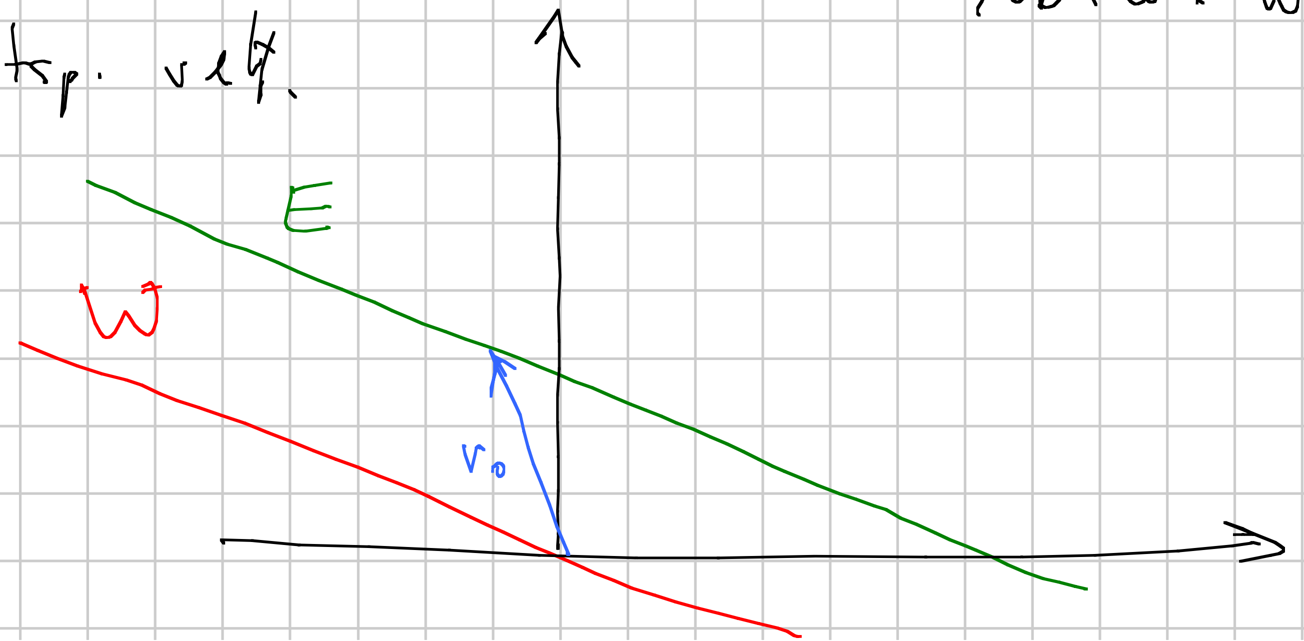
Algebra Lineare 6/12/15

V sp. vett. $E \subset V$ stsp. affine se $E = v_0 + W$

$= \{v_0 + w : w \in W\}$

$W \subset V$ stsp. vett.



Visto: W unico (giacitura); v_0 va bene qualsiasi el. di E.

$V = \mathbb{R}^n$

Presentaz. cart. di E:

$E = \{ x \in \mathbb{R}^n : A \cdot x = b \}$; due casi

$\underset{\mathcal{M}_{m \times n}}{\underset{\Uparrow}{A}}$

$\Rightarrow E = \emptyset$ (non è stsp. aff)

$\Rightarrow E \neq \emptyset$

$\Rightarrow E = v_0 + \mathrm{Ker}(A)$

$\Rightarrow \dim(E) = \dim(\mathrm{Ker}(A))$

$= n - \mathrm{rank}(A)$

$\geq n - m$

Presentaz. param. di E

$$E = v_0 + \mathrm{Span}(w_1, \dots, w_h)$$

$$\Rightarrow \dim(E) = \dim(\mathrm{Span}(w_1, \dots, w_h)) \leq h$$

Prop: se $E \subset \mathbb{R}^n$ è un sttsp. aff. di dim. p allora ammette
- presentaz. param. con p parametri
- presentaz. cartesiana con $n-p$ equazioni.

Dim: $E = v_0 + W$, $\dim W = p$;
prendo $w_1, \dots, w_p$ base di W

$\Rightarrow E = v_0 + \mathrm{Span}(w_1, \dots, w_p)$ ✓

Cerco $A \in M_{(n-p)\times n}(\mathbb{R})$ t.c. $\ker(A) = \mathrm{Span}(w_1, \dots, w_p)$;

Trovata tale A sono a posto:

$$E = \{x \in \mathbb{R}^n : A \cdot x = b\} \quad \text{con } b = A \cdot v_0.$$

Basterà prendere $A = [f]^{\mathcal{E}[n-p]}_{\mathcal{E}[n]}$ dove $f: \mathbb{R}^n \to \mathbb{R}^{n-p}$ ha nucleo $\mathrm{Span}(w_1, \dots, w_p)$. Per trovare f: completo $w_1, \dots, w_p$ a base $w_1, \dots, w_n$ di $\mathbb{R}^n$ e pongo:

$$f(w_i) = \begin{cases} 0 & i = 1 \dots p \\ e_{i-p} & i = p+1, \dots, n \end{cases}$$

$$[f]^{\mathcal{E}^{[n-p]}}_{(w_1, \dots, w_n)} = \underbrace{\left(\underbrace{\;0\;}_{p} \,\middle|\, \underbrace{I_{n-p}}_{n-p} \right)}_{n} \Bigg\} {n-p}$$

$\blacksquare$

Passaggi :

- cart $\rightsquigarrow$ param : risolvere le equaz.
- param $\rightsquigarrow$ cart : "devo eliminare i parametri trovando equazioni soddisfatte dai pti di E"

Es :

$$E = \begin{pmatrix} 1 \\ -1 \\ 2 \\ 4 \end{pmatrix} + \underbrace{\mathrm{Span}\left(\begin{pmatrix} 2 \\ 1 \\ -3 \\ 5 \end{pmatrix}, \begin{pmatrix} 3 \\ -4 \\ 1 \\ -2 \end{pmatrix} \right)}_{W}$$

$$\dim(W) = 2$$

$\Rightarrow \dim(E) = 2$

$\Rightarrow$ serviranno $4 - 2 = 2$ equazioni

$$E:\begin{cases} x_1 = 1 + 2t + 3s \\ x_2 = -1 + t - 4s \\ x_3 = 2 - 3t + s & \checkmark \\ x_4 = 4 + 5t - 2s & \checkmark \end{cases}$$

$$\text{III}: \quad s = x_3 + 3t - 2 \qquad s = x_3 - 6x_3 - 3x_4 + 24 - 2 = -5x_3 - 3x_4 + 22$$

$$\text{IV}: \quad x_4 = 4 + 5t - 2x_3 - 6t + 4 \,, \qquad t = -2x_3 - x_4 + 8$$

$$E:\begin{cases} x_1 = 1 - 4x_3 - 2x_4 + 16 - 15x_3 - 9x_4 + 66 \\ x_2 = -1 - 2x_3 - x_4 + 8 + 20x_3 + 12x_4 - 88 \end{cases}$$

$$E:\begin{cases} x_1 + 19x_3 + 11x_4 = 83 \\ x_2 - 18x_3 - 11x_4 = -81 \end{cases}$$

Rette in $\mathbb{R}^2$

param : $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + \mathrm{Span}\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ α, β non entrambi 0

cart : $ax + by = c$ a, b non entrambi 0

param $\rightsquigarrow$ cart : $\beta x - \alpha y = \beta x_0 - \alpha y_0$

cart $\rightsquigarrow$ param : $\frac{c}{a^2+b^2}\begin{pmatrix} a \\ b \end{pmatrix} + \mathrm{Span}\begin{pmatrix} b \\ -a \end{pmatrix}$

funziona anche se $a=0$ o $b=0$

mentre

$$\begin{pmatrix} c/a \\ 0 \end{pmatrix} \quad o \quad \begin{pmatrix} 0 \\ c/b \end{pmatrix}$$

non sempre

Rette in $\mathbb{R}^3$

param: $\begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} + \mathrm{Span}\begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}$ con α, β, γ non tutti 0

cart: $\begin{cases} a_1 x + b_1 y + c_1 z = d_1 \\ a_2 x + b_2 y + c_2 z = d_2 \end{cases}$ con $\mathrm{rank}\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{pmatrix} = 2$

ovvero uno dei det di

$$\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}, \begin{pmatrix} a_1 & c_1 \\ a_2 & c_2 \end{pmatrix}, \begin{pmatrix} b_1 & c_1 \\ b_2 & c_2 \end{pmatrix} \grave{e} \neq 0$$

param $\rightsquigarrow$ cart:

$$r = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} + \operatorname{Span}\begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} \Longrightarrow r: \begin{cases} \beta x - \alpha y = \beta x_0 - \alpha y_0 \\ \gamma x - \alpha z = \gamma x_0 - \alpha z_0 \\ \gamma y - \beta z = \gamma y_0 - \beta z_0 \end{cases}$$

Sono tre equazioni ma

$$\det\begin{pmatrix} \beta & -\alpha & 0 \\ \gamma & 0 & -\alpha \\ 0 & \gamma & -\beta \end{pmatrix} = 0 + 0 + 0 - 0$$

$$- (-\alpha)\beta(-\gamma) - \beta(-\alpha)\gamma = 0$$

$\Rightarrow$ sono lin. dip.

almeno una 2×2 ha sempre $\det \neq 0 \Rightarrow \text{rank} = 2$.

cart $\leadsto$ param : π : $\begin{cases} a_1 x + b_1 y + c_1 z = d_1 \\ a_2 x + b_2 y + c_2 z = d_2 \end{cases}$

* giacitura: vogliamo un vettore che genera le soluz. di

$$\begin{cases} a_1 x + b_1 y + c_1 z = 0 \\ a_2 x + b_2 y + c_2 z = 0 \end{cases}$$

Oss: $\det\begin{pmatrix} a_1 & b_1 & c_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{pmatrix} = \det\begin{pmatrix} a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{pmatrix} = 0$

$$\Rightarrow \quad a_1 \cdot \det\begin{pmatrix} b_1 & c_1 \\ b_2 & c_2 \end{pmatrix} + b_1 \cdot \left(-\det\begin{pmatrix} a_1 & c_1 \\ a_2 & c_2 \end{pmatrix}\right) + c_1 \cdot \det\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} = 0$$

$$a_2 \cdot \quad // \quad + b_2 \quad // \quad + c_2 \quad // \quad = 0$$

$$\Rightarrow \begin{pmatrix} b_1c_2 - b_2c_1 \\ -(a_1c_2 - a_2c_1) \\ a_1b_2 - a_2b_1 \end{pmatrix} \text{ è la soluz. cercata}$$

(è $\neq 0$ poiché $\operatorname{rank}\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{pmatrix} = 2$).

* soluz. part. non omogeneo: qualsiasi metodo;

ad es. se $\det\begin{pmatrix} b_1 & c_1 \\ b_2 & c_2 \end{pmatrix} \neq 0$ so che

$$\begin{cases} b_1 y + c_1 z = d_1 \\ b_2 y + c_2 z = d_2 \end{cases} \text{ ha soluz. unica } \begin{pmatrix} y_0 \\ z_0 \end{pmatrix}$$

e posso prendere $v_0 = \begin{pmatrix} 0 \\ y_0 \\ z_0 \end{pmatrix}$.

Es:
$$\begin{cases} 3x - 5y + 2z = 7 \\ 4x + y - 2z = 6 \end{cases}$$

- giacitura: Span
$$\begin{pmatrix} (-5)\cdot(-2) - 1\cdot 2 \\ -(3\cdot(-2) - 4\cdot 2) \\ 3\cdot 1 - 4\cdot(-5) \end{pmatrix} = \begin{pmatrix} 8 \\ 14 \\ 23 \end{pmatrix}$$

Verifica:
$$24 - 70 + 46 = 0 \quad \checkmark$$
$$32 + 14 - 46 = 0 \quad \checkmark$$

- pto base : scelgo $x=0$ e risolvo

$$\begin{cases} -5y + 2z = 7 \\ y - 2z = 6 \end{cases} \qquad y = -13/4 \qquad z = -11/8$$

$$\Longrightarrow \quad r = \begin{pmatrix} 0 \\ -13/4 \\ -11/8 \end{pmatrix} + \mathrm{Span}\begin{pmatrix} 8 \\ 14 \\ 23 \end{pmatrix}$$

———— o ————

Piani in $\mathbb{R}^3$:

Param: $\begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} + \operatorname{Span}\Bigg(\overbrace{\begin{pmatrix} \alpha_1 \\ \beta_1 \\ \gamma_1 \end{pmatrix}\begin{pmatrix} \alpha_2 \\ \beta_2 \\ \gamma_2 \end{pmatrix}}^{\text{rank}=2}\Bigg)$

Cart $\quad ax+by+cz=d \qquad$ a, b, c non tutti 0

cart $\rightsquigarrow$ param: $\dfrac{d}{a^2+b^2+c^2}\begin{pmatrix} a \\ b \\ c \end{pmatrix} + \operatorname{Span}\Bigg(\underbrace{\begin{pmatrix} b \\ -a \\ 0 \end{pmatrix}\begin{pmatrix} c \\ 0 \\ -a \end{pmatrix}\begin{pmatrix} 0 \\ c \\ -b \end{pmatrix}}\Bigg)$

$$\begin{cases} \det\begin{pmatrix} \beta_1 & \beta_2 \\ \gamma_1 & \gamma_2 \end{pmatrix} \cdot \alpha_1 - \det\begin{pmatrix} \alpha_1 & \alpha_2 \\ \gamma_1 & \gamma_2 \end{pmatrix} \cdot \beta_1 + \det\begin{pmatrix} \alpha_1 & \alpha_2 \\ \beta_1 & \beta_2 \end{pmatrix} \cdot \gamma_1 = 0 \\ \qquad '' \qquad \cdot \alpha_2 \qquad '' \qquad \cdot \beta_2 \qquad '' \qquad \cdot \gamma_2 = 0 \end{cases}$$

$$\Rightarrow (\beta_1\gamma_2 - \beta_2\gamma_1)x - (\alpha_1\gamma_2 - \alpha_2\gamma_1)y + (\alpha_1\beta_2 - \alpha_2\beta_1)z = 0$$

$\star$ termine noto: