Algebra Lineare 13/12/16

$\boxed{36}$ (1) $\sqrt{2}+\sqrt{7} \notin \mathbb{Q}$?

Sappiamo: $\sqrt{2} \notin \mathbb{Q}$; analog. $\sqrt{k}$ non è raz.

se la decomposiz. in primi di k ne contiene uno

con esponente 1.

Se $\sqrt{2}+\sqrt{7}$ fosse raz. allora $(\sqrt{2}+\sqrt{7})^2$ sarebbe raz.

ovvero $2 + 2\sqrt{14} + 7 \in \mathbb{Q} \Rightarrow 2\sqrt{14} \in \mathbb{Q} \Rightarrow \sqrt{14} \in \mathbb{Q}$:

Falso. <u>No</u>

(2) 13 generatori di $X = \{ p(t) \in \mathbb{R}_{\leq 9}[t] : p(1) = p''(-7) = 0 \}$

Quanti ne scarto per avere base?

$\dim X = 10 - 2 = 8 \Rightarrow 13 - 8 = 5$

③ $f: \mathbb{C}^8 \to \mathbb{C}^3$ lin. map. $\quad W \subset \mathbb{C}^8 \quad \dim W = 6.$

$\dim(W \cap \operatorname{Ker} f) = ?$

$$8 = \dim \operatorname{Ker} f + 3 \implies \dim \operatorname{Ker} f = 5$$

$$\underbrace{\dim(W \cap \operatorname{Ker} f)}_{3 \le \cdot \le 5} + \underbrace{\dim(W + \operatorname{Ker} f)}_{\le 8} = \underbrace{\dim W}_{6} + \underbrace{\dim \operatorname{Ker} f}_{5}$$

(4) Per quale $z \in \mathbb{C}$ si ha $\det\begin{pmatrix} 0 & i & 2 \\ 1-i & 0 & 1+z \\ -2i & 1 & 0 \end{pmatrix} = 0$?

$$i(1+z)(-2i) + 2(1-i)\cdot 1 = 0$$

$$2(1+z) + 2(1-i) = 0$$

$$z = i - 2$$

(5) Risolvere $\begin{cases} 2x - 3y + 5z = 4 \\ 3x + 2y - 4z = 1 \\ 5x - 14y + 24z = 15 \end{cases}$

$$\begin{cases} 2x = 3y - 5z + 4 \\ 9y - 15z + 12 + 4y - 8z = 2 \\ 15y - 25z + 20 - 28y + 48z = 30 \end{cases}$$

$$\begin{cases} 2x = 3y - 5z + 4 \\ 13y - 23z = -10 \\ -13y + 23z = 10 \end{cases}$$

✓

$$\Rightarrow \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \mathrm{Span} \begin{pmatrix} 2 \\ 23 \\ 13 \end{pmatrix}$$

Altri modi:

$\Rightarrow$ det(incompleta) = 0

si trova: III (equaz omog) = $\alpha \cdot$ (I omog) + $\beta \cdot$ (II omog)

si verifica che $\alpha \cdot 4 + \beta \cdot 1 = 15$

(cioè III (non omog) = $\alpha \cdot$ (I non) + $\beta \cdot$ (II non)

Dunque equivale a
$$\begin{cases} 2x - 3y + 5z = 4 \\ 3x + 2y - 4z = 1 \end{cases}$$

Soluz:
$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \mathrm{Span}\begin{pmatrix} (-3)(-4) - 2 \cdot 5 \\ -(2 \cdot (-4) - 3 \cdot 5) \\ 2 \cdot 2 - (-3) \cdot 3 \end{pmatrix}$$

$$\begin{array}{c} \| \\ \begin{pmatrix} 2 \\ 23 \\ 13 \end{pmatrix} \end{array}$$

Andava bene anche
$$\begin{pmatrix} -1 \\ -22 \\ -12 \end{pmatrix} + \mathrm{Span}\begin{pmatrix} -2\sqrt{7} \\ -23\sqrt{7} \\ -13\sqrt{7} \end{pmatrix}.$$

⑥ Calcolare det orlate di $\begin{pmatrix} 2 & 0 \\ -1 & -2 \end{pmatrix}$ in $\begin{pmatrix} 2 & 1 & 0 & -2 \\ -3 & 2 & 1 & 0 \\ -1 & 3 & -2 & 5 \end{pmatrix}$

$$\det \begin{pmatrix} 2 & 1 & 0 \\ -3 & 2 & 1 \\ -1 & 3 & -2 \end{pmatrix} = -8 - 1 - 6 - 6 = -21$$

$$\det \begin{pmatrix} 2 & 0 & -2 \\ -3 & 1 & 0 \\ -1 & -2 & 5 \end{pmatrix} = \ldots$$

⑦ $W = \mathrm{Span}(2e_1 - e_2 + e_3 ,\ e_2 - e_3 + e_4)$

$Z = \mathrm{Span}(3e_1 - e_3 , 2e_2 - e_4)$

Calcolare la proiez. di e_1 su W rispetto alla deco $\mathbb{R}^4 = W \oplus Z$.

