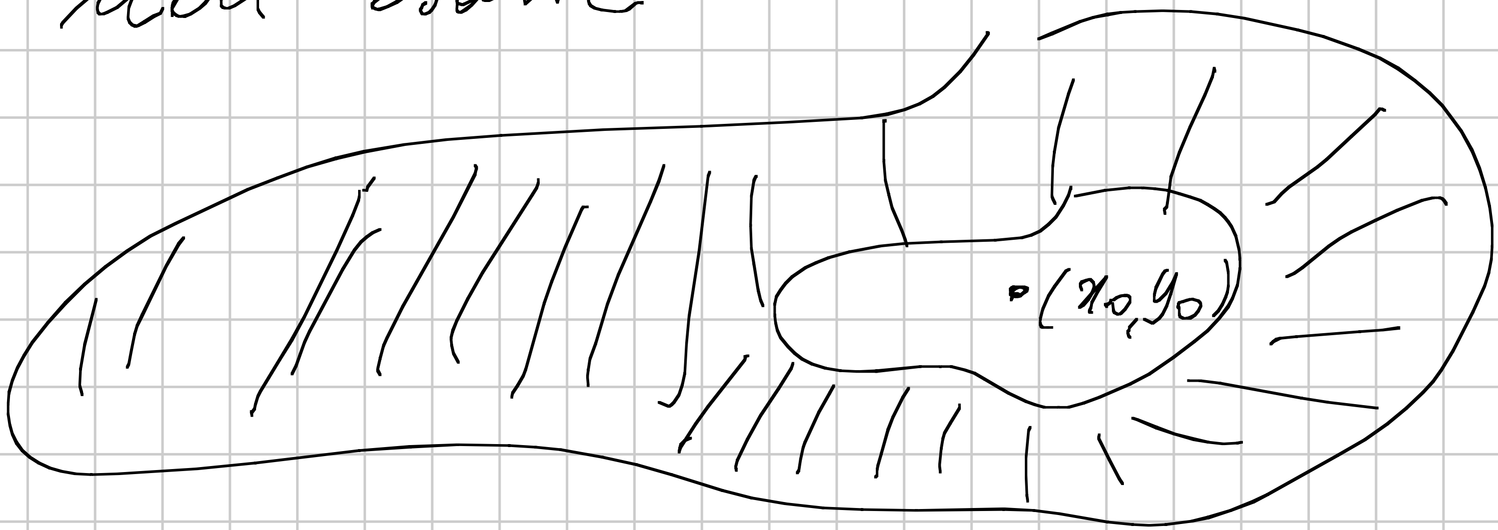
Dim: Basta vedere che $\int_\alpha \omega = 0$ $\forall \alpha$ semplice chiusa

Ma $\alpha = \partial A$, $A \subset \Omega$

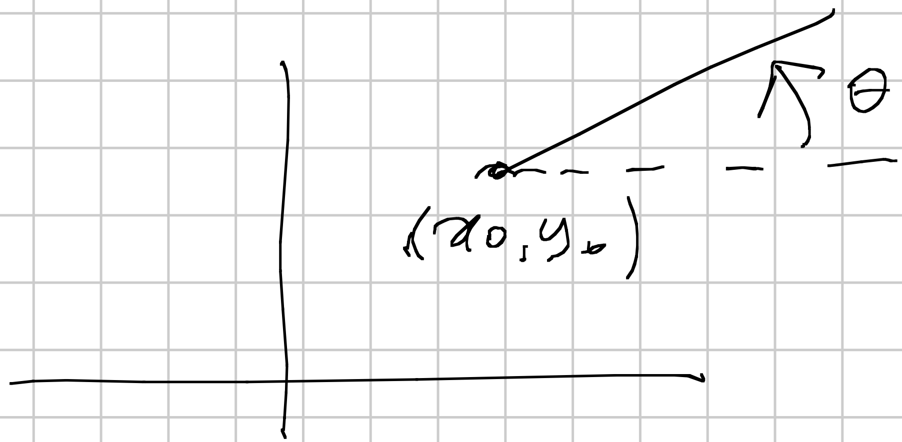
$$\Longrightarrow \int_\alpha \omega = \int_{\partial A} \omega = \int_A d\omega = \int_A 0 = 0.$$

