ETA 17/12/15

UCT : $H_* \Rightarrow H^*$. Però: H^* ha un prodotto.

R = anello con 1.

<u>Def</u>: $\cup : C^k(X;R) \times C^\ell(X;R) \to C^{k+\ell}(X;R)$

$$(\varphi \cup \psi)([v_0, \dots, v_{k+\ell}]) = \varphi([v_0, \dots, v_k]) \cdot \psi([v_k, \dots, v_{k+\ell}])$$

(sia PL sia SING).

Lem: $\delta_{k+\ell}(\varphi \cup \psi) = (\delta_k \varphi) \cup \psi + (-1)^k \varphi \cup (\delta_\ell \psi)$

(come $\otimes$)

Dim: esempio $k = \ell = 1$.

$\varphi, \psi \in C^1 \qquad \varphi \cup \psi \in C^2 \qquad \delta_2(\varphi \cup \psi) \in C^3$

$$(\delta_2(\varphi \cup \psi))(0123) = (\varphi \cup \psi)(\partial_3(0123)) = (\varphi \cup \psi)(123 - 023 + 013 - 012)$$

$$= \underbrace{\varphi(12) \cdot \psi(23)}_{1} - \varphi(02) \cdot \psi(23) + \varphi(01) \cdot \psi(13) - \varphi(01) \cdot \psi(12)$$

$$((\delta_1 \varphi) \cup \psi - \varphi \cup (\delta_1 \psi))(0123) = (\delta_1 \varphi)(012) \cdot \psi(23) - \varphi(01)(\delta_1 \psi)(123)$$

$$= \varphi(12 - 02 + 01)\,\psi(23) - \varphi(01)\cdot\psi(12 - 13 + 23)$$

$$= \varphi(12)\cdot\psi(23) - \varphi(02)\,\psi(23) + \cancel{\varphi(01)\cdot\psi(23)}$$
$$- \varphi(01)\cdot\psi(12) + \varphi(01)\cdot\psi(13) - \cancel{\varphi(01)\cdot\psi(23)}$$

$$\delta(\varphi\cup\psi) = \pm(\delta\varphi)\cup\psi \pm \varphi\cup(\delta\psi)$$

$$\Longrightarrow\ Z^k \cup Z^\ell \subset Z^{k+\ell} \qquad Z^k \cup B^\ell \subset B^{k+\ell}$$
$$B^k \cup Z^\ell \subset B^{k+\ell}$$

$\Longrightarrow\ \cup : H^k \times H^\ell \longrightarrow H^{k+\ell}$ ben definito.

$$H^* = \bigoplus_{k=0}^{+\infty} H^k \qquad R\text{-modulo prodotto} \qquad (X \text{ a.c.})$$

Proprietà di $\cup : H^* \times H^* \longrightarrow H^*$

- rispetta la graduazione
- associativo
- distributivo
- $\exists\, \underline{1} \in R = H^0(X;R)$
- "commut" $\quad \varphi \in H^k,\ \psi \in H^\ell \qquad \psi \cup \varphi = (-1)^{k \cdot \ell}\, \varphi \cup \psi$.

Dim: Definisco $\rho_n : C_n(X) \circlearrowleft$

$$\rho_n([v_0, \ldots, v_n]) = \varepsilon_n \cdot [v_n, \ldots, v_0]$$

$$\varepsilon_n = \frac{n(n+1)}{2}$$

Affermo che $r_n : \Delta_n \to \Delta_n$ $\quad e_j \mapsto e_{n-j}$

è omotopa $\varepsilon_n \cdot id_{\Delta_n}$. Ne segue che ρ_n è omotopa (come mappe tra c.c.) a id.

$$e_0, \ldots, e_n \longrightarrow e_n \ldots e_0$$

$$e_1 - e_0, e_2 - e_0, \ldots, e_{n-1} - e_0, e_n - e_0 \to e_{n-1} - e_n, e_{n-2} - e_n, \ldots, e_0 - e_n$$

Matrice di cambio:

$$\begin{pmatrix} 0 & 0 & \cdots & 0 & 1 & 0 \\ \vdots & \vdots & & & 0 & \vdots \\ & & & & \vdots & \vdots \\ 0 & 1 & & & & \vdots \\ 1 & 0 & & & 0 & \vdots \\ -1 & -1 & & & -\underline{1} & -1 \end{pmatrix}$$

$$\det = (-1) \cdot \det \overbrace{\begin{pmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{pmatrix}}^{n-1}$$

$$= (-1)\,(-1)^{n-1+1} \cdot \det \overbrace{\begin{pmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{pmatrix}}^{n-2}$$

$$= \cdots = \varepsilon_n .$$

$$(\psi \cup \varphi)([v_0, \ldots, v_{k+\ell}]) = \varepsilon_{k+\ell}(\psi \circ \varphi)[v_{k+\ell}, \ldots, v_0]$$

$$= \varepsilon_{k+\ell} \quad \psi([v_{k+\ell} \ldots v_k]) \cdot \varphi([v_k, \ldots, v_0])$$

$$= \underbrace{\varepsilon_{k+l} \cdot \varepsilon_k \cdot \varepsilon_l}_{(-1)^{k \cdot l}} \cdot \underbrace{\psi([v_k, \dots, v_{k+l}]) \cdot \varphi([v_0, \dots, v_l])}_{(\varphi \cup \psi)[v_0, \dots, v_{k+l}]} \qquad \boxed{/\!/}$$