Devo scrivere $e_1 = \underset{W}{\underset{\cap}{v}} + \underset{Z}{\underset{\cap}{z}}$ cioè

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \underbrace{\left(\alpha \cdot \begin{pmatrix} 2 \\ -1 \\ 1 \\ 0 \end{pmatrix} + \beta \cdot \begin{pmatrix} 0 \\ 1 \\ -1 \\ 1 \end{pmatrix} \right)}_{\text{lui è il vettore cercato}} + \left(\delta \cdot \begin{pmatrix} 3 \\ 0 \\ -1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 0 \\ 2 \\ 0 \\ -1 \end{pmatrix} \right)$$

$$\begin{cases} 2\alpha + 3s = 1 \\ -\alpha + \beta + 2t = 0 \\ \alpha - \beta - s = 0 \\ \beta - t = 0 \end{cases} \qquad \begin{cases} s = \alpha - \beta \\ t = \beta \\ 2\alpha + 3\alpha - 3\beta = 1 \\ -\alpha + \beta + 2\beta = 0 \end{cases}$$

$$\begin{cases} 5\alpha - 3\beta = 1 \\ -\alpha + 3\beta = 0 \end{cases} \qquad \alpha = \frac{1}{4} \qquad \beta = \frac{1}{12}$$

$$\Longrightarrow \quad \frac{1}{4}\begin{pmatrix} 2 \\ -1 \\ 1 \\ 0 \end{pmatrix} + \frac{1}{12}\begin{pmatrix} 0 \\ 1 \\ -1 \\ 1 \end{pmatrix} = \frac{1}{12}\begin{pmatrix} 6 \\ -2 \\ 2 \\ 1 \end{pmatrix}$$

$\boxed{38}$ (1) 9 vett. lin. indip. in $X = \{x \in \mathbb{R}^7 : x_1 = x_3 = x_7\}$.

Quanti ne elimino per avere una base?

$$9 - (7 - 2) = 4$$

(2) $f : \mathbb{C}^5 \to \mathbb{C}^{13}$ lin

$f(e_2) = f(e_3 + e_5)$ $\qquad$ $\exists$? su $\dim W$?

$\mathbb{C}^{13} = W + \operatorname{Im}(f)$

$\mathcal{H}_0$ $f: \mathbb{C}^5 \to \mathbb{C}^{13}$ con $f(e_2 - e_3 - e_5) = 0$

$\Rightarrow \dim \operatorname{Ker} f \geq 1 \Rightarrow \dim \operatorname{Im} f \leq 4$

$\Rightarrow \dim W \geq 9$

③ $f: \mathbb{R}^2 \to \mathbb{R}^2$, $f\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 3x_1 - x_2 \\ -2x_1 + 5x_2 \end{pmatrix}$

$\mathcal{B} = \left(\begin{pmatrix} 7 \\ 5 \end{pmatrix}, \begin{pmatrix} 4 \\ 3 \end{pmatrix} \right)$. $[f]_{\mathcal{B}}^{\mathcal{B}} = ?$

$$f = f_A \qquad A = \begin{pmatrix} 3 & -1 \\ -2 & 5 \end{pmatrix}$$

$$\begin{pmatrix} 3 & -1 \\ -2 & 5 \end{pmatrix} \cdot \begin{pmatrix} 7 \\ 5 \end{pmatrix} = \begin{pmatrix} 16 \\ 11 \end{pmatrix} = \boxed{4} \begin{pmatrix} 7 \\ 5 \end{pmatrix} + \boxed{-3} \begin{pmatrix} 4 \\ 3 \end{pmatrix}$$

$$\begin{pmatrix} 3 & -1 \\ -2 & 5 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ 3 \end{pmatrix} = \begin{pmatrix} 9 \\ 7 \end{pmatrix} = \boxed{-1} \begin{pmatrix} 7 \\ 5 \end{pmatrix} + \boxed{4} \begin{pmatrix} 4 \\ 3 \end{pmatrix}$$

da cui $[f]_{\mathcal{B}}^{\mathcal{B}} = \begin{pmatrix} \boxed{4} & \boxed{-1} \\ \boxed{-3} & \boxed{4} \end{pmatrix}$

