Es. geom. 10/3/2017

(9.1.8) Data $A \in M_{n \times n}(\mathbb{R})$

B definita da $b_{ij} = (-1)^{i+j} a_{ij}$

A determina un p. scalare sse lo fa B.

- A simmetrica $\Longleftrightarrow$ B simmetrica

$a_{ij} = a_{ji}$

$$b_{ij} = b_{ji}$$

$$\parallel \qquad\qquad \parallel$$

$$(-1)^{i+j} a_{ij} \qquad (-1)^{i+j} a_{ji}$$

• A def. pos $\iff$ B def. pos.

$$\langle x | x \rangle_A = \sum_{i=1}^{n} x_i \, (A \cdot x)_i = $$

$$= \sum_{i=1}^{n} x_i \sum_{j=1}^{n} a_{ij} \, x_j = $$

$$= \sum_{i,j=1}^{n} a_{ij} \, x_i \, x_j$$

$$(-1)^i \cdot (-1)^j$$

$$\langle x | x \rangle_B = \sum_{i,j=1}^{n} b_{ij} \, x_i \, x_j = \sum_{i,j=1}^{n} (-1)^{i+j} a_{ij} \, x_i \, x_j$$

Pongo $T : \mathbb{R}^n \to \mathbb{R}^n$

$$(Tx)_i = (-1)^i x_i$$

$$T : \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \longmapsto \begin{pmatrix} -x_1 \\ x_2 \\ -x_3 \\ x_4 \\ \vdots \\ (-1)^n x_n \end{pmatrix}$$

$$\langle x | x \rangle_B = \langle Tx | Tx \rangle_A$$

dicome T invertibile $\left(T^{-1} = T \right)$

$\langle x | x \rangle_B > 0 \quad \forall x \neq 0 \qquad \langle x | x \rangle_A > 0 \quad \forall x \neq 0$

(9.1.9) Dati $v_1, \dots, v_n \in \mathbb{R}^n$ $A \in M_{n \times n}(\mathbb{R})$

$a_{ij} = \langle v_i | v_j \rangle_{\mathbb{R}^n}$ (A è determ. dai vettori)

For. quali ipotesi per i vett. $v_1, \dots, v_n$
$\langle \cdot | \cdot \rangle_A$ è p. scalare in $\mathbb{R}^n$?

- $a_{ij} = \langle v_i | v_j \rangle_{\mathbb{R}^n} = \langle v_j | v_i \rangle_{\mathbb{R}^n} = a_{ji}$

$\Rightarrow$ A è sempre simm.

- Def pos $\langle x | x \rangle_A = \sum_{i,j=1}^{n} a_{ij} x_i x_j =$

$$= \sum_{i,j=1}^{n} \langle v_i | v_j \rangle_{\mathbb{R}^n} \, x_i x_j =$$

$$= \Big\langle \underbrace{\sum_{i=1}^{n} v_i x_i}_{} \;\Big|\; \underbrace{\sum_{j=1}^{n} v_j x_j}_{} \Big\rangle =$$

$$= \overbrace{\Big\| \sum_{i=1}^{n} x_i v_i \Big\|^2_{\mathbb{R}^n}}^{=0} \geq 0 \quad \underline{\text{sempre}}$$

sse $\sum_{i=1}^{n} x_i v_i = 0$

A def. pos. quando se $0 \in \mathbb{R}^n$ l'unico $x \in \mathbb{R}^n$ t.c. $\sum_{i=1}^{n} x_i v_i = 0$ cioè se $v_1, \dots, v_n$ sono lin. indip., cioè se sono <u>base</u>.