$\blacksquare$

<u>Oss</u>: se Ω non è semplicemente connesso esistono su Ω delle forme chiuse non esatte



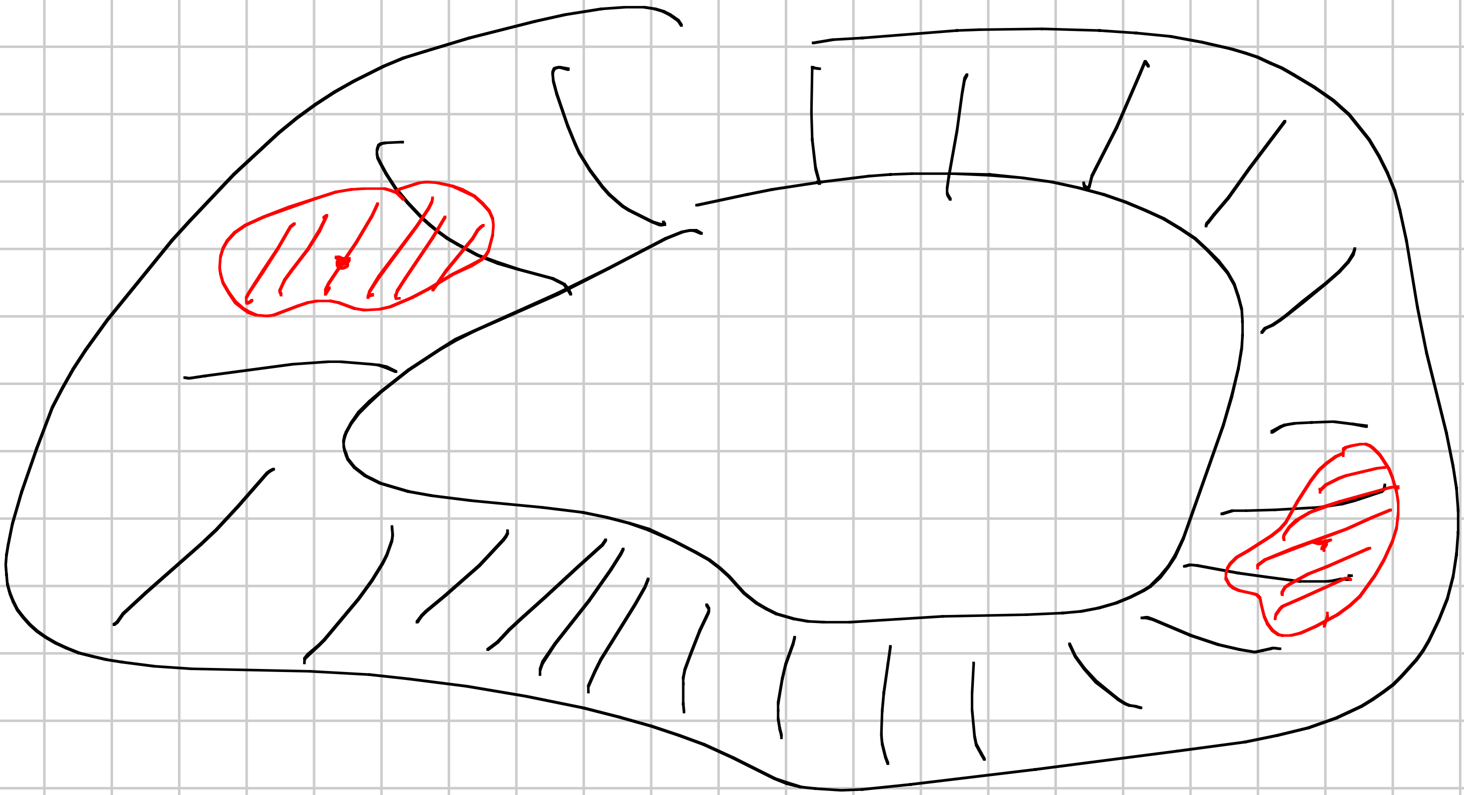
$$\omega_{(x_0,y_0)} = d\theta = \frac{-(y-y_0)\,dx + (x-x_0)\,dy}{(x-x_0)^2+(y-y_0)^2}$$