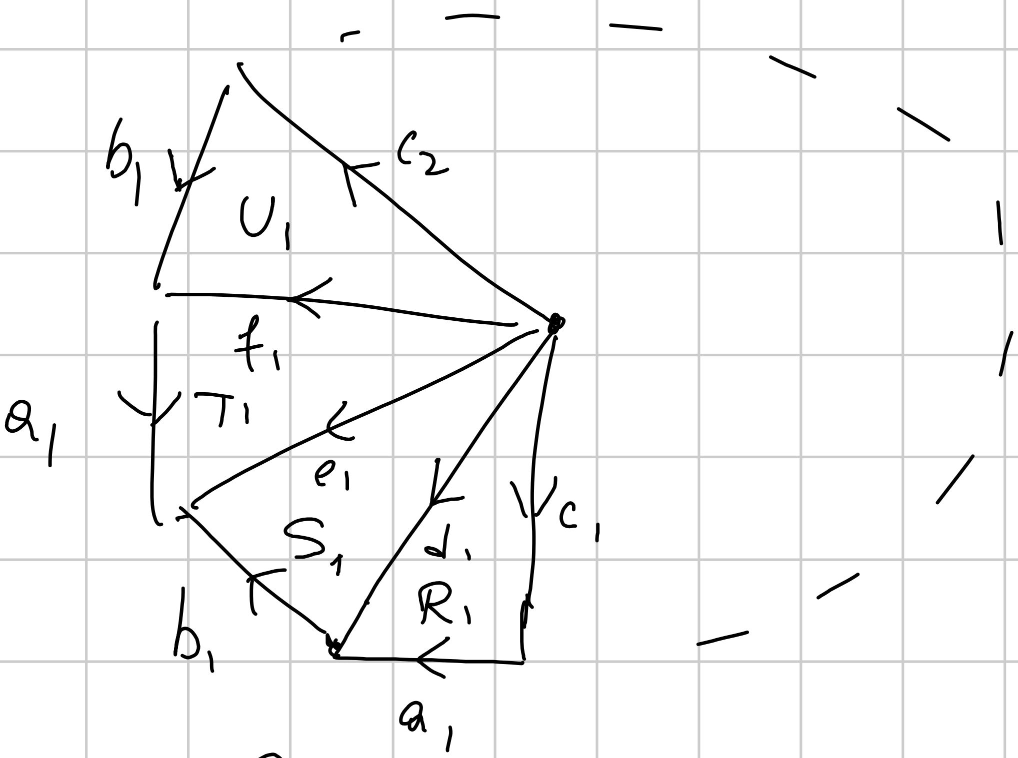
$\cup$ è funtoriale: $\quad f^*([\varphi] \cup [\psi]) = f^*([\varphi]) \cup f^*([\psi])$.

Es. 1: S = superficie di genere g.

$H_0 = \mathbb{Z} \qquad H_1 = \mathbb{Z}\langle a_1, b_1, \dots, a_g, b_g \rangle \qquad H_2 = \mathbb{Z}\langle S \rangle$

UCT $\Rightarrow$ $H^0 = \mathbb{Z} \qquad H^1 = \mathbb{Z}\langle \hat{a}_1, \hat{b}_1, \dots, \hat{a}_g, \hat{b}_g \rangle \qquad H^2 = \mathbb{Z}\langle \hat{S} \rangle$

Per $\cup$ devo triangolare (accettando adiacenze multiple/auto) –



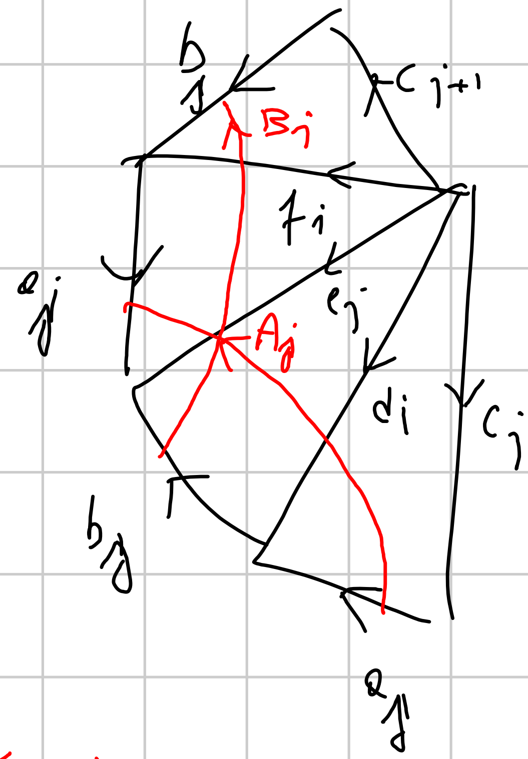
Interessa solo

$\cup : H^1 \times H^1 \to H^2$

Realizzo $\hat{a}_j, \hat{b}_j$ geometricamente come

$$\alpha_j(\ell) = \#_{alg}(A_j \cap \ell)$$

$$\beta_j(\ell) = \#_{alg}(B_j \cap \ell)$$



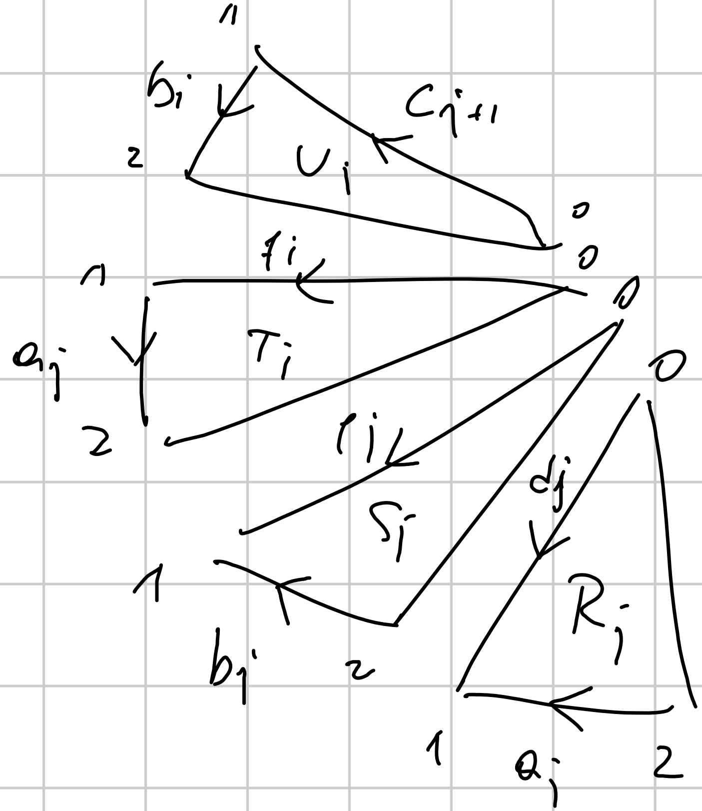
Oss: α_j, β_j sono cocicli.