(9.2.1) V sp. vett su $\mathbb{R}$, se $f : V \times V \to \mathbb{R}$ forma bilineare simmetrica, definiamo la forma quadratica $q : V \longrightarrow \mathbb{R}$ associata ad f $\quad q(v) := f(v, v)$

Dato $q(v)$ trovare f associata

ovvero se conosco $\langle v | v \rangle$ voglio trovare $\langle v | w \rangle$

(e) $V = \mathbb{R}^2$ $\quad q(x) = 3 x_1^2 + 2 x_1 x_2 - x_2^2$

$$A = \begin{pmatrix} 3 & \boxed{1} \\ \boxed{1} & -1 \end{pmatrix}$$

divido per 2

$$v = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \qquad \langle v | v \rangle_A = 3x_1^2 + \underline{x_1 x_2} + \underline{x_1 x_2} - x_2^2 = q(v)$$

(b) $U = \mathbb{R}^3$

$$q(x) = -2x_1^2 + x_2^2 - 5x_3^2 - x_1 x_2 + 3x_1 x_3 + 2 x_2 x_3$$

$$A = \begin{pmatrix} -2 & -1/2 & 3/2 \\ -1/2 & 1 & 1 \\ 3/2 & 1 & -5 \end{pmatrix}$$

(c) $V = M_{2\times 2}(\mathbb{R})$ $\qquad q(A) = \det(A) =$

$$= a_{11}a_{22} - a_{12}a_{21}$$

$$f(A,B) = \frac{1}{2}(a_{11}b_{22} + a_{22}b_{11}) + -\frac{1}{2}(a_{12}b_{21} + a_{21}b_{12})$$

f è simm. e $f(A,A) = q(A)$

(e) $V = \mathbb{R}_{\leq 2}[t]$ $\qquad q(p(t)) = p(1)\cdot p(-2)$

$$g(p(t), q(t)) = \frac{1}{2}\left(p(1)\,q(-2) + p(-2)\,q(1)\right)$$

(9.2.2) Su $V = \mathbb{R}_{\leq 2}[t]$, p. scalare:

$$\langle p(t) \mid q(t) \rangle = p(0)\,q(0) + p(1)\,q(1) + p(2)\,q(2)$$

Oss.: è def. positivo: $\langle p(t) \mid p(t) \rangle = 0$

$\Rightarrow \; p(0)^2 + p(1)^2 + p(2)^2 = 0$

$\Rightarrow \; p(0) = p(1) = p(2) = 0 \qquad \Rightarrow \; p = 0$

(un pol. di grado n è determinato dal valore in $n+1$ punti)

Trovare un polinomio $p(t)$ di norma $\sqrt{5}$, ortogonale a $1+t$, $1+t^2$.

Intanto lo cerco ortogonale, poi sistemiamo la norma

$$\langle p(t) \mid 1+t \rangle = 0$$

$$\langle p(t) \mid 1+t^2 \rangle = 0$$

$$p(t) = a_0 + t a_1 + t^2 a_2$$

$$p(0) = a_0$$

$$p(1) = a_0 + a_1 + a_2$$

$$p(2) = a_0 + 2a_1 + 4a_2$$

$$\begin{cases} a_0 \cdot 1 + (a_0 + a_1 + a_2) \cdot 2 + (a_0 + 2a_1 + 4a_2) \cdot 3 = 0 \\ a_0 \cdot 1 + (a_0 + a_1 + a_2) \cdot 2 + (a_0 + 2a_1 + 4a_2) \cdot 5 = 0 \end{cases}$$

$$\begin{cases} a_0 + 2a_1 + 4a_2 = 0 \\ 3a_0 + 2a_1 + 2a_2 = 0 \end{cases} \qquad a = \cancel{\begin{pmatrix} 4 \\ -10 \\ 4 \end{pmatrix}} \quad \begin{pmatrix} 2 \\ -5 \\ 2 \end{pmatrix}$$

$$p(t) = 2 - 5t + 2t^2$$

$$\|p(t)\|^2 = p(0)^2 + p(1)^2 + p(2)^2 = 4 + 1 + 0 = 5$$

$$\|p(t)\| = \sqrt{5}$$

$p(t)$ è il pol. cercato.