<u>Attenzione</u>: non è vero che ogni forma chiusa su Ω è non esatta

(cioè: esistono forme esatte)

<u>Con</u>: ogni forma chiusa è "localmente esatta"
cioè ammette un potenziale su piccoli insiemi vicino a ogni pto:



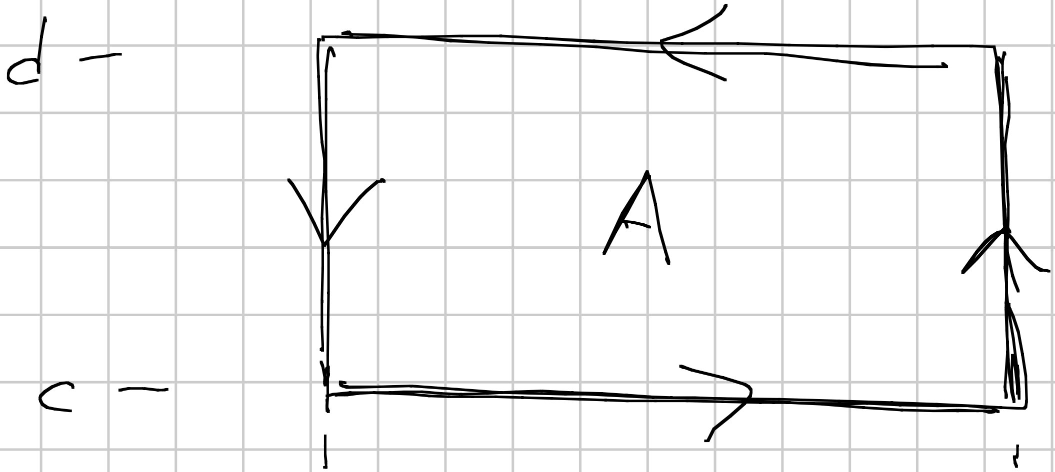
Dim (G-G):

$$\int_A d\omega = \int_{\partial A} \omega$$

$\omega = f\,dx + g\,dy$

$d\omega = \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dx\,dy$

Passo 1: $A = [a,b] \times [c,d]$



a $\qquad\qquad\qquad$ b

$$\int_{\partial A} \omega = \underset{1}{\int_a^b f(t,c)\,dt} + \underset{2}{\int_c^d g(b,t)\,dt}$$