4 $A = \begin{pmatrix} 1-i & 1 & 0 \\ -1 & i & 2 \\ 0 & 1+i & 1 \end{pmatrix}$. $(A^{-1})_{2,3} = ?$

$$(A^{-1})_{i,j} = (-1)^{i+j} \frac{\det(A^{ji})}{\det(A)}$$

$$\det A = i+1+1-1-2\underbrace{(1+i)(1-i)}_{2} = i-3$$

$$(A^{-1})_{2,3} = -\frac{2(1-i)}{i-3} = 2\cdot\frac{1-i}{3-i} = \frac{2}{10}(1-i)(3+i)$$

$$= \frac{1}{5}(4-2i)$$

(7) trovare $[7e_1-5e_2]_\beta$ con $\beta = (2e_1-e_2\ ,\ -e_1+e_2)$

$$\begin{pmatrix} 7 \\ -5 \end{pmatrix} = \boxed{2}\begin{pmatrix} 2 \\ -1 \end{pmatrix} \boxed{-3}\begin{pmatrix} -1 \\ 1 \end{pmatrix} \qquad \begin{pmatrix} \boxed{2} \\ \boxed{-3} \end{pmatrix}$$

$\boxed{39}$ (3) $A = (v_1 \; v_2 \; v_3) \qquad \det(A) = -\frac{1}{5}$.

$$\det(3v_3 - 2v_2,\; v_1 + 4v_3,\; 2v_1 - 3v_2) = \,?$$

- $3 \cdot 1 \cdot (-3) \det(v_3, v_1, v_2) + (-2) \cdot 4 \cdot 2 \det(v_2, v_3, v_1)$

$= -9 \cdot \det(v_1, v_2, v_3) - 16 \cdot \det(v_1, v_2, v_3)$

$= -25\left(-\frac{1}{5}\right) = 5$

- $(v_1 \; v_2 \; v_3) \cdot \begin{pmatrix} 0 & 1 & 2 \\ -2 & 0 & -3 \\ 3 & 4 & 0 \end{pmatrix}$

$$\begin{pmatrix} -1/3 \end{pmatrix} \cdot \begin{pmatrix} -3 - 16 \end{pmatrix} = 5$$

(4) $f : \{ x \in \mathbb{R}^7 : x_3 = x_5 \} \longrightarrow \mathbb{R}^4$ lin. non surj.

dim Ker f?

$$(7-1) = \dim(\mathrm{Ker} f) + \dim(\mathrm{Im} f)$$

$$\underbrace{3 \leq \cdot \leq 6} \qquad 0 \leq \cdot \leq 3$$

(5) Risolvere $\begin{cases} 3x - 2y + 5z = 10 \\ 4x + y + 2z = 5 \\ -6x - 7y + 4z = 5 \end{cases}$

Sol. part $\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$.

Omog. $\begin{cases} y = -4x - 2z \\ 3x + 8x + 4z + 5z = 0 \\ -6x + 28x + 14z + 4z = 0 \end{cases}$

$\begin{cases} y = -4x - 2z \\ 11x + 9z = 0 \\ 22x + 18z = 0 \end{cases}$ ✓

$$\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + \operatorname{Span} \begin{pmatrix} -9 \\ 14 \\ 11 \end{pmatrix}$$

⑥ Sapendo che $2z^3 + (2i-11)z^2 + 3(7-i)z + i - 8 = 0$ ha soluz. $z = 1/2$ trovare le altre.

$$\begin{array}{c|ccc|c}
 & 2 & 2i-11 & 21-3i & i-8 \\
1/2 & & 1 & i-5 & 8-i \\
\hline
 & 2 & 2i-10 & 16-2i & \checkmark
\end{array}$$

$$(\dots) = (2z-1)\left(z^2 - (5-i)z + 8 - i\right)$$

$$\Delta = 25 - 10i - 1 - 32 + 4i = -8 - 6i$$

$$\begin{cases} a^2 - b^2 = -8 \\ ab = -3 \end{cases} \qquad a = 1 \quad b = -3$$

$$z_{2,3} = \frac{5 - i \pm (1 - 3i)}{2} = \begin{cases} 3 - 2i \\ 2 + i \end{cases}$$

$\boxed{40}$ (1) $\mathbb{R}[t]$ con solite operaz. è un campo?

Visto a lezione: valgono le 9 proprietà algebriche tranne esistenza inverso. Ed esso non vale:

$t \in \mathbb{R}[t]$ non ha inverso:

$$q(t)\cdot t \;\begin{cases} \to 0 \neq 1 & \text{se } q(t) = 0 \\ \to \deg \geq 1 & \text{se } q(t) \neq 0 \end{cases}$$

2 Dati 7 gen. di $\{x \in \mathbb{R}^5 : x_1 + 2x_2 + 3x_3 + 4x_4 + 5x_5 = 0\}$ quanti ne scarto per base?

$$7 - (5 - 1) = 3$$

③ Risolvere

$$\begin{cases} 3x + 2y - 2z = 1 \\ -2x + 5y + 3z = -15 \\ 4x + 3y + 5z = -7 \end{cases}$$

$$\det \begin{pmatrix} 3 & 2 & -2 \\ -2 & 5 & 3 \\ 4 & 3 & 5 \end{pmatrix} = \begin{array}{r} 75 \\ 48 \\ 12 \\ 40 \\ 20 \\ \hline 195 \end{array} \quad -27 \quad = 168$$

$$x = \frac{1}{178} \det \begin{pmatrix} 1 & 2 & -2 \\ -15 & 5 & 3 \\ -7 & 3 & 5 \end{pmatrix} =$$

$$= \frac{1}{168} \left(\cdot \quad \cdot \quad - \quad \right)$$

$$\begin{cases} 3x + 2y - 2z = 1 \\ -2x + 5y + 3z = -15 \\ 4x + 3y + 5z = -7 \end{cases}$$

$$\begin{cases} 2z = 3x + 2y - 1 \\ -4x + 10y + 9x + 6y - 3 = -30 \\ 8x + 6y + 15x + 10y - 5 = -14 \end{cases}$$

$$\begin{cases} 2z = 3x + 2y - 1 \\ 5x + 16y = -27 \\ 23x + 16y = -9 \end{cases}$$

$18x = 18$ $\qquad$ $x = 1$ $\quad$ $y = -2$ $\quad$ $z = -1$

4 $\quad A = (v_1 \; v_2 \; v_3)$ $\quad \det(A) = -i$

$\det(v_1 + iv_2,\ v_1 - iv_3,\ 2iv_2 + v_3) = ?$

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$$(-i)\cdot\det\begin{pmatrix} 1 & 1 & 0 \\ -i & 0 & 2i \\ 0 & -i & 1 \end{pmatrix}$$

$$= (-i)(-i-2) = i(i+2) = 2i-1$$

5 $f: \mathbb{R}^7 \longrightarrow \{x \in \mathbb{R}^4 : x_1 + x_2 = x_3 + x_4\}$

lin. surj.; $W \cap \operatorname{Ker} f = \{0\}$; $\dim W = ?$

$\dim \operatorname{Ker} f = 7 - 3 = 4 \implies 0 \leq \dim W \leq 3$

$\boxed{4.1}$ (3) $X = \{x \in \mathbb{R}^3 : x_1 + x_2 + x_3 = 0\}$

$\mathcal{B} = (e_1 + e_2 - 2e_3 , e_1 - e_3)$

$f : X \to X \qquad f(x) = \begin{pmatrix} 2x_1 - x_2 \\ x_2 + x_3 \\ -x_1 + x_2 \end{pmatrix}$

$[f]_{\mathcal{B}}^{\mathcal{B}} = ?$

$$f(x) = \underbrace{\begin{pmatrix} 2 & -1 & 0 \\ 0 & 1 & 1 \\ -1 & 1 & 0 \end{pmatrix}}_{A} \cdot x$$

non è vero che $f = f_A$ perché $f_A : \mathbb{R}^3 \to \mathbb{R}^3$.

Invece avremo $f = f_A\big|_X^X$ se verifichiamo che $f_A(X) \subset X$: infatti:

se $x \in X$ allora $f_A(x) \in X$, cioè che

se $x_1 + x_2 + x_3 = 0$ allora

$$\underbrace{(2x_1 - x_2) + (x_2 + x_3) + (-x_1 + x_2)}_{x_1 + x_2 + x_3} = 0$$

Vero

$$\begin{pmatrix} 2 & -1 & 0 \\ 0 & 1 & 1 \\ -1 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = \boxed{-1} \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} \quad \boxed{2} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

$$\begin{pmatrix} 2 & -1 & 0 \\ 0 & 1 & 1 \\ -1 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix} = \boxed{-1} \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} \quad \boxed{3} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

$$\Rightarrow [f]_{\mathcal{B}}^{\mathcal{B}} = \begin{pmatrix} \boxed{-1} & \boxed{-1} \\ \boxed{2} & \boxed{3} \end{pmatrix}$$

$\boxed{42}$ ② $f: \mathbb{R}^8 \to \mathbb{R}^3$ lin. surj.

$X \subset \mathbb{R}^8$ $\dim X = 5$

$\dim(X \cap \operatorname{Ker} f) = ?$

$\dim(\operatorname{Ker} f) = 5$

$2 \le \dim(X \cap \operatorname{Ker} f) \le 5$

$\boxed{43}$ ① Completare $e_1 + e_2 - e_4$ a base di

$$X = \{ x \in \mathbb{R}^4 : 3x_1 + 2x_2 - 4x_3 + 5x_4 = 0 \}$$

$$\begin{pmatrix} 1 \\ 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 5 \\ 4 \end{pmatrix} \quad \checkmark$$