(9.2.3) Date le funzioni $\mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$

$$SNCF(x,y) := \begin{cases} \|x-y\| & \text{se } x,y \text{ lin. dip.} \\ \|x\|+\|y\| & \text{se } x,y \text{ indip.} \end{cases}$$

$$NYC(x,y) := \sum_{j=1}^{n} |x_j - y_j|$$

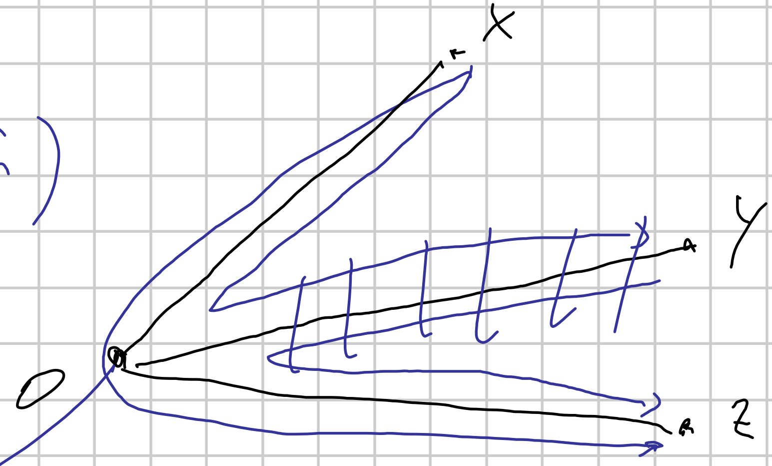
SNCF e NYC sono distanze

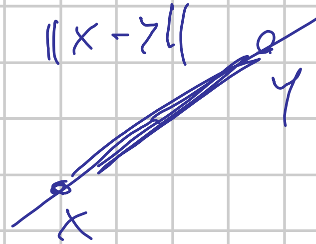
Rifl. , simm. OK

Disuguaglianza triangolare

SNCF

(ferrovie francesi)

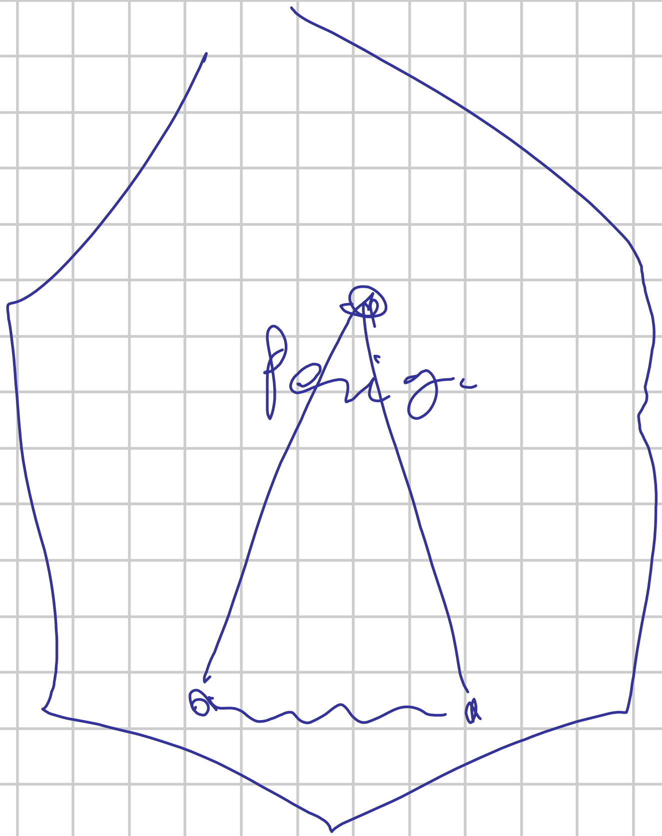




triangolare.

dividiamo per casi: x, y allineati e z no, x, y, z non allineati, ...

$$SNCF(x,y) + SNCF(y,z) = SNCF(x,z)$$



NYC