$$- \underset{3}{\int_a^b f(t,d)\,dt} - \underset{4}{\int_c^d g(a,t)\,dt}$$

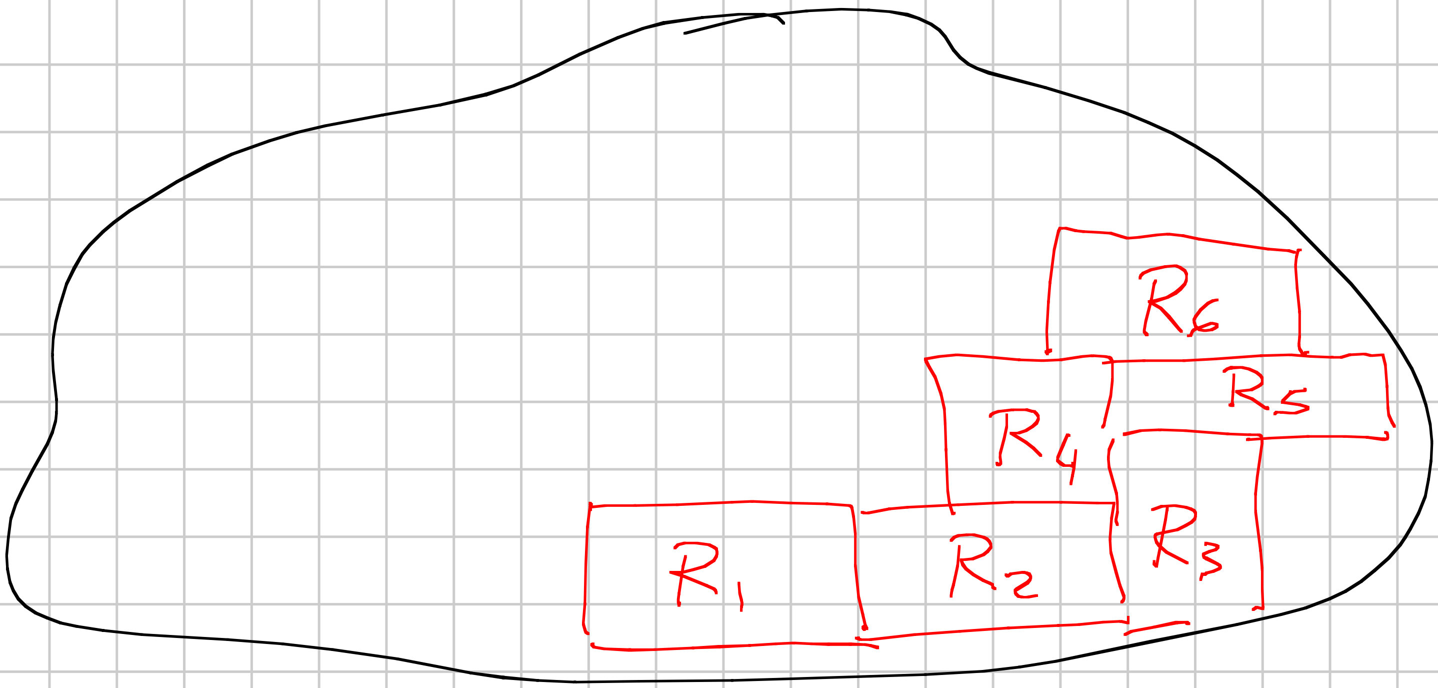
$$\int_A d\omega = \int_A \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dx\,dy$$

$$= \int_c^d \left(\int_a^b \frac{\partial g}{\partial x}\, dx \right) dy - \int_a^b \left(\int_c^d \frac{\partial f}{\partial y}\, dy \right) dx$$

$$= \int_c^d \underbrace{g(b,y)}_{2}\, dy - \int_c^d \underbrace{g(a,y)}_{4}\, dy$$

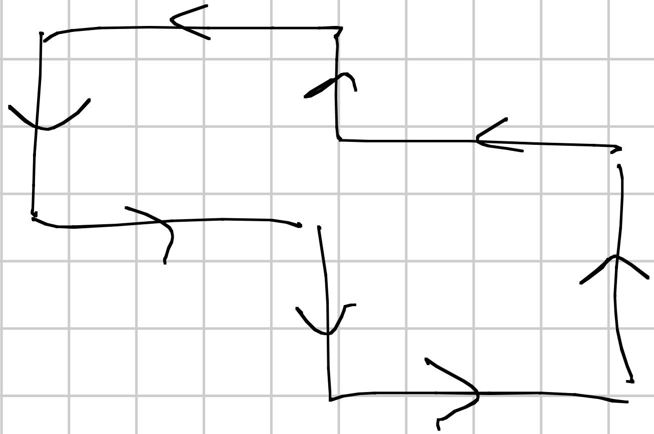
$$- \int_a^b \underbrace{f(x,d)}_{3}\, dx + \int_a^b \underbrace{f(x,c)}_{1}\, dx$$

Passo 2: approssimare A con rettangoli:



$$\int_A d\omega \cong \sum \int_{R_i} d\omega = \sum \int_{\partial R_i} \omega = \cdots$$