se ω è curve chiuse, $\delta\omega = 0$

poiché $(\delta\omega)(T) = \omega(\partial T)$

α_j vale 1 su a_j, d_j, e_j, e 0 su tutti gli altri segmenti
β_j vale 1 su b_j, d_j, f_j, e 0 — ________

Parametrizzo i triangoli così:



$$
\begin{aligned}
(\varphi \cup \psi): \; R_j &\longmapsto \varphi(d_j)\cdot\psi(-a_j) \\
S_j &\longmapsto \varphi(e_j)\cdot\psi(-b_j) \\
T_j &\longmapsto \varphi(f_j)\cdot\psi(a_j) \\
U_j &\longmapsto \varphi(c_{j+1})\cdot\psi(b_j)
\end{aligned}
$$

Ci interessa $H^1 \times H^1 \longrightarrow H^2 = \mathbb{Z}$

$$(\varphi \cup \psi)(S) \qquad \forall \varphi, \psi \in \{\alpha_1, \beta_1, \dots, \alpha_g, \beta_g\}$$

$$\| $$

$$\sum_{j=1}^{g} (R_j + S_j + T_j + U_j)$$

Mettendo insieme: $\alpha_i \cup \alpha_j = 0 \qquad \beta_i \cup \beta_j = 0$

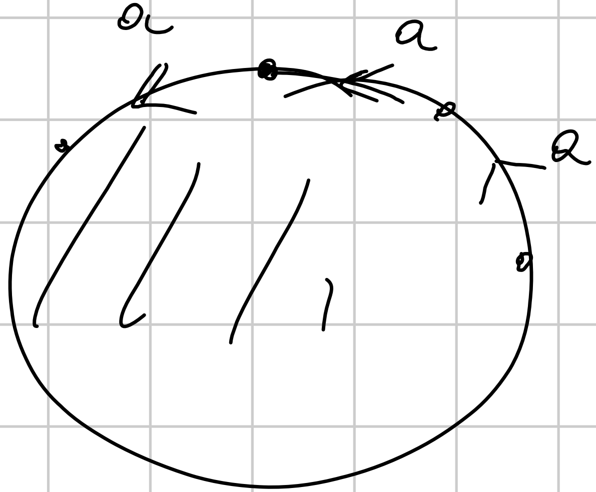
$$\alpha_i \cup \beta_j = 0 \qquad \forall i \neq j$$

$$\alpha_j \cup \beta_j = 1 \qquad \beta_j \cup \alpha_j = -1 \quad (= (-1)^{1 \cdot 1} \cdot 1)$$

$\Rightarrow \exists\, H^1 \times H^1 \longrightarrow \mathbb{Z}$ è $\langle \cdot | \cdot \rangle$

$$\begin{pmatrix} \begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix} & & \\ & \begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix} & \\ & & \ddots \end{pmatrix}$$

Es2: $P_m = \Delta \Big/ \begin{matrix} z \sim z^m \\ \text{per } z \in \partial\Delta \end{matrix}$



= attaccamento di 1 2-cella alla circonf. con $z \mapsto z^m$

$$\Rightarrow H_0 = \mathbb{Z} \qquad H_1 = \mathbb{Z}/m \langle a \rangle \qquad H_2 = 0$$

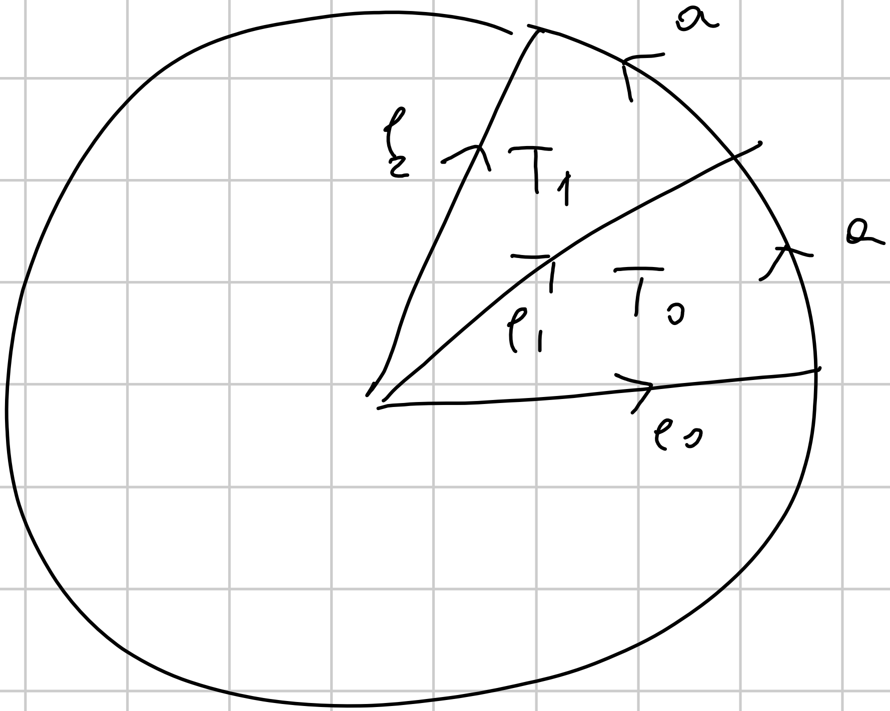
UCT $\Rightarrow H^0_{\mathbb{Z}/m} = \mathbb{Z}/m \qquad H^1_{\mathbb{Z}/m} = \mathbb{Z}/m \langle \hat{a} \rangle \qquad H^2_{\mathbb{Z}/m} = \mathbb{Z}/m$

