ETA 27/10/15

$M^{(m)}, N^{(m)}$ m-var chiuse orientate (DIFF)

$f : M \to N$ DIFF , y val reg

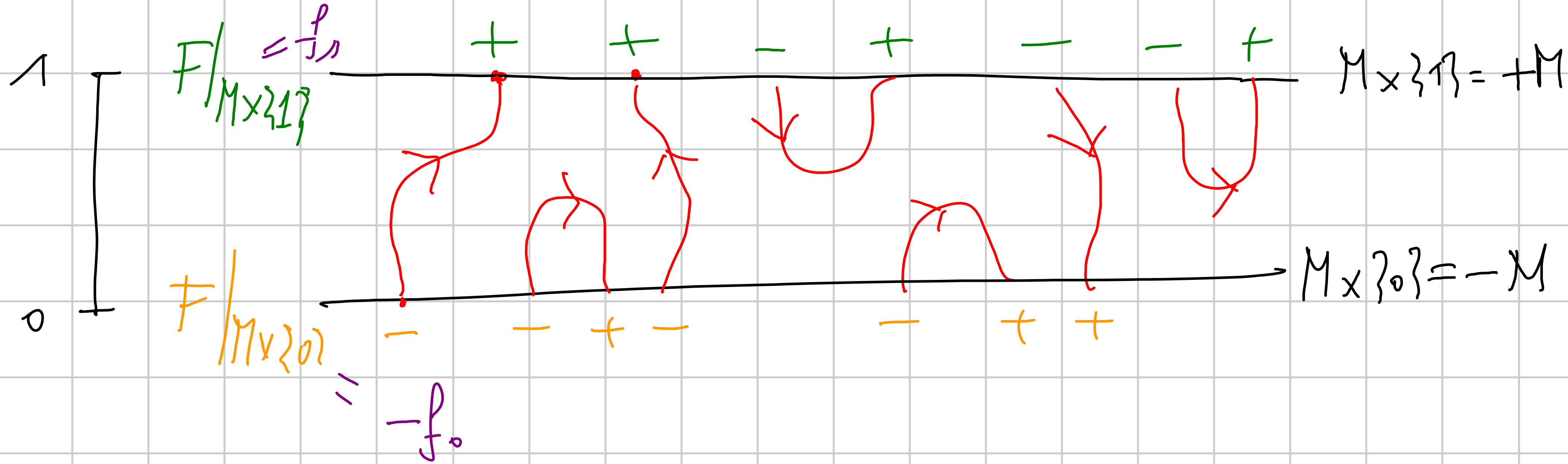
$$\deg(f, y) = \underset{alg}{\#} f^{-1}(y)$$

Oss: val critici sono chiuso e $\deg(f, y)$ loc. cost w.r.t. y.

__Prop__: $\deg(f,y)$ indip da y + inv. per $\simeq$
($\Longrightarrow$ è ben def $\deg(f)$ per f continua).

__Dim__: __Passo 1__: $f_0 \simeq f_1$ via F e y reg. per F
$\Longrightarrow \deg(f_0,y) = \deg(f_1,y)$.

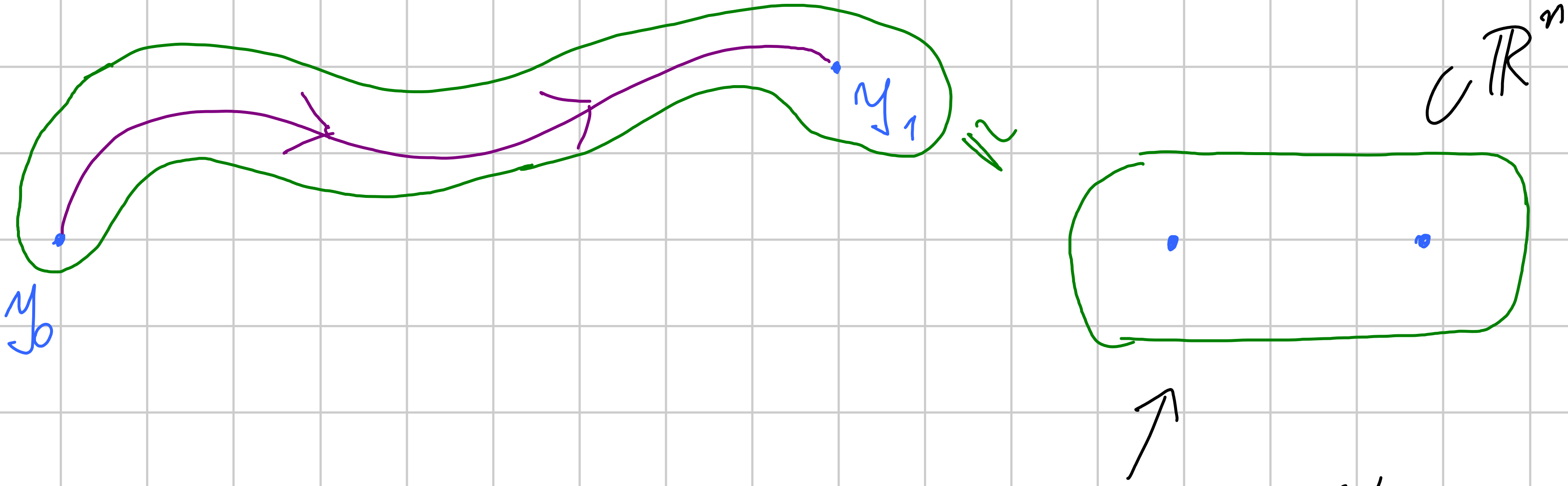
Sappiamo: $F^{-1}(y)$ è sottovar. orientata di $M \times [0,1]$
con orientaz. coerenti su $\partial F^{-1}(y)$.



<u>Passo 2</u>: dati $y_0, y_1 \in N$ $\quad \exists\, G : N \times [0,1] \to N$ (isotopia)

$g_t = G(\cdot, t)$ ognuno è diffeo

$g_0 = id$ $\quad g_1(y_0) = y_1$



qui è zero rel ∂

<u>Passo 3</u> : y_0, y_1 val. reg. per f

↳ come nel passo 2

Uso il passo 1 con $g_1 \circ f \simeq f$ via $F(x,t) = g_t(f(x))$
$= G(f(x), t)$

e wlog uso y_1 come val reg. (perturbo in modo che sia regolare per F).

$$(g_1 \circ f)^{-1}(y_1) = f^{-1}(g_1^{-1}(y_1)) = f^{-1}(y_0)$$

$\Rightarrow y_1$ è reg. anche per $g_1 \circ f$

Ora: $\deg(f, y_1) \underset{P.I}{=} \deg(g_1 \circ f, y_1) = \deg(g_1 \circ f, g_1(y_0)) = \deg(f, y_0)$.

Ho la buona def. Per P.I. ho inv. per $\simeq$. $\boxed{//}$

<u>Oss</u>: se $M^{(n)}, N^{(n)}$ chiuse orientate o PL o DIFF

$f : M \to N$ continua, ho $\deg^{PL}(f) = \deg^{DIFF}(f)$.

(Ovviamente diciamo solo tecnico.)

Applicaz. della nozione di grado.

1. Immersioni $S^1 \hookrightarrow \mathbb{R}^2 \;/\; \simeq_{reg}$

|

mappe $\alpha : S^1 \to \mathbb{R}^2$
con $\alpha' \neq 0$

\

$\alpha : S^1 \times [0,1] \to \mathbb{R}^2$
con $\alpha_t' \neq 0 \ \forall t$
(omotopie via immersioni).

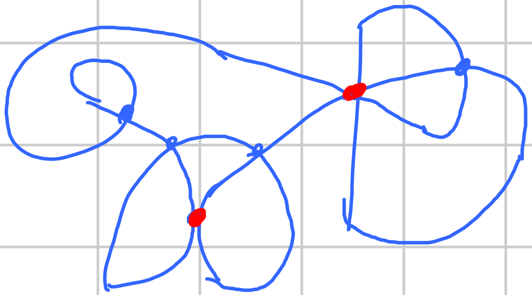
Teo: $\left\{ S^1 \looparrowright \mathbb{R}^2 \right\} \big/ \simeq_{\text{reg}} \xrightarrow{\quad w \quad} \mathbb{Z}$ biiettiva

(writhe) $\qquad w(\alpha) = \deg\left(\dfrac{\alpha'}{\|\alpha'\|} : S^1 \longrightarrow S^1 \right)$.

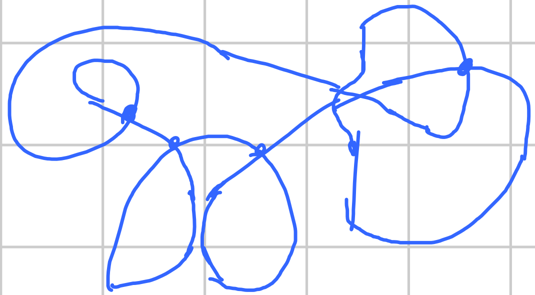
Dim: buona def: $(\alpha_t)_{t\in[0,1]}$ omotopie reg

$$\left(\frac{\alpha'_t}{\|\alpha'_t\|}\right)_{t\in[0,1]}$$ omotopie —

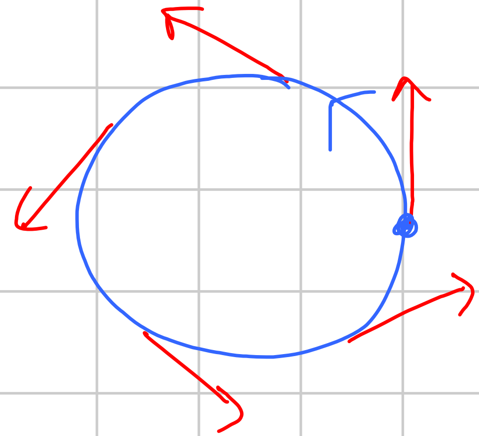
Osservo che $\alpha: S^1 \to \mathbb{R}^2$ è rappresentata a meno di $\simeq_{reg}$ da $In(\alpha)$ come curve singolare orientata e wlog che abbia ~~solo~~ sing. del tipo punti doppi trasversali:

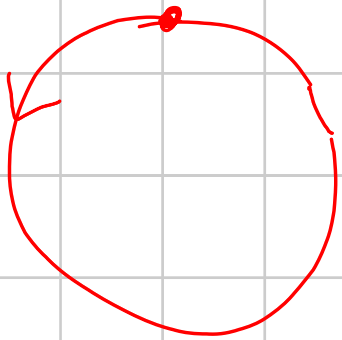


$\simeq$

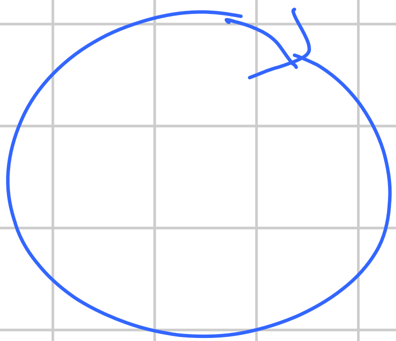


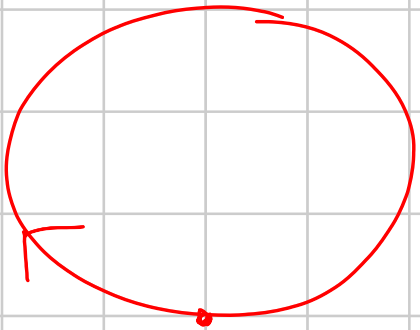
Suriettività :



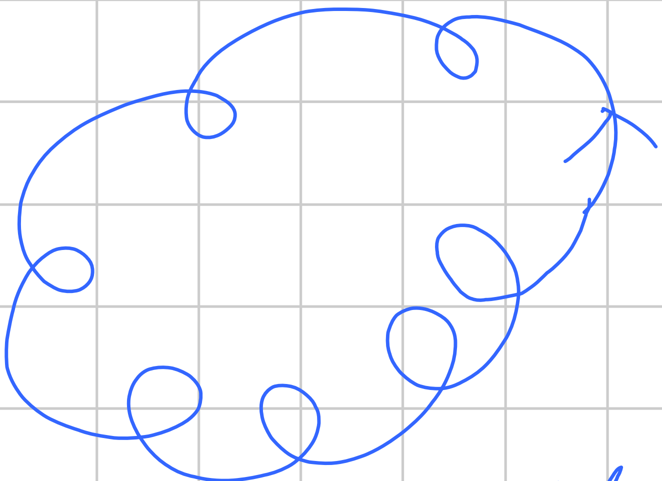


+1

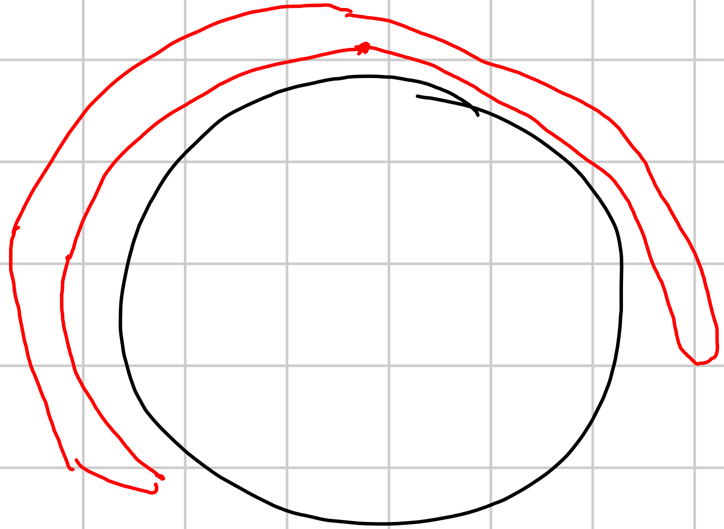




−1



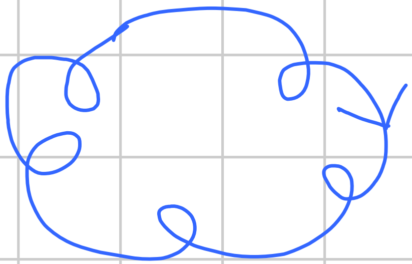
$n-1$ riccioli



0

$\alpha'/\|\alpha'\|$

$+n$

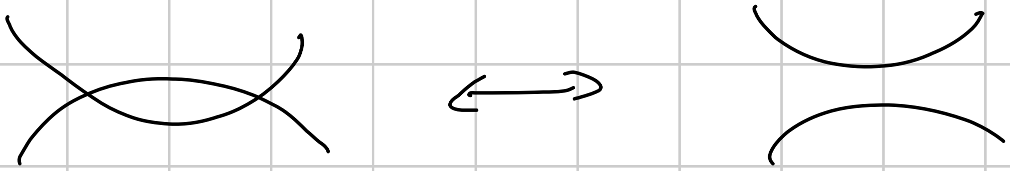


$- m$

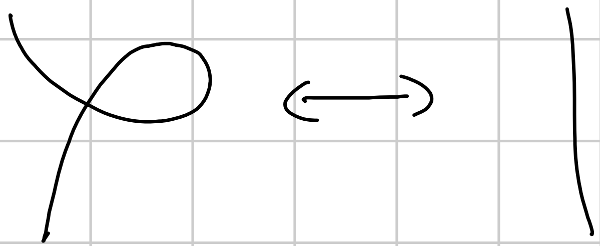
Iniettività: provo che $/ \simeq_{rep}$ mi riconduco a uno dei modelli di sopra

<u>Oss</u>: sono certamente generate da $\simeq_{rep}$ le mosse

(R_{III})



(R_{II})

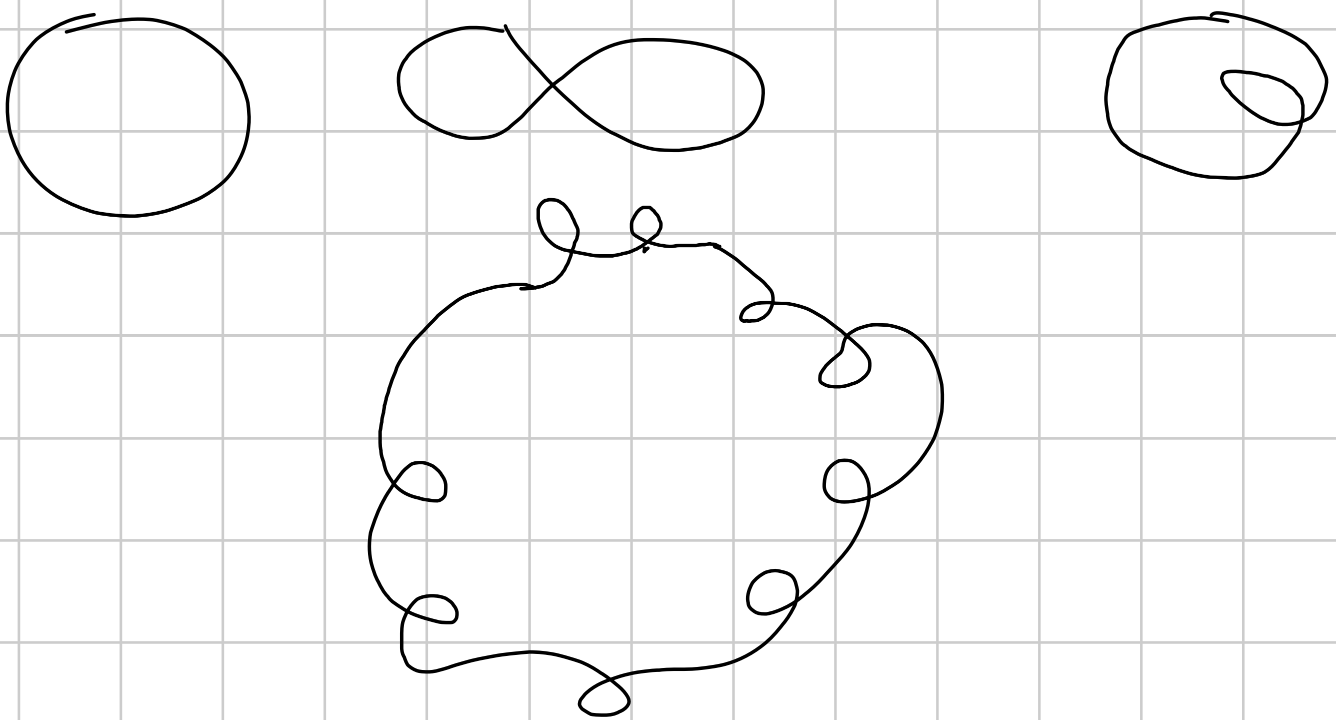
(NON:  (R_I)).

<u>Passo 1</u>: se α è iniettiva ho uno dei modelli ± 1

Se non lo è prendo due pti consecutivi su S^1 con stessa immagine (prendo immagine di un segmento innermost per α^{-1}(sing))

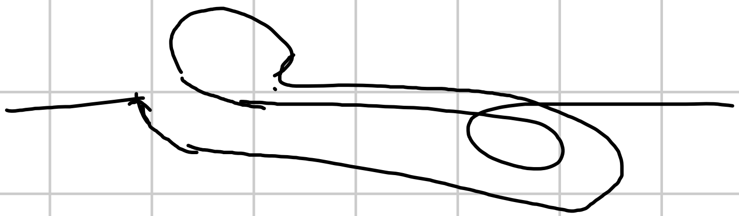
$\underset{\text{nep}}{\simeq}$

Passo 2 : dopo iteraz. passo 1 ho solo una circonf.
con riccioli: dentro/fuori

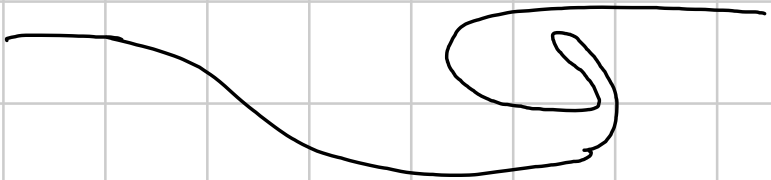


Affermo che posso supporli tutti dentro o tutti fuori.
Altrimenti:

$\simeq_{reg}$

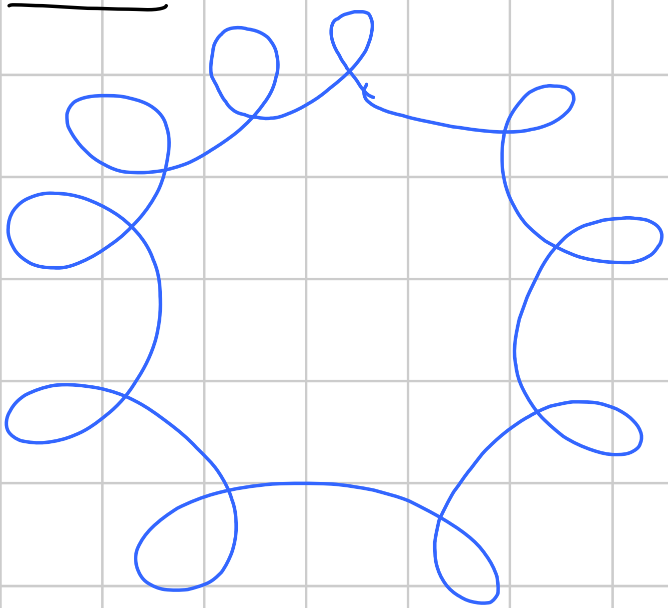


$\simeq_{reg}$



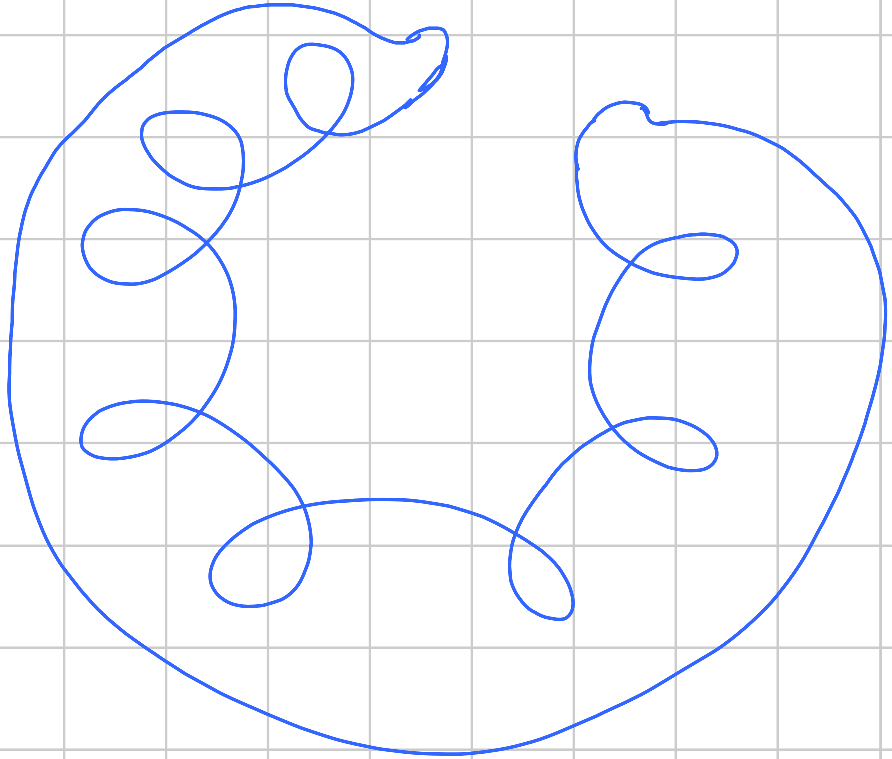
$=$

Passo 3 : tutti dentro $\longleftrightarrow$ tutti fuori



$k \geq 2$ fuori

$\simeq_{reg}$



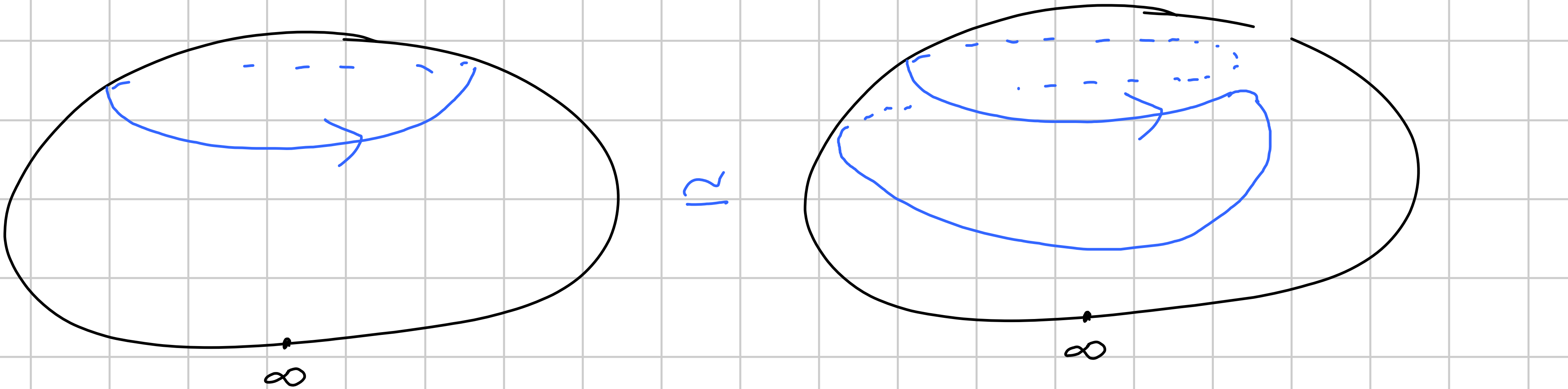
$k-2$ dentro

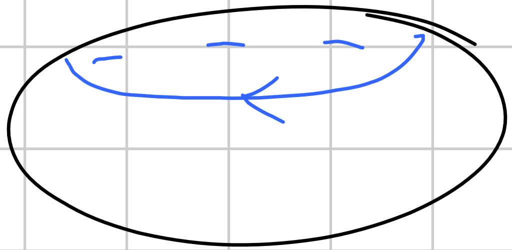
$\blacksquare$

Cor: $\{S^1 \hookrightarrow S^2\}/\simeq_{reg} \longleftrightarrow \mathbb{Z}/2$

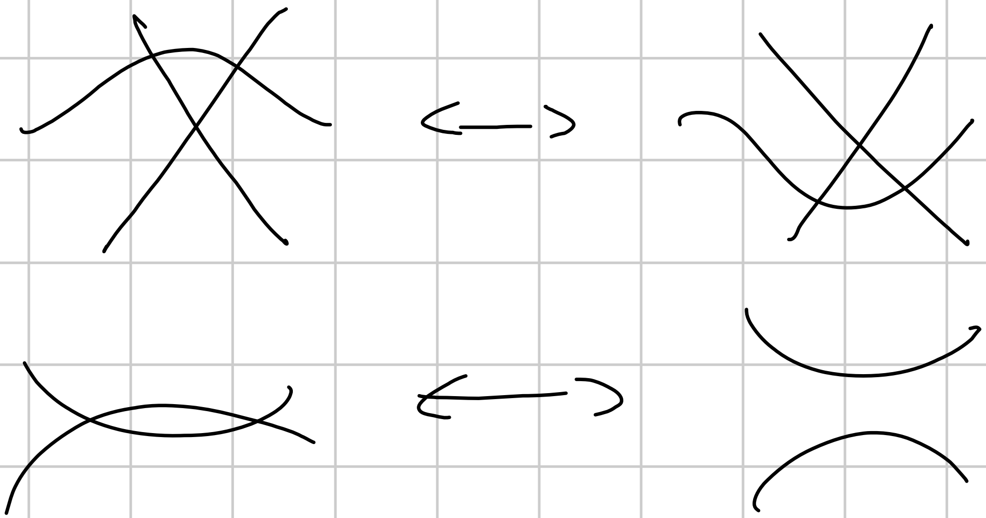
via $[w]_2$

La differenza è che una omotopia non può attraversare ∞



$=$ 

Dunque sopravvive al più $[W]_2$. Ma



non nulla $[\#\text{incroci}]_2 = [W]_2$. $\boxed{//}$

2. **Teo**: $\{f : S^1 \to S^1\}/_{\simeq} \longleftrightarrow \mathbb{Z}$ via deg.

Dim: ben def.

$\deg(z \mapsto z^n) = n \Rightarrow$ surj.

Devo provare che $\deg(f_0) = \deg(f_1) \Longrightarrow f_0 \simeq f_1$.

Idea: prendo l'anello $S^1 \times [0,1]$ dove ho f_j su $S^1 \times \{j\}$; cerco di estendere a $F : S^1 \times [0,1] \to S^1$.

Prendo y val. reg. per entrambe:

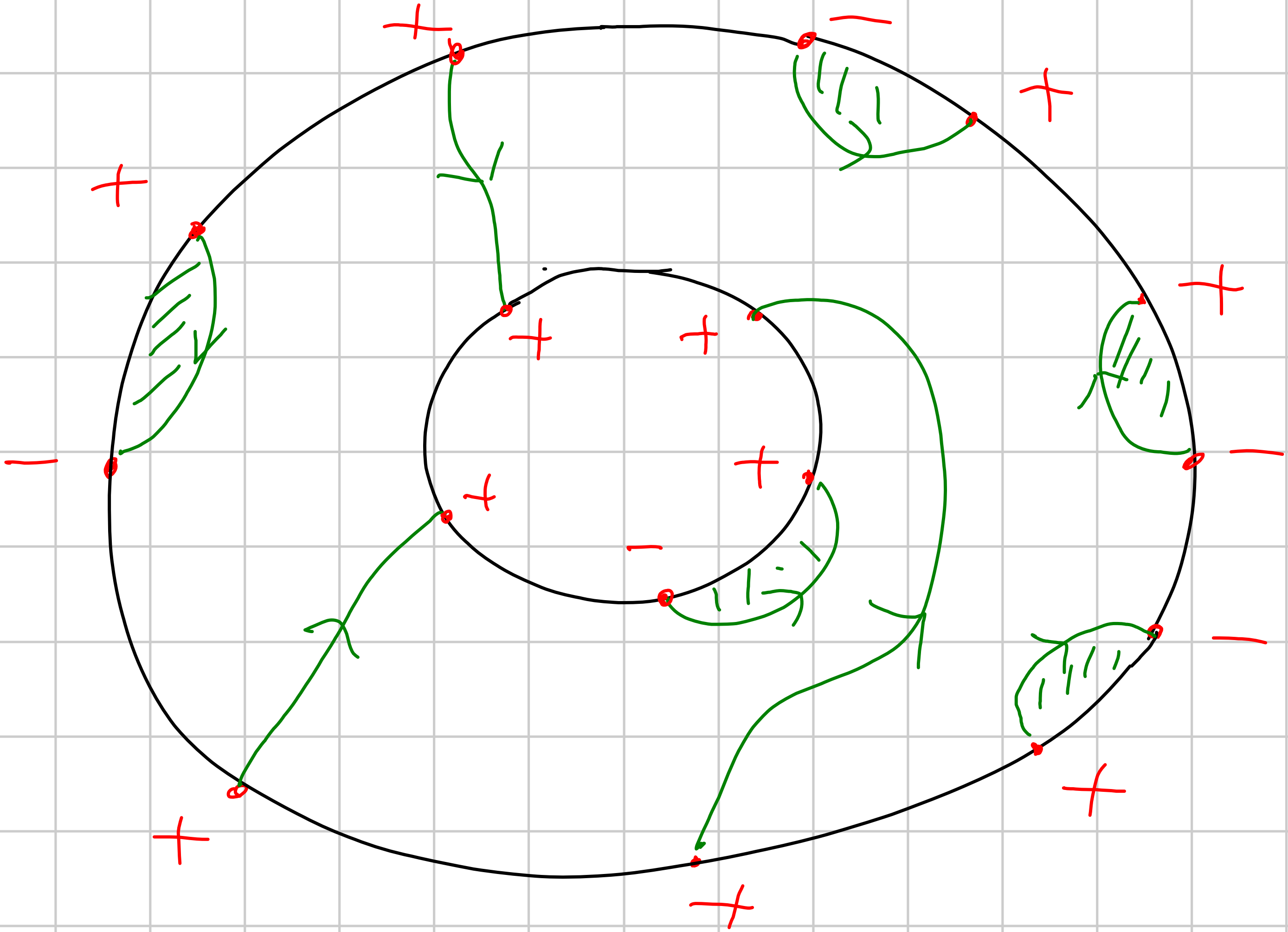
$$m_+^{(0)} / m_-^{(0)} \quad \text{pti pos/neg per } f_0^{-1}(y)$$

$$m_+^{(1)} / m_-^{(1)} \quad \text{pti pos/neg per } f_1^{-1}(y)$$

So: $m_+^{(0)} - m_-^{(0)} = m_+^{(1)} - m_-^{(1)}$.

Affermo che esistono archi $\beta_1, \dots, \beta_q$ orientati propriamente embedded in $S^1 \times [0,1]$ t.c.

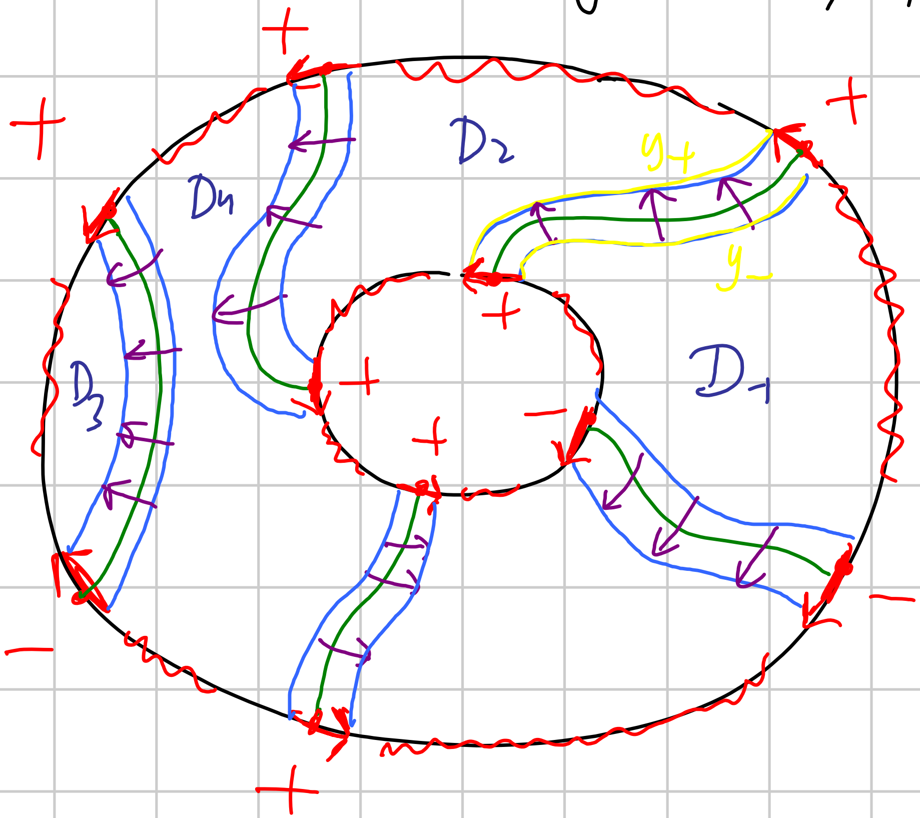
$$\partial \beta_i = \begin{cases} \text{pti concord. su basi distinte} \\ \text{pti discord. su stessa base} \end{cases}$$



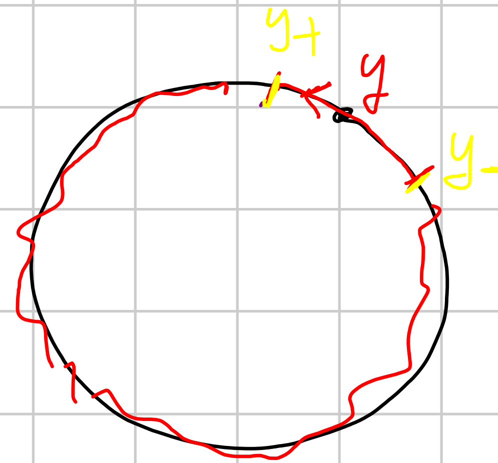
t.c. ognuno dei

$f_j^{-1}(g)$ sia in $\partial\beta_i$

Suppongo di avere almeno un β_i con estremi su bordi diversi (automatico se $\deg \neq 0$; per $\deg = 0$ ... esercizio).



$$\xrightarrow[F]{f_0,\, f_1}$$



Ho già def. $F/\partial D_k$ ed è a valori in $S^1 \setminus [y_-, y_+] = $

Dunque $h_0 : \partial D^2 \longrightarrow [0,1]$ che posso studiare razionalmente.

$\boxed{/\!/}$

3. Teo. fond. algebra: $f(z) = a_d z^d + \dots + a_0 \in \mathbb{C}[z]$, $a_d \neq 0$, $d \geq 1$ ha radici.

$f : \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$ $\qquad f(\infty) = \infty$

Affermo che ha grado $d \Longrightarrow$ surgettiva $\Longrightarrow$ ok.

Esamino f vicino a ∞ usando $\Delta \ni w \xmapsto{\alpha} \begin{cases} \infty & w = 0 \\ 1/w & w \neq 0 \end{cases}$.

$$(\alpha^{-1} \circ f \circ \alpha)(w) = \frac{1}{a_d \frac{1}{w^d} + \dots + a_0} = \frac{w^d}{a_d + a_{d-1} \cdot w + \dots + a_0 \cdot w^d}$$

$$= w^d \cdot h(w) \qquad h \in Hol(\Delta_\varepsilon), \; h(0) \neq 0$$

Fatto: $\exists g \in Hol(\Delta_\varepsilon) \quad g(w)^d = h(w)$

$$\Longrightarrow (\alpha^{-1} \circ f \circ \alpha)(w) = (\underbrace{w \cdot g(w)}_{t(w)})^d$$

$t \in \mathcal{H}ol(\Delta_\varepsilon) \qquad t(0) = 0 \qquad t'(0) = g(0) \neq 0$

$\Longrightarrow$ posso usare t come carta

$\Longrightarrow$ f letta in carte opportune è $\quad w \longmapsto w^d$

che ha grado d (0 unico valore non reg.). $\boxed{///}$

4. Hairy ball : $\underbrace{\not\exists v : S^n \to \mathbb{R}^{n+1}}_{n \text{ pari}}$ mai nullo con

$v(x) \perp x \quad \forall x$.

Infatti: p.a.

1. $w = \frac{v}{\|v\|} : S^n \to S^n$, $w(x) \neq x, -x \quad \forall x$

2. $f_0, f_1 : S^n \to S^n$ mai antipodali $\Rightarrow f_0 \simeq f_1$

$$F(t) = \frac{(1-t)f_0 + t f_1}{\| \quad " \quad \|}$$

3. Da 1+2 deduco $id_{S^n} \simeq -id_{S^n}$; ma $\deg(id_{S^n}) = 1$

$\deg(-id_{S^n})$: in e_0 , $T_{e_0} S^n = \mathrm{Span}(\underbrace{e_1, \dots, e_n}_{\text{pos}})$

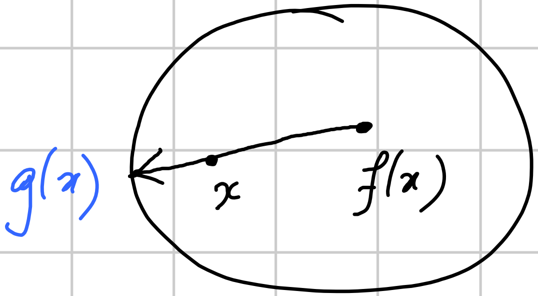
$d(id_{S^n}) : (e_1, \dots, e_n) \longmapsto (-e_1, \dots, -e_n)$

in $-e_0$ il segno $(-e_1,\dots,-e_n)$ è quello di

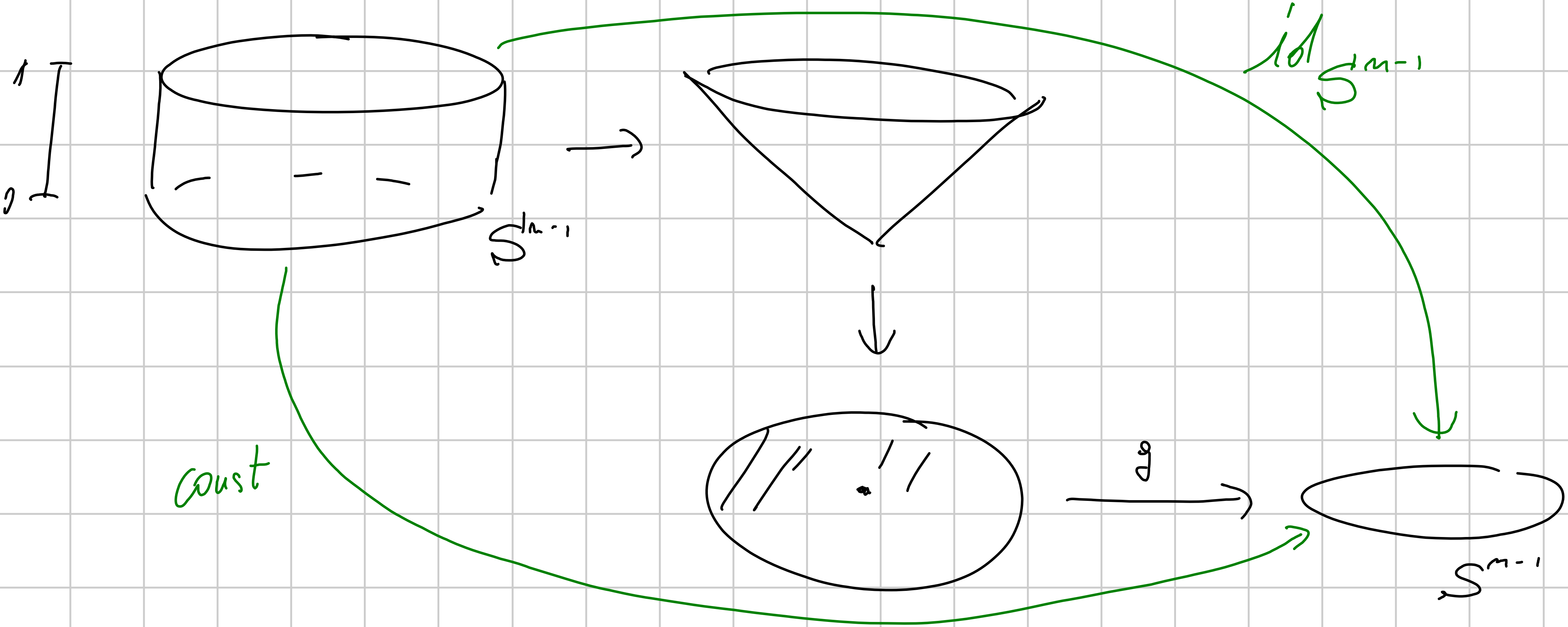
$$(-e_0,-e_1,\dots,-e_n) = (-1)^{n+1}$$

impossibile per n pari.

5. Brouwer: $f : D^4 \circlearrowleft$ continua $\Rightarrow \mathrm{Fix}(f) \neq \emptyset$.



$g : D^m \to S^{m-1}$ $\qquad g|_{S^{m-1}} = id$



$\Rightarrow \quad id_{S^{n-1}} \simeq const$

deg = 1   deg = 0

<u>ab</u> . $\blacksquare$