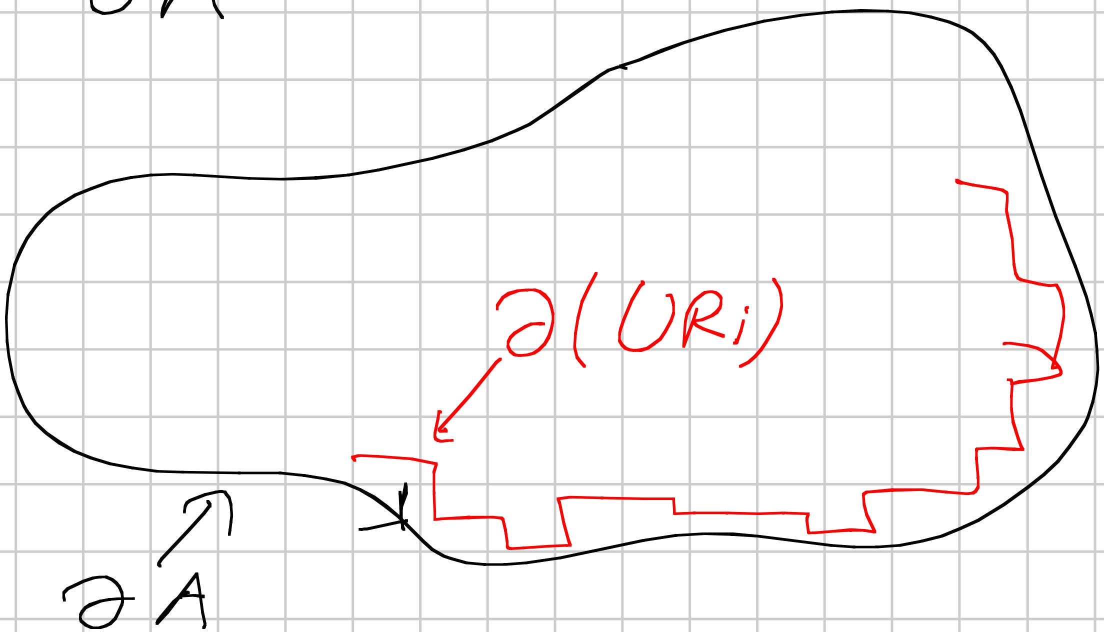
$=$



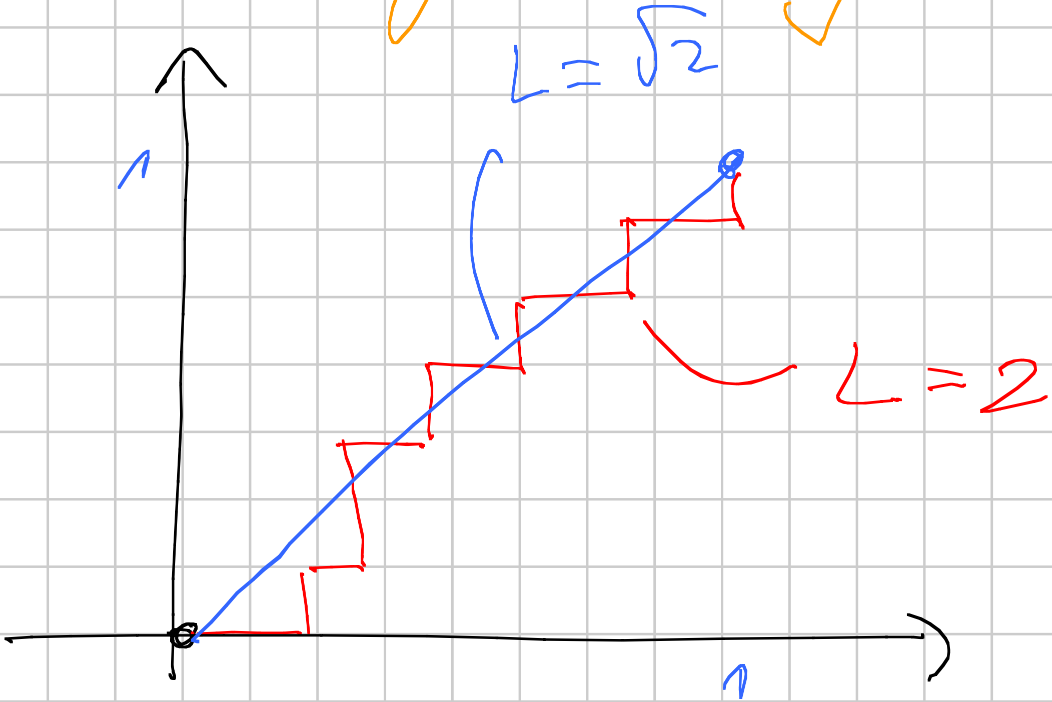
$$\int_{\partial R_1} + \int_{\partial R_2} = \int_{\partial(R_1 \cup R_2)}$$

$$\ldots = \int_{\partial(\cup R_i)} \omega \cong \int_{\partial A} \omega$$

non ovvio



Vero per l'integrale di una forma ma non per l'integrale di funzione:



ATTENZIONE:

$$\int_\alpha h = \int_a^b h(\alpha(t)) \cdot \|\alpha'(t)\| \, dt$$

$$\int_\alpha f\,dx + g\,dy = \int_a^b f(x(t), y(t)) \cdot x'(t) + g(x(t), y(t)) \cdot y'(t)$$

14.1.11

$$\int_\alpha xy \qquad \alpha : [0, \tfrac{\pi}{2}] \to \mathbb{R}^2 \quad \alpha(t) = \begin{pmatrix} \sin t \\ 2\cos t \end{pmatrix}$$

$$\int_\alpha xy = \int_0^{\pi/2} 2 \sin t \cos t \cdot \sqrt{\cos^2 t + 4 \sin^2 t} \; dt$$

$$= \int_0^{\pi/2} \sin(2t) \sqrt{1 + 3 \cdot \frac{1 - \cos(2t)}{2}} \; dt$$

$$\cos(2t) = \cos^2 t - \sin^2 t$$
$$= 1 - 2\sin^2 t$$

$$= \int_0^{\pi/2} \sin(2t)\sqrt{\frac{5}{2} - \frac{3}{2}\cos(2t)}\, dt$$

$$= \frac{2}{3} \cdot \frac{2}{3} \cdot \frac{1}{2} \left(\frac{5}{2} - \frac{3}{2} \cos 2t \right)^{3/2} \Bigg|_0^{\pi/2}$$

$$= \frac{2}{9}(8-1) = \frac{14}{9}.$$

14.2.2. Parametrizzare α il luogo in $\mathbb{R}^2$ di equazione $\rho = r \cdot e^{\lambda\theta}$ nelle coord polari ρ, θ ($r, \lambda > 0$ fissati).

Trovare τ t.c. $\alpha \circ \tau$ sia in p.d'a.
e calcolare curvatura.

$$\begin{cases} x = r e^{\lambda\theta} \cos\theta \\ y = r e^{\lambda\theta} \sin\theta \end{cases} \qquad \alpha(\theta) = \begin{pmatrix} r e^{\lambda\theta} \cos\theta \\ r e^{\lambda\theta} \sin\theta \end{pmatrix}$$

La T cercata è σ^{-1} dove

$$\sigma(\theta) = \int_0^\theta \| \alpha'(u) \| \, du = \dots$$

$$\alpha'(u) = \begin{pmatrix} r\lambda e^{\lambda u} \cos(u) - r e^{\lambda u} \sin(u) \\ r\lambda e^{\lambda u} \sin(u) + r e^{\lambda u} \cos(u) \end{pmatrix}$$

$$\dots = \int_{0}^{\theta} r e^{\lambda u} \cdot \sqrt{1+\lambda^2}\, du = \frac{r}{\lambda}\sqrt{1+\lambda^2}\, e^{\lambda u}\Big|_{0}^{\theta}$$

$$= \frac{r}{\lambda}\sqrt{1+\lambda^2}\left(e^{\lambda\theta} - 1\right)$$

$\tau(s)$ lo ricavo risolvendo $\frac{r}{\lambda}\sqrt{1+\lambda^2}\cdot\left(e^{\lambda\theta}-1\right) = s$

$$\Rightarrow \tau(\lambda) = \frac{1}{\lambda}\log\left(1 + \frac{\lambda \cdot \lambda}{r\sqrt{1+\lambda^2}}\right)$$

$$\alpha(t) = r e^{\lambda t}\begin{pmatrix}\cos(t)\\ \sin(t)\end{pmatrix}$$

$$\alpha'(t) = \begin{pmatrix} r\lambda e^{\lambda t}\cos t - r e^{\lambda t}\sin t \\ r\lambda e^{\lambda t}\sin t + r e^{\lambda t}\cos t \end{pmatrix} = r e^{\lambda t}\begin{pmatrix}\lambda c - s\\ \lambda s + c\end{pmatrix}$$

$$\alpha''(t) = \begin{pmatrix} \pi\lambda^2 e^{\lambda t}\cos t - 2\pi\lambda e^{\lambda t}\sin t - \pi e^{\lambda t}\cos t \\ \pi\lambda^2 e^{\lambda t}\sin t + 2\pi\lambda e^{\lambda t}\cos t - \pi e^{\lambda t}\sin t \end{pmatrix}$$

$$= \pi e^{\lambda t}\begin{pmatrix} (\lambda^2-1)c - 2\lambda s \\ (\lambda^2-1)s + 2\lambda c \end{pmatrix}$$

$$\kappa(t) = \frac{\det(\alpha'(t), \alpha''(t))}{\|\alpha'(t)\|^3}$$

$$= \frac{r^2 e^{2\lambda t} \det \begin{pmatrix} \lambda c - s & (\lambda^2 - 1)c - 2\lambda s \\ \lambda s + c & (\lambda^2 - 1)s + 2\lambda c \end{pmatrix}}{\ldots} = \ldots$$

14.2.3 $\quad \alpha(t) = \rho(t) \begin{pmatrix} \cos(t) \\ \sin(t) \end{pmatrix}$

P. d'a. + K.

$$\alpha'(t) = \begin{pmatrix} \rho' \cdot \cos - \rho \cdot \sin \\ \rho' \cdot \sin + \rho \cdot \cos \end{pmatrix}$$

$$\tau = \sigma^{-1}$$

$$\sigma(t) = \int_0^t \|\alpha'(u)\| \, du = \int_0^t \sqrt{\rho(u)^2 + \rho'(u)^2} \, du$$

$$\alpha''(t) = \begin{pmatrix} \rho'' \cdot c - 2\rho' s - \rho c \\ \rho'' \cdot s + 2\rho' c - \rho s \end{pmatrix}$$

$$\kappa = \frac{\det(\alpha' \ \alpha'')}{\|\alpha'\|^3}$$

14.3.3

Calcolare t, n, b, κ, τ per α in s_0.

(a) $\alpha(s) = \begin{pmatrix} s \cdot \cos(s) \\ 1 + s^2 \\ \log(1+s) \end{pmatrix}$ $\quad s_0 = 0$

$$\begin{pmatrix} \cos(s) - s \cdot \sin(s) \\ 2s \\ \frac{1}{1+s} \end{pmatrix} \begin{pmatrix} -2\sin s - s\cos(s) \\ 2 \\ -\frac{1}{(1+s)^2} \end{pmatrix} \begin{pmatrix} -3\cos(s) + s \cdot \sin(s) \\ 0 \\ \frac{2}{(1+s)^3} \end{pmatrix}$$

$$\alpha'(0) = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \qquad \alpha''(0) = \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix} \qquad \alpha'''(0) = \begin{pmatrix} -3 \\ 0 \\ 2 \end{pmatrix}$$

$$\alpha'(0) \wedge \alpha''(0) = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \wedge \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix}$$

$$t = \frac{\alpha'(0)}{\|\alpha'(0)\|} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$$n = \frac{\alpha'' - \langle \alpha'' | t \rangle \cdot t}{\| \quad \dots \quad \|} = \dots \text{complicato}$$

$$b = \frac{\alpha'(0) \wedge \alpha''(0)}{\| \quad \cdots \quad \|} = \frac{1}{3}\begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix}$$

$$n = b \wedge t = \frac{1}{3}\begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix} \wedge \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \frac{1}{3\sqrt{2}}\begin{pmatrix} 1 \\ 4 \\ -1 \end{pmatrix}$$

$$\kappa = \frac{\|\alpha' \wedge \alpha''\|}{\|\alpha'\|^3} = \frac{3}{2\sqrt{2}}$$

$$\tau = \frac{\langle \alpha' \wedge \alpha'' \mid \alpha''' \rangle}{\| \alpha' \wedge \alpha'' \|^3}$$

$$= \frac{\left\langle \begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix} \middle| \begin{pmatrix} -3 \\ 0 \\ 2 \end{pmatrix} \right\rangle}{27} = \frac{10}{27}.$$