Devo calcolare $\hat{a} \cup \hat{a}$.

$$\hat{a} \cup \hat{a} = (-1)^{1 \cdot 1} \cdot \hat{a} \cup \hat{a} \quad \Rightarrow \quad 2\, \hat{a} \cup \hat{a} = 0 \in \mathbb{Z}/m$$

$\Rightarrow \hat{a} \cup \hat{a} = 0$ se m è dispari

$\hat{a} \cup \hat{a} \in \{0, k\}$ se $m = 2k$



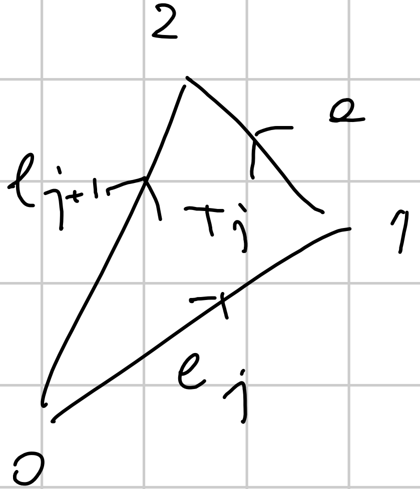
Cerco cocatene che rappresenti $\hat{a}$:

$\hat{a}(a) = 1$

$\delta_1(\hat{a}) = 0$

$$0 = (\delta_1 \hat{a})(T_j) = \hat{a}(\partial T_j) = 1 - \hat{a}(e_{j+1}) + \hat{a}(e_j)$$

$$\hat{a}(e_{j+1}) = 1 + \hat{a}(e_j)$$



Prendo $\hat{a}(e_j) = j$

$$(\varphi \cup \psi)(T_j) = \varphi(e_j) \cdot \psi(a)$$

$$\Rightarrow \hat{a} \cup \hat{a}\,(T_j) = j$$

$$\Rightarrow (\hat{a} \cup \hat{a})(P_m) = \sum_{j=0}^{m-1} j = \frac{m(m-1)}{2} = \begin{cases} 0 & m \text{ disp.} \\ k & m = 2k \end{cases}$$

$m = 2 \qquad \mathbb{P}^2(\mathbb{R})$

$$H^0_{\mathbb{Z}/2} \qquad H^1_{\mathbb{Z}/2} \qquad H^2_{\mathbb{Z}/2}$$

$$\mathbb{Z}/_2\langle 1\rangle \qquad \mathbb{Z}/_2\langle u\rangle \qquad \mathbb{Z}/_2\langle u\cup u\rangle$$

$$\Longrightarrow H^*(\mathbb{P}^2(\mathbb{R}); \mathbb{Z}/_2) = \mathbb{Z}/_2[u]\big/_{u^3}$$

<u>Teorema</u> (dualità di Poincaré):

$M^{(m)}$ chiusa orientata $\Longrightarrow H_p(M;\mathbb{Z}) \cong H^{m-p}(M;\mathbb{Z})$

$M^{(n)}$ chiusa $\Longrightarrow H_p(M; \mathbb{Z}/_2) \cong H^{n-p}(M; \mathbb{Z}/2)$.

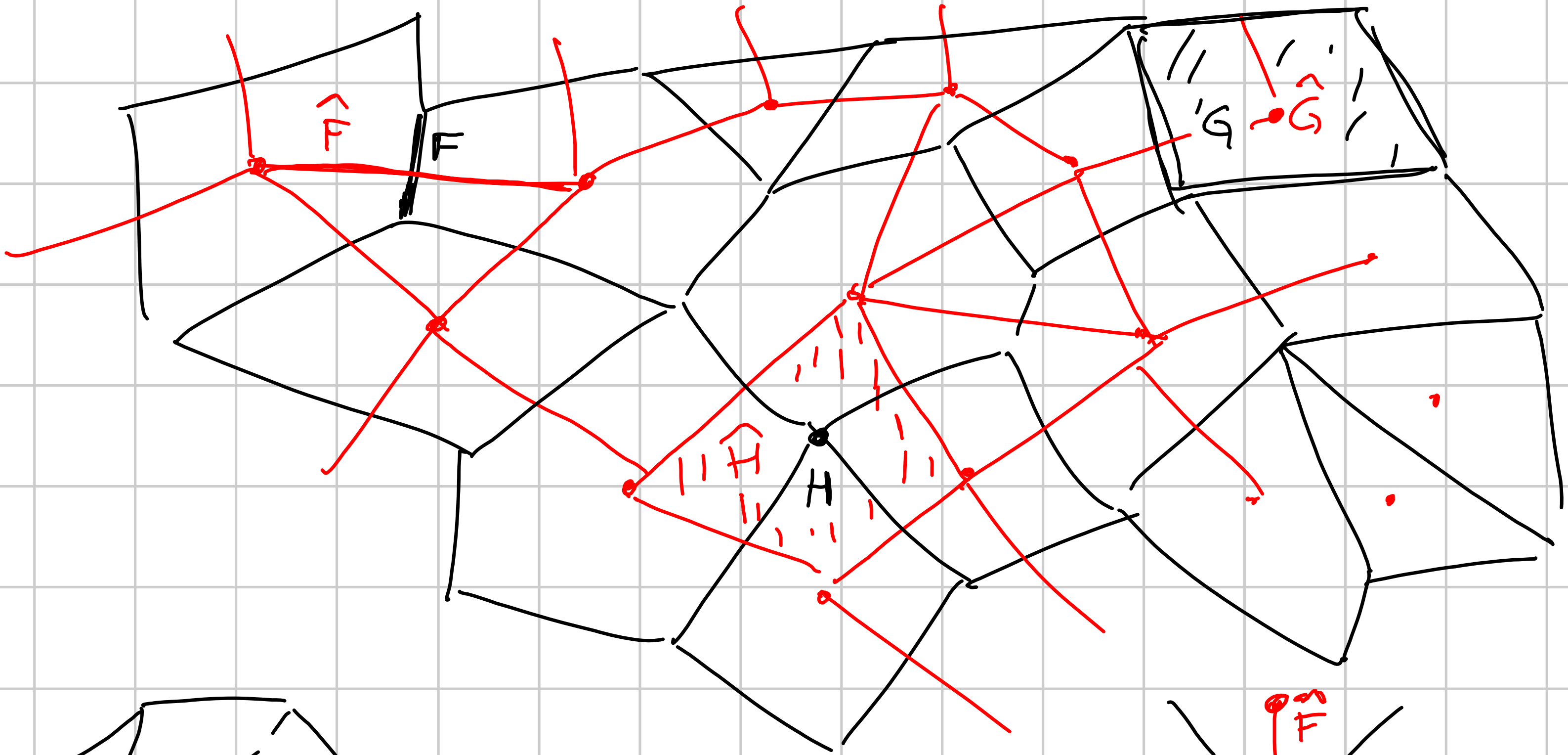
<u>Dimo</u>(PL): K realizzazione di M come complesso politopale.

$\hat{K}$ realizzazione duale;

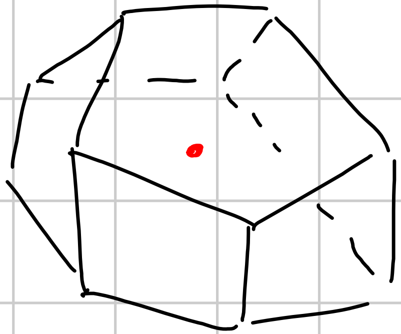
$$K^{[p]} \ni F \longleftrightarrow \hat{F} \in \hat{K}^{[n-p]}$$

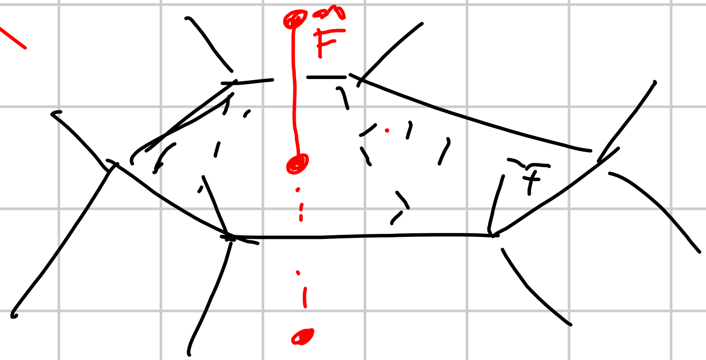
t.c. $F \pitchfork \hat{F}$ nel centro di entrambe.

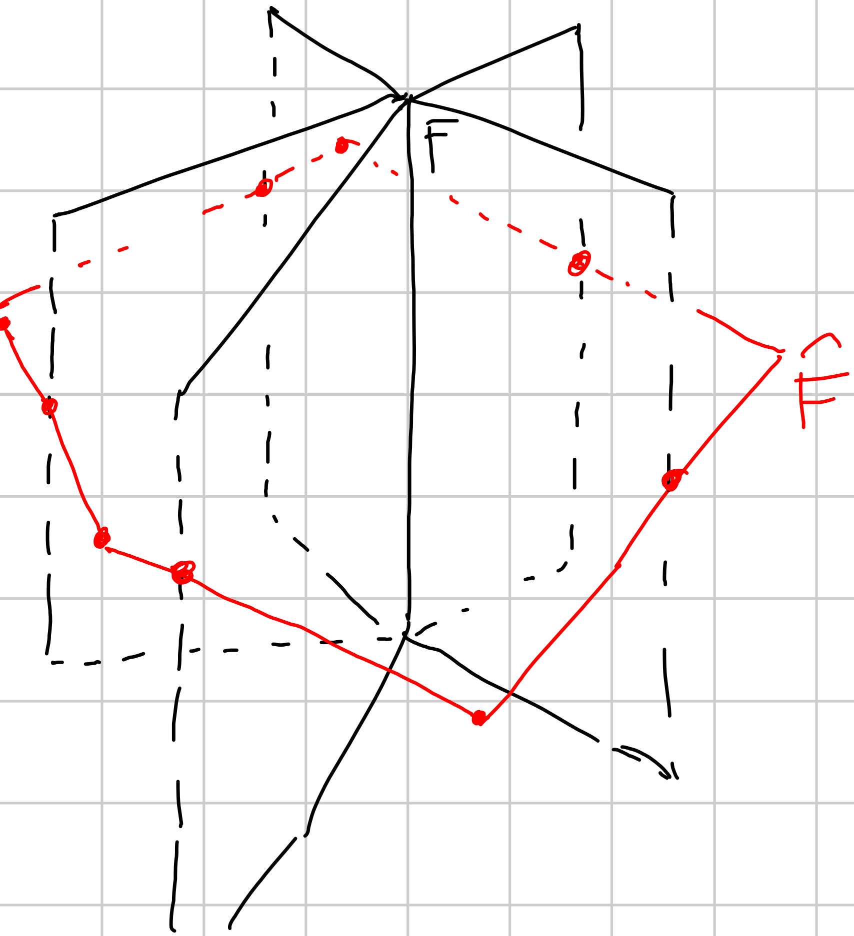
$n = 2$



$n = 3$







Per M orientata orientiamo K e $\hat{K}$ in modo che

$$(\text{base pos di } F),\,(\text{base pos di } \hat{F}) = (\text{base pos. di } M)$$

In K e $\hat{K}$ orientiamo tutti gli n-politopi come M e i vertici come $(-1)^n$ _

Chiamo $\overline{F}$ duale algebrico di F

Affermo che $\varphi_p : C_p(\hat{K}) \longrightarrow C^{n-p}(K)$

$$\hat{F} \longmapsto \overline{F}$$

soddisfa le

$$\varphi_{p-1} \circ \partial_p^{\hat{K}} = \delta_{n-p}^{K} \circ \varphi_p .$$

Questo dà la conclusione:

φ è un isomorfismo di (C_*, ∂_*) con (C^*, δ_*) che cambia i gradi $p \longleftrightarrow n-p$.

$$\hat{F} \in \hat{K}^{(p)}$$

$$G \in K^{[m-p+1]}$$

$$\left(\varphi_{p-1}\left(\partial_p^k \hat{F}\right)\right)(G)$$

$$= \sum_{\hat{Q}^{[p-1]} \subset \partial_p^k \hat{F}} \pm \overline{Q}(G) = \begin{cases} \pm 1 & \text{sc } \hat{G} \subset \partial \hat{F} \\ 0 & \text{d}\sim \end{cases}$$

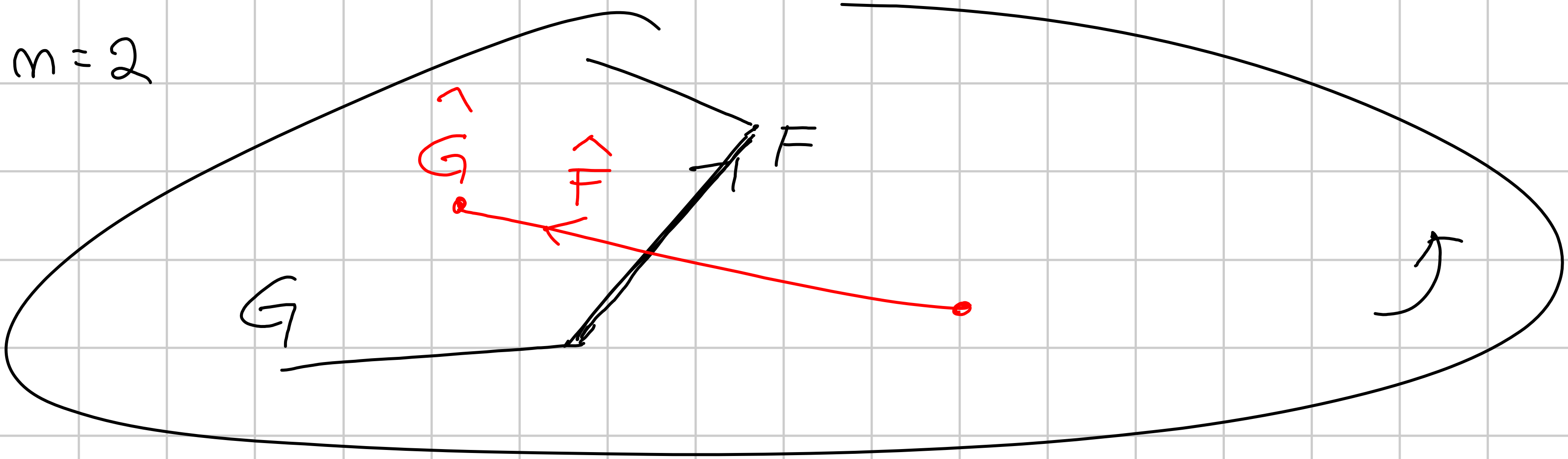
$$\left(\delta^K_{m-p}\, \varphi_p(\hat{F})\right)(G) = \varphi_p(\hat{F})\left(\partial^k_{m-p+1} G\right)$$

$$= \overline{F}\left(\sum_{R \in K^{[m-p]},\, R \subset \partial G} \pm R\right) = \begin{cases} \pm 1 & F \subset \partial G \\ 0 & \text{d}\sim \end{cases}$$

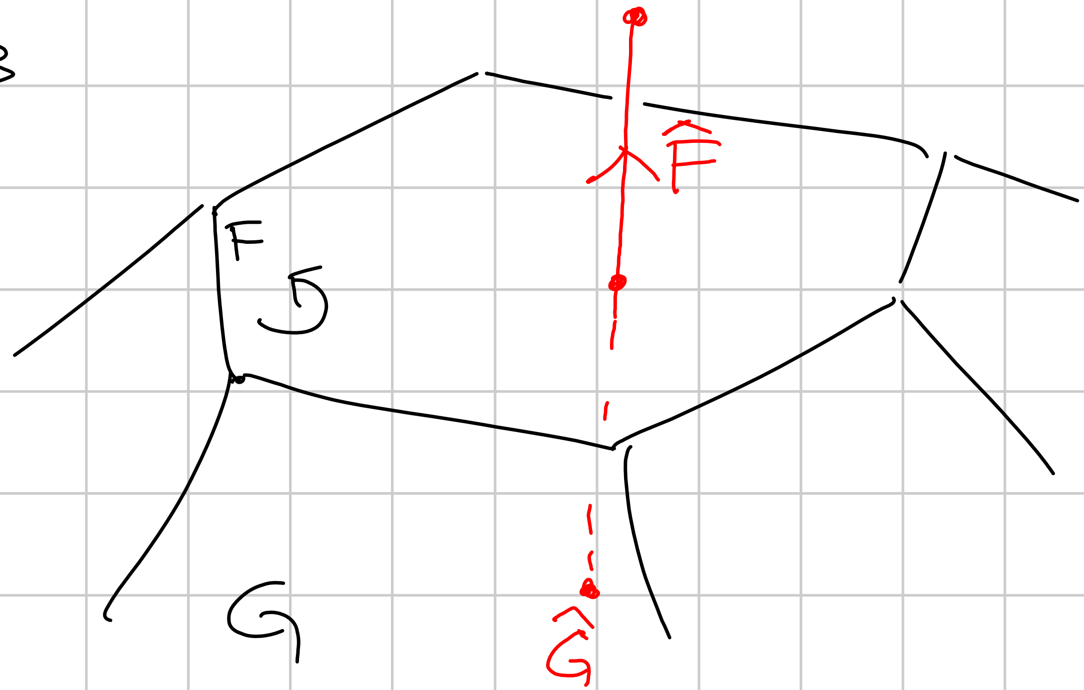
<u>Chiaro</u>: $F \subset \partial G \Longleftrightarrow \hat{G} \subset \partial \hat{F}$ su $\mathbb{Z}/2$ fine.

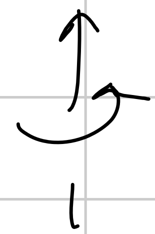
Su $\mathbb{Z}$ bisogna vedere che $+F \subset \partial G \Longleftrightarrow +\hat{G} \subset \partial \hat{F}$

Es: $n=2$

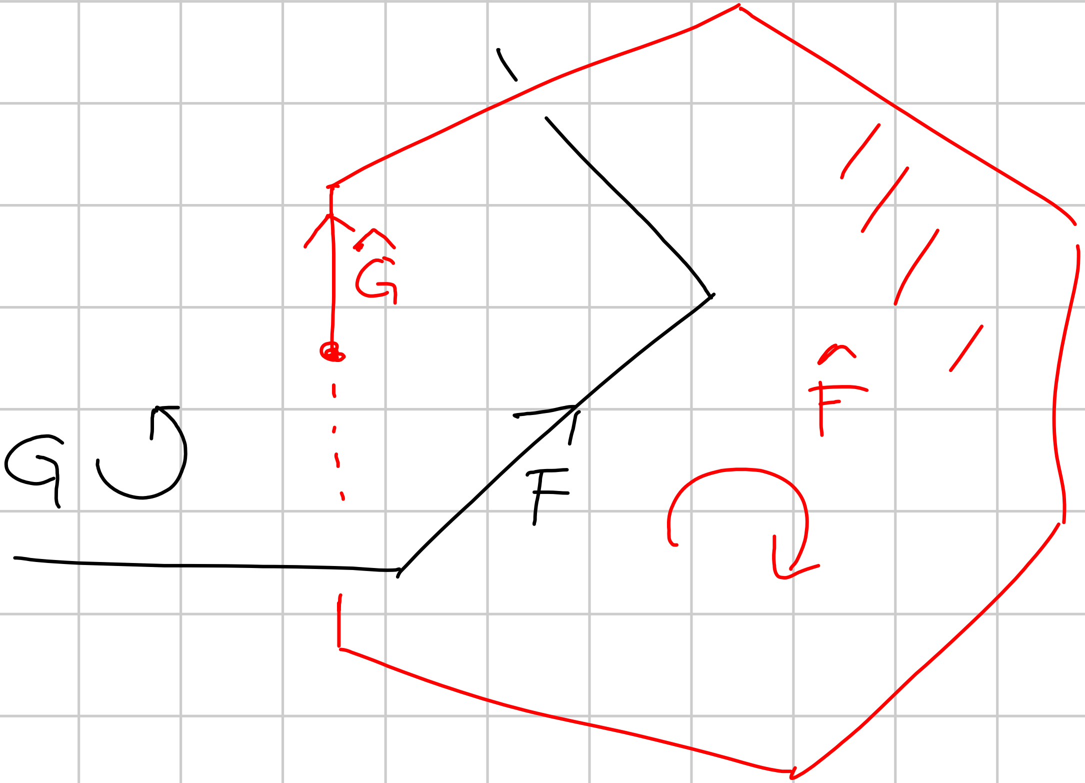


$n = 3$





$+F \subset \partial G$



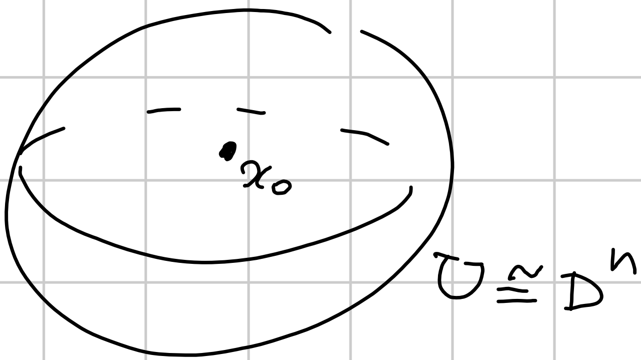
$$+F \subset \partial G$$
$$\Downarrow$$
$$+\hat{G} \subset \partial \hat{F}$$

$\blacksquare$

<u>Oss</u>: $M^{(n)}$ TOP $\partial M = \emptyset$ (chiusa)

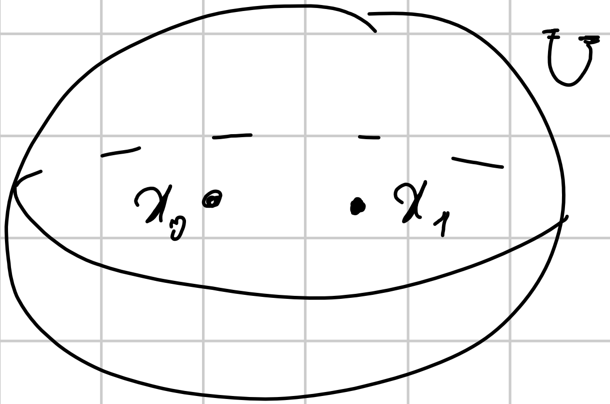
$$H_n(M, M \setminus \{x_0\}; \mathbb{Z}) \underset{\text{escissione}}{\cong} H_n(D^n, D^n \setminus \{0\}; \mathbb{Z}) \underset{\text{omotop}}{\cong} H_n(D^n, S^{n-1}; \mathbb{Z}) = \mathbb{Z}$$

canonico



non canonico!
dipende $U \xrightarrow{\cong} D^n$

Def: una orientazione di $M^{(n)}$ è la scelta di un generatore di $H_n(M, M \setminus \{x_0\}; \mathbb{Z})$ $\forall x_0 \in M$ che sia loc costante;



$$\begin{array}{c} H_n(M, M \setminus \{x_0\}) \\ \wr\Vert \\ H_n(U, \partial U) \\ \wr\Vert \\ H_n(M, M \setminus \{x_1\}) \end{array} \qquad \text{Voglio:} \quad \begin{array}{c} 1 \\ \big\downarrow \\ 1 \end{array}$$

Classe fondamentale: $[M] \in H_n(M; \mathbb{Z}) \cong \mathbb{Z}$

il generatore che viene mandato in quello scelto dall'inclusione $(M, \emptyset) \hookrightarrow (M, M \setminus \{x_0\})$.

<u>Def</u>: Se $p \leq k$

$$C_k(X) \times C^p(X) \longrightarrow C_{k-p}(X)$$

$$\sigma \cap \varphi = \varphi\left(\sigma|_{[e_0,\dots,e_p]}\right) \cdot \underbrace{\sigma|_{[e_p,\dots,e_k]}}$$

riparametrizzato via Δ_{k-p} con

$e_j \longmapsto e_{p+j}$

Lem: $\partial_{k-p}(\sigma \cap \varphi) = (-1)^p \left((\partial_k \sigma) \cap \varphi - \sigma \cap (\delta_p \varphi) \right)$

Dim: es. $k=2$ $\quad p=1$ $\qquad \sigma \in C_2 \quad \varphi \in C^1$

$$\partial_{2-1}(\sigma \cap \varphi) = \partial_1 \left(\varphi(\sigma|_{[e_0,e_1]}) \cdot \sigma|_{[e_1,e_2]} \right)$$

$$= \varphi(\sigma|_{[e_0,e_1]}) \left(\sigma(e_2) - \sigma(e_1) \right)$$

$$(-1)^1 \left((\partial_2 \sigma) \cap \varphi - \sigma \cap (\delta_1 \varphi) \right)$$

$$= -\left(\sigma|_{[e_1,e_2]} - \sigma|_{[e_0,e_2]} + \sigma|_{[e_0,e_1]}\right) \cap \varphi$$

$$\left(+\sigma|_{[e_2,e_0]}\right) \qquad + (\delta_1\varphi)\cdot\sigma(e_2)$$

$$= -\varphi(\sigma|_{[e_1,e_2]})\cdot\sigma(e_2) + \varphi(\sigma| \quad \ldots \qquad \boxed{\;}$$

<u>Con</u>:

$$Z_k \cap Z^p \subset Z_{k-p}$$
$$B_k \cap Z^p \subset B_{k-p} \quad \Longrightarrow \quad \cap : H_k \times H^p \to H_{k-p}$$
$$Z_k \cap B^p \subset B_{k-p}$$

Thm (dualità di Poincaré TOP): $[M]$ classe fondam. $\mathbb{Z}$ se M orientata

in $\mathbb{Z}/2$ se no

$$H^p(M) \longrightarrow H_{n-p}(M)$$
$$[\varphi] \longmapsto [M] \cap [\varphi]$$

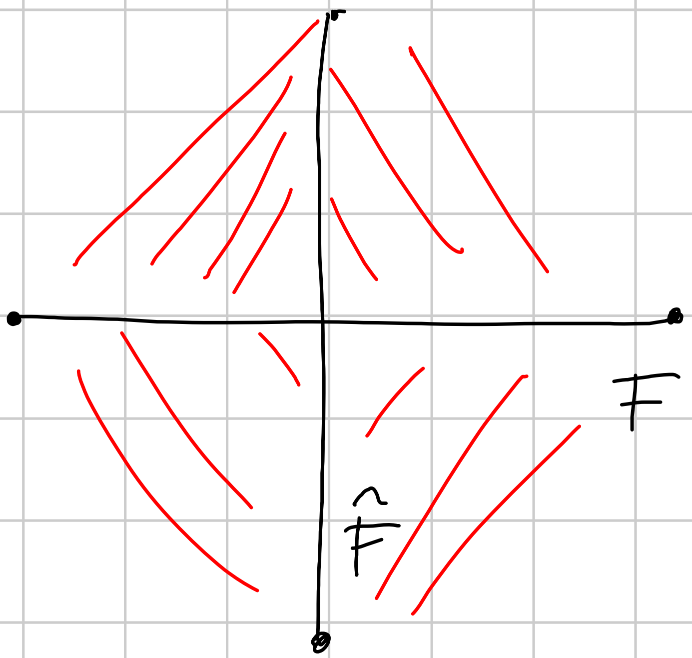
è un isomorfismo.

Coerente con prima: $H^p(M) \longrightarrow H_{n-p}(M)$

$$\overline{F} \longmapsto \hat{F} \in \hat{K}^{[n-p]}$$

$$F \in K^{[p]}$$

Ora $[M] \cap \bar{F}$



$$F \cdot \hat{F} = \{ t x + (1-t) y : t \in [0,1] \\ x \in F ,\ y \in \hat{F} \}$$

$F \cdot \bar{F}$ può essere visto come un politopo in M

$\Rightarrow [M] =$ somme di essi

$$[M] \cap \bar{F} = \bar{F}(F) \cdot \hat{F} = \hat{F